Allocation of risks and equilibrium in markets with finitely many traders

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Abstract

The optimal risk allocation problem, equivalently the optimal risk sharing problem, in a market with n traders endowed with risk measure $\varrho_1, \dots, \varrho_n$ is a classical problem in insurance and mathematical finance. This problem however makes only sense under a condition motivated from game theory which is called *Pareto equilibrium*. There are many situations of practical interest, where this condition does not hold. This is the case if the risk measures are based on essential different views towards risk. In this paper we introduce and analyze a meaningful extension of the optimal risk allocation (risk sharing) problem without assuming the equilibrium condition. The main point of this is to introduce a suitable and well motivated restriction on the class of admissible allocations which prevents effects of artificial 'risk arbitrage'. As a result we obtain a new coherent risk measure which describes the inherent risk which remains after using admissible risk exchange in an optimal way.

1 Introduction

In this paper we consider the problem of allocation of risk and characterizing equilibrium in a market with n traders endowed with risk measures $\varrho_1, \dots, \varrho_n$. A market is defined to be in a Pareto equilibrium if in a balance of supply and demand it is not possible to lower the risk of some traders without increasing the risk of some other traders. Based on the duality theory of linear programming Heath and Ku (2004) characterized the Pareto equilibrium condition in terms of the representing scenarios of the risk measures for the case of a finite space Ω of possible outcomes.

In the first part of this paper we give a new derivation of the characterization of Pareto equilibrum of Heath and Ku (2004) based on properties of related risk measures derived from $\varrho_1, \dots, \varrho_n$. In particular the infimal convolution is naturally associated to this problem. We also give an extension to the case of incomplete markets, where trading and allocation is possible only in linear subsets.

The optimal allocation of risk and the construction of reinsurance treaties is a classical problem in insurance and is of considerable practical and theoretical interest. It has a long history in the insurance literature going back to the construction of linear reinsurance treaties based upon minimizing individual and aggregate variance of risk. (For references see Seal (1969).)

In a series of important papers Borch (1960a, 1960b, 1962), Du Mouchel (1968), and Gerber (1978) showed that based on utility functions Pareto optimal risk exchanges can be characterized and in many cases lead to familiar linear quota-sharing of the total pooled losses or to stop loss contracts and to mixtures of both. Solutions are however typically not uniquely determined which may lead to the necessity to arrange substantial side payments in order to make these solutions acceptable. In several papers authors have added game theoretic considerations or additional concepts (like the concept of fairness) to arrive at a specific element in the set of Pareto optimal rules (see Borch (1960b), Lemaire (1977), Bühlmann and Jewell (1979)).

Since risk pools redistribute only actual losses and possibly the associated premiums but not the individual wealth of the company it is natural to include side constraints in the exchange protocol of the form $Y_i \geq A_i$ for the components Y_i of the allocation and some constant or random bounds A_i, to limit negative charges or payouts of company i. Similarly also upper constraints of the form $Y_i \leq A_i + B_i$ have been introduced to protect the liquidity of the individual companies. The importance of side constraints has been suggested by Borch (1968) and has formally been introduced and applied in Gerber (1978, 1979).

Several authors have extended the framework to include the presence of background risk and have considered the allocation problem also in the context of financial risks (see Leland (1980), Chavallier and Müller (1994), and Barrieu and El Karoui (2004, 2005), Dana and Scarsini (2005), Chateauneuf, Dana, and Tallon (2000), Denault (2001) and references therein). Also more general types of risk measures (distortion type, coherent, convex, comonotone risk measures) have been considered for the allocation problem. For the background literature on risk measures and their applications to finance and insurance we refer to Deprez and Gerber (1985), Kaas, Goovaerts, Dhaene, and Denuit (2001), Delbaen (2000), Delbaen (2002), and Föllmer and Schied (2004).

Our present paper is based on these developments. We consider the risk allocation problem for a market where ϱ_i are coherent risk measures. This is the frame for which in the paper of Heath and Ku (2004) the Pareto equilibrium was characterized (even if not explicitely stated in that paper). In Remark 2.9 we comment on extensions of our results to the more general case of convex risk measures. We show that the general formulation of the optimal allocation problem in the sense of minimizing the sum of risks is only well defined when the Pareto equilibrium condition holds. The main new part of this paper is concerned with the allocation problem in the case that the equilibrium condition does not hold. In this case the above formulation of the optimal allocation problem leads to inconsistencies. We introduce a suitable class of restrictions on the set of allocations which we call admissible allocations and consider the problem of optimal allocations with respect to this restricted class. In comparison to the constraints as dealt with in Gerber (1978) we postulate essentially constraints on the compensation structure of the form $|X_i| \leq |X|$ for the allocation X_i motivated as above to limit negative charges or payouts and to protect the liquidity. The bounds depend on the absolute size of the total risk X. From a mathematical point of view our side constraints are connected with a similar idea in portfolio theory, where one considers (lower bounded) admissible strategies in order to exclude strategies which allow arbitrage. As consequence we obtain a new coherent risk measure – called the *coherent admissible infimal convolution risk measure* – which describes the optimal total admissible risk $\sum_{i=1}^n \varrho_i(X_i)$ in the market. We could call this part of the risk the *inherent risk* of the allocation problem, which remains even after optimally allocating the risk to the n traders. The risk measure can be characterized as the largest coherent risk measure ϱ such that $\varrho \leq \min_i \varrho_i$ (Theorem 3.5). This result justifies our choice of restrictions as minimal kind of restrictions leading to a senseful optimal allocation problem. Based on a general version of the minimax theorem we are able to derive a simplified dual representation of this risk measure (Theorem 3.1).

The risk sharing problem is a problem where the traders minimize the total risk by some kind of exchange contracts. This can be considered as an *'optimistic attitude'* towards risk. It aims to construct an optimal admissible exchange which is typical for insurance and reinsurance contracts. In the final part of our paper we consider the opposite view from the perspective of a regulatory agent in a financial market who takes care that the individual agents (traders) have enough capital reserves to cover their part of the risk X_i in any allocation $X = \sum_{i=1}^n X_i$ to the n traders. The regulatory agent considers any possible (admissible) allocation and determines the total risk in the worst case which is the necessary total capital reserve. Therefore, we describe this situation as a situation as a *'cautious risk attitude'*. Again as a result we obtain a new coherent risk measure describing the worst case total admissible risk.

Risk measures have a long tradition in the actuarial literature – denoted there as premium calculation principles. They also received considerable attention in the financial mathematics literature more recently. Here a major need for risk measures is related to pricing in incomplete models. Complete hedging of a claim is generally not possible but even after hedging there remains a risky position. Thus the price of the claim depends on the price of the hedging portfolio but also on the attitude of the agent (trader) towards risk. There are essentially two ways to define risk measures. One way is to pose axioms on a risk functional on the set of all risks (random variables). A second way is more economically motivated and based on the preference structure of the decision maker. These two ways are essentially two equivalent ways of describing risk measures. For a presentation of these views and a decription of their connections we refer to the recent informative survey in Denuit et al. (2006).

In the final part of the introduction we give a short review of the basic notions on coherent risk measures needed throughout the paper. A coherent risk measure ϱ on a general probability space $(\Omega, \mathfrak{A}, P)$, i.e. $\varrho : L^\infty(P) \to \mathbb{R}$ is a monotone, translation invariant, subadditive homogeneous functional (see Delbaen (2002)). The associated acceptance set $\mathcal{A} = \mathcal{A}_\varrho$ is given by

$$\mathcal{A}_\varrho = \{X \in L^\infty(P);\ \ \varrho(X) \le 0\}. \tag{1.1}$$

ϱ is a coherent risk measure if and only if $\mathcal{A}_\varrho$ is a monotone convex cone and $\inf\{m \in \mathbb{R}; m \in \mathcal{A}_\varrho\} > -\infty$. Further, the following basic relation holds

$$\varrho(X) = \inf\{m \in \mathbb{R};\ \ X + m \in \mathcal{A}_\varrho\} \tag{1.2}$$

and (1.2) allows to define coherent risk measures from convex, monotone cones $\mathcal{A}$. The following representation result in terms of *scenario sets* $\mathcal{P} \subset \mathrm{ba}(P)$ – the finitely additive measures absolutely continuous to P – is essential: ϱ is a coherent risk measure if and only if there exists a convex $\sigma(\mathrm{ba}(P), L^\infty(P))$-closed set $\mathcal{P} \subset \mathrm{ba}(P)$ such that

$$\varrho(X) = \sup_{Q \in \mathcal{P}} E_Q(-X), \quad \forall X \in L^\infty(P). \tag{1.3}$$

Further, the representation scenario set $\mathcal{P}$ can be chosen in the class of probability measures absolutely continuous to P if and only if any of the following equivalent conditions hold:

a) $\mathcal{A} = \{X \in L^\infty(P);\ \ \varrho(X) \le 0\}$ is $\sigma(L^\infty(P), L^1(P))$ closed (1.4)

b) ϱ has the Fatou property, i.e. for any uniformly bounded sequence (X_n) of random variables s.t. $X_n \xrightarrow{P} X$ it holds

$$\varrho(X) \le \underline{\lim}\, \varrho(X_n). \tag{1.5}$$

c) For any uniformly bounded sequence $(X_n) \subset L^\infty(P), X_n \downarrow X$ implies

$$\varrho(X) = \lim \varrho(X_n). \tag{1.6}$$

(1.3)–(1.6) are in this general form due to Delbaen (2002).

In the following part of the paper we will mostly restrict to the representation by the finitely additive measures even if extensions to a discussion of the Fatou-property are possible.

2 Optimal allocation of risks and the related Pareto equilibrium

In this section we establish that the Pareto equilibrium notion and its characterization by Heath and Ku (2004) is naturally associated with properties of some risk measures which describe the optimal risk sharing problem in a market with n traders as introduced in section 1 (optimistic view towards risks). The main result in this section shows that the optimal risk sharing problem without constraints is well defined if and only if the Pareto equilibrium condition holds. In the second part of this section we briefly indicate an extension of these results to the case of incomplete markets.

2.1 Pareto equilibrium and related risk measures

Let $(\Omega, \mathfrak{A}, P)$ be the underlying probability space and consider a market with n traders with coherent risk measures $\varrho_1, \dots, \varrho_n$, $\varrho_i : L^\infty(P) \to \mathbb{R}$, acceptance sets $\mathcal{A}_{\varrho_i} = \{X \in L^\infty(P); \varrho_i(X) \le 0\}$ and convex, $\sigma(\mathrm{ba}(P), L^\infty(P))$-closed representing scenario sets $\mathcal{P}_i \subset \mathrm{ba}(P)$, such that

$$\varrho_i(X) = \sup_{Q \in \mathcal{P}_i} E_Q(-X), \quad 1 \le i \le n.$$

One can consider the risk allocation problem as a game in the sense of game theory. From this point of view Heath and Ku (2004) introduced and characterized the notion of Pareto equilibrium for the allocation problem.

Definition 2.1 (Pareto equilibrium) *A market model with risk measures $\varrho_1, \dots, \varrho_n$ is in Pareto equilibrium, if*

$$\text{(E)} \qquad X_i \in L^\infty(P) \textit{ with } \sum_{i=1}^n X_i = 0 \textit{ and } \varrho_i(X_i) \le 0, \quad 1 \le i \le n,$$
$$\textit{implies } \varrho_i(X_i) = 0, \quad 1 \le i \le n.$$

In a balance of supply and demand it is in equilibrium not possible to lower the risk of some traders without increasing that of others. Vaguely one could say that there is *no arbitrage* situation concerning risk. Equivalently, the equilibrium condition says that the trivial decomposition $0 = 0 + \ldots + 0$ is a Pareto optimal decomposition of zero. We define as **optimal risk allocation problem** the problem to determine the set of allocations (X_i) of X that minimize $\sum_{i=1}^n \varrho_i(X_i)$. Under the Pareto equilibrium condition this infimum is finite and the set of solutions coincides with the set of Pareto optimal allocations (see Gerber (1979, page 89, 90)). Thus minimizing the total risk of allocations is equivalent with determining Pareto optimal allocations under the equilibrium condition.

In this section we give a new derivation of the characterization of the Pareto equilibrium condition (E) due to Heath and Ku (2004). We derive this result from properties of risk measures which are naturally associated to this problem; in particular we make use of the inf-convolution risk measure $\widehat{\varrho} = \varrho_1 \wedge \dots \wedge \varrho_n$ and of a risk measure Ψ defined in terms of the acceptance sets of ϱ_i. To derive this connection we first introduce a seemingly stronger version of the Pareto equilibrium condition (E).

(SE) **Strong equilibrium**

$$\text{If } X_i \in L^\infty(P) \text{ with } \sum_{i=1}^n X_i = 0, \text{ then } \sum_{i=1}^n \varrho_i(X_i) \ge 0.$$

It is immediate to see that (SE) $\Rightarrow$ (E). Therefore, we call this condition strong equilibrium. But in fact both conditions are equivalent.

Proposition 2.2 *The equilibria conditions (E) and (SE) are equivalent.*

Proof: Assume that for some $X_i \in L^\infty(P)$ with $\sum_{i=1}^n X_i = 0$ holds $\sum_{i=1}^n \varrho_i(X_i) =: c < 0$. Then with $c_i := \varrho_i(X_i)$ and $Z_i := X_i + c_i - \frac{c}{n}$ holds: $\sum_{i=1}^n Z_i = 0$ and

$$\varrho_i(Z_i) = \varrho_i(X_i) - c_i + \frac{c}{n} = \frac{c}{n} < 0, \quad 1 \le i \le n.$$

Thus we obtain a contradiction to (E). □

Thus under the Pareto equilibrium condition (E) the sum of all risks in a balance of supply and demand situation is nonnegative. The equilibrium condition (E) is closely connected with the following risk measure Ψ defined by the acceptance set

$$\mathcal{A} := \operatorname{cone}\Big(\bigcup_{i=1}^{n} \mathcal{A}_{\varrho_i}\Big) \tag{2.1}$$

where

$$\Psi(X) := \inf\{m \in \mathbb{R};\ \ X + m \in \mathcal{A}\} \tag{2.2}$$

is the associated risk measure. All risk positions are acceptable if they are acceptable for any of the traders in the market. Thus it seems natural that Ψ is connected with an optimistic view towards risk (as introduced in section 1) and thus with the optimal risk sharing problem to minimize $\sum_{i=1}^n \varrho_i(X_i)$ over all allocations (X_i). The equilibrium condition (E) can be described in terms of the risk measure Ψ.

Proposition 2.3 Ψ *is a coherent risk measure* $\Leftrightarrow \Psi(0) = 0$
$\Leftrightarrow$ *The equilibrium condition (E) holds.*

Proof: The first equivalence is easy to check. For the second one assume that (E) holds. Then by Proposition 2.2 also (SE) holds. By definition

$$\Psi(0) = \inf\left\{m \in \mathbb{R};\ \ \exists X_i \in \mathcal{A}_{\varrho_i}, 1 \le i \le n, m = \sum_{i=1}^n X_i\right\}.$$

By (SE) for any $X_i \in \mathcal{A}_{\varrho_i}$ with

$$\sum_{i=1}^n X_i - m = (X_1 - m) + \sum_{i=2}^n X_i = 0$$

holds

$$\varrho_1(X_1 - m) + \sum_{i=2}^n \varrho_i(X_i) = \sum_{i=1}^n \varrho_i(X_i) + m \ge 0,$$

i.e. $m \ge -\sum_{i=1}^n \varrho_i(X_i) \ge 0$, since $\varrho_i(X_i) \le 0$. Thus we obtain $\Psi(0) = 0$.

Conversely, if $\Psi(0) = 0$ and if $X_i \in \mathcal{A}_{\varrho_i}$ are in balance, $\sum_{i=1}^n X_i = 0$. Then using that

$$\Psi \le \varrho_i, \quad 1 \le i \le n, \tag{2.3}$$

we obtain

$$\begin{aligned} 0 \ &= \ \Psi\Big(\sum_{i=1}^n X_i\Big) \le \sum_{i=1}^n \Psi(X_i) \\ &\le \sum_{i=1}^n \varrho_i(X_i) \le 0. \end{aligned}$$

This implies $\varrho_i(X_i) = 0, 1 \le i \le n$, i.e. (E) holds. □

As corollary we obtain a characterization of Ψ under the Pareto equilibrium condition as largest coherent risk measure below $\min \varrho_i(X)$.

Corollary 2.4 *Let the market* $((\Omega, \mathfrak{A}, P), \varrho_1, \dots, \varrho_n)$ *with coherent risk measures* ϱ_i *be in equilibrium (i.e. condition (E) holds). Then* Ψ *is the largest coherent risk measure* ϱ *such that*

$$\varrho(X) \le \min\{\varrho_i(X);\ \ 1 \le i \le n\}. \tag{2.4}$$

Proof: Let ϱ be a coherent risk measure $\varrho \le \min_{1\le i\le n} \varrho_i$. Then $X \in \mathcal{A}_{\varrho_i}$ implies that $X \in \mathcal{A}_\varrho$ and thus

$$\mathcal{A}_\varrho \supset \operatorname{cone}\left(\bigcup_{i=1}^n \mathcal{A}_{\varrho_i}\right) \tag{2.5}$$

This implies that $\varrho \le \Psi$. □

The equivalence of (E) and (SE) suggests to consider the infimal convolution $\widehat{\varrho} = \varrho_1 \wedge \cdots \wedge \varrho_n$ defined by

$$\widehat{\varrho}(X) := \inf\left\{\sum_{i=1}^n \varrho_i(X_i);\ \ \sum_{i=1}^n X_i = X\right\}. \tag{2.6}$$

$\widehat{\varrho}$ is the risk measure that describes the value of the *optimal* allocation of the total risk X to the traders in the market, such that the sum of the allocated risks is minimal.

It is easy to check that $\widehat{\varrho}$ satisfies all axioms of a coherent risk measure except possibly the conditions $\widehat{\varrho}(0) = 0$.

Proposition 2.5

1) *$\widehat{\varrho}$ is a coherent risk measure* $\Leftrightarrow \widehat{\varrho}(0) = 0 \Leftrightarrow$ *(SE) holds*
2) *Under the equilibrium condition (E) holds:*

$$\widehat{\varrho} = \Psi. \tag{2.7}$$

Proof:

1) The first equivalence has been mentioned already. The second equivalence follows from the definition of $\widehat{\varrho}$ since

$$\widehat{\varrho}(0) = \inf\left\{\sum_{i=1}^n \varrho_i(X_i);\ \ \sum_{i=1}^n X_i = 0\right\} = 0$$

is equivalent to

$$\sum_{i=1}^n X_i = 0 \ \ \Rightarrow \ \ \sum_{i=1}^n \varrho_i(X_i) \ge 0$$

i.e. to condition (SE).

2) Assume that (E) holds true. Then by 1) $\widehat{\varrho}$ is a coherent risk measure and thus by Corollary 2.4 we have $\widehat{\varrho} \le \Psi$. Conversely, for any decomposition $X = \sum_{i=1}^n X_i$ holds – using $\Psi \le \min\{\varrho_i\}$ –

$$\sum_{i=1}^n \varrho_i(X_i) \ge \sum_{i=1}^n \Psi(X_i) \ge \Psi\left(\sum_{i=1}^n X_i\right) = \Psi(X),$$

i.e. $\widehat{\varrho}(X) \ge \Psi(X)$. □

Remark 2.6 *As consequence of Proposition 2.5 we obtain that the optimal (unconstrained) allocation problem makes only sense under the Pareto equilibrium condition (E). Without condition (E) the optimal risk allocation problem leads to the inconsistency that* $\widehat{\varrho}(0) = -\infty$. *In particular without the Pareto equilibrium condition (E) it is not possible to determine Pareto optimal allocations from the optimal allocation problem.*

By the optimal proof of the general representation theorem of Delbaen (2002) (see (1.3)) there exists a $\sigma(\mathrm{ba}(P), L^\infty(P))$-closed representation set of scenarios $\mathcal{P} \subset \mathrm{ba}(P)$ such that $\varrho(X) = \sup_{Q \in \mathcal{P}} E_Q(-X)$. $\mathcal{P}$ can be chosen as the set of normed elements of the polar set $\mathcal{A}^0$ of the acceptance set $\mathcal{A} = \mathcal{A}_\varrho$ of ϱ, i.e.

$$\mathcal{P} = \{Q \in \mathrm{ba}(P);\ \ Q1 = 1, E_Q X \geq 0,\ \ \forall X \in \mathcal{A}\} = \mathcal{P}_\varrho. \tag{2.8}$$

Our aim is to describe the equilibrium condition (E) in terms of the scenario set $\mathcal{P}$ of Ψ. The following proposition says that $\mathcal{P}$ contains exactly those scenario measures which are common to all risk measures.

Proposition 2.7 *Consider the market model with coherent risk measures* ϱ_i *and representation sets* $\mathcal{P}_i \subset \mathrm{ba}(P)$, $1 \leq i \leq n$ *and assume that the Pareto equilibrium condition (E) holds: Then the representation set* $\mathcal{P} = \mathcal{P}_{\widehat{\varrho}} = \mathcal{P}_\Psi$ *is nonempty and is given by*

$$\mathcal{P} = \bigcap_{i=1}^n \mathcal{P}_i. \tag{2.9}$$

Proof: It holds

$$\begin{aligned}
\mathcal{P} &= \{Q \in \mathrm{ba}(P);\ \ E_Q X \geq 0,\ \ \forall X \in \mathrm{cone}\Big(\bigcup_{i=1}^n \mathcal{A}_{\varrho_i}\Big)\} \\
&= \{Q \in \mathrm{ba}(P);\ \ \forall i \leq n \text{ holds: } E_Q X \geq 0,\ \ \forall X \in \mathcal{A}_{\varrho_i}\} \\
&= \bigcap_{i=1}^n \{Q \in \mathrm{ba}(P);\ \ E_Q X \geq 0,\ \ \forall X \in \mathcal{A}_{\varrho_i}\} \\
&= \bigcap_{i=1}^n \mathcal{P}_i.
\end{aligned}$$

□

As consequence of Propositions 2.5 and 2.7 we obtain the characterization result of Heath and Ku (2004) for equilibrium in terms of the scenarios of the risk measures ϱ_i. This result was stated in Heath and Ku (2004) for finite models Ω and finitely generated scenario sets $\mathcal{P}_i$.

Theorem 2.8 (Characterization of equilibrium) *Consider the market with coherent risk measures* $\varrho_1, \dots, \varrho_n$. *Then the equilibrium condition (E) is equivalent to the condition*

$$\bigcap_{i=1}^n \mathcal{P}_i \neq \emptyset, \tag{2.10}$$

i.e., there exists a scenario measure Q *which is shared by all traders in the market.*

Proof: If condition (E) holds, then by Proposition 2.7 we obtain $\mathcal{P} = \bigcap_{i=1}^n \mathcal{P}_i \neq \emptyset$. Conversely, if $\bigcap_{i=1}^n \mathcal{P}_i \neq \emptyset$ and $Q \in \bigcap_{i=1}^n \mathcal{P}_i$, then

$$\widehat{\varrho}(0) = \inf\left\{\sum_{i=1}^n \varrho_i(X_i);\ \sum_{i=1}^n X_i = 0\right\} \geq \inf\left\{\sum_{i=1}^n E_Q(-X_i);\ \sum_{i=1}^n X_i = 0\right\} = 0.$$

This implies that $\widehat{\varrho}(0) = 0$ and thus by Proposition 2.5 condition (E) holds. □

Remark 2.9 *a) In comparison to the proof of Heath and Ku (2004) who reduce the characterization problem to an application of the duality theorem of linear programming, our proof is based on properties of the derived risk measures Ψ and $\widehat{\varrho}$ which are naturally associated to the problem.*

b) The infimal convolution $\widehat{\varrho}$ and the risk measure Ψ have already been introduced in the literature and applied to the problem of risk transfer (see Barrieu and El Karoui (2004, 2005), and Delbaen (2000)). In these papers also related results on the acceptance set and representation set of $\widehat{\varrho}$ are given. In particular one finds there an investigation of the Fatou-property of $\widehat{\varrho}$. The application of these risk measures to derive the equilibrium characterization result is given in our paper for the first time.

c) Based on this paper (which has been circulated in march (2005)) we have extended in a follow up paper (see Burgert and Rüschendorf (2006)) the results to the more general case of convex risk measures. In this framework the Pareto equilibrium condition is no longer equivalent to the fact that the optimal risk allocation problem is well defined, but it is a stronger condition (see Proposition 3.3 in that paper). The intersection property in Proposition 2.10 is in this case equivalent with the well posedness of the optimal allocation problem. This equivalence is also derived independently in a paper by Jouini, Schachermayer, and Touzi (2005) for the case of monetary utility functions the proof there is based on an application of convex duality theory. Under this condition also existence of Pareto optimal allocations is proved in that paper for law invariant monetary utility functions.

d) An important early paper on the allocation problem is Deprez and Gerber (1985). In that paper Deprez and Gerber characterize for convex premium prinicples (corresponding to monetary utility functions) Pareto optimal allocations (generalization of Borch's Theorem). Moreover, for the class of those premium principles, which are based on a generalized principle of utility in that paper a no trade equilibrium *premium notation is introduced. The existence of a no trade equilibrium premium is equivalent to the Pareto equilibrium notion in Definition 2.1 (see Theorem 16, 17 in Deprez and Gerber (1985)). So their paper can be considered as an original source of the notion of convex risk measure and as an early relevant contribution to the allocation problem. We thank a reviewer for a hint to this paper.*

2.2 Incomplete models

The results of section 2.1 extend directly to incomplete models where trading of the i-th trader is only possible in linear subspaces $M_i \subset L^\infty(P)$, $1 \leq i \leq n$. There are various motivations for considering restricted classes of trading sets in the literature like restricted resources, regulatory restrictions, technical restrictions. For some motivation and further references in the context of the related assignment problem we refer to (Ramachandran and Rüschendorf (2002)). We assume that the constants are contained in the trading sets $\mathbb{R} \subset M_i$ and define risk measures on the trading space $M := \sum_{i=1}^n M_i$, by defining

$$\overline{\mathcal{A}_i} := \{X_i \in M_i;\ \ \varrho_i(X_i) \leq 0\} \tag{2.11}$$

$$\overline{\mathcal{A}_M} := \operatorname{cone}\left(\bigcup_{i=1}^n \overline{\mathcal{A}_i}\right) \tag{2.12}$$

For $X \in M$ we introduce the modified version of the risk measure Ψ defined by

$$\Psi_M(X) := \inf\{m \in \mathbb{R};\ \ m + X \in \overline{\mathcal{A}_M}\}, \tag{2.13}$$

$$\widehat{\varrho}_M(X) := \inf\left\{\sum_{i=1}^n \varrho_i(X_i);\ \ X_i \in M_i, \sum_{i=1}^n X_i = X\right\}. \tag{2.14}$$

Then we obtain as in section 2.1:

$$\Psi_M \text{ is a coherent risk measure on } M \Leftrightarrow \Psi_M(0) = 0 \tag{2.15}$$

$$\widehat{\varrho}_M \text{ is a coherent risk measure on } M \Leftrightarrow \widehat{\varrho}_M(0) = 0. \tag{2.16}$$

The Pareto equilibrium condition for the incomplete market case is defined by

$$(\mathrm{E_M}) \quad X_i \in M_i \text{ with } \sum_{i=1}^n X_i = 0 \text{ and } \varrho_i(X_i) \le 0 \text{ for all } i \quad \text{implies } \varrho_i(X_i) = 0 \text{ for all } i. \tag{2.17}$$

$$\text{The Pareto equilibrium condition}(\mathrm{E_M}) \text{ is equivalent to } \Psi_M(0) = 0. \tag{2.18}$$

The corresponding strong equilibrium condition is defined by

$$(\mathrm{SE_M}) \quad X_i \in M_i \text{ and } \sum_{i=1} X_i = 0 \text{ implies } \sum_{i=1}^n \varrho_i(X_i) \ge 0. \tag{2.19}$$

$$\text{The strong equilibrium condition } (\mathrm{SE_M}) \text{ is equivalent to } \widehat{\varrho}_M(0) = 0. \tag{2.20}$$

As consequences we obtain for the incomplete case the following conclusions in a similar way as in section 2.1 for the complete case.

Proposition 2.10 *Under the Pareto equilibrium condition (E_M) holds for the incomplete model:*

1) Ψ_M *is the largest coherent risk measure on* M *such that* $\Psi_{M/M_i} \le \varrho_{i/M_i}, 1 \le i \le n$, *where* Ψ_{M/M_i} *and* ϱ_{M/M_i} *denote the restrictions of* Ψ_M *and* ϱ_i *to the trading set* M_i.

2) *The risk measures* $\widehat{\varrho}_M$ *and* Ψ_M *are identical. The corresponding scenario sets are given by*

$$\mathcal{P}_{\widehat{\varrho}_M} = \mathcal{P}_{\Psi_M} = \{Q \in \overset{\mathrm{s}}{\mathrm{ba}}(P);\ Q_{/M_i} \in \mathcal{P}_{i/M_i}, 1 \le i \le n\} \tag{2.21}$$

where $\overset{\mathrm{s}}{\mathrm{ba}}(P)$ *is the set of signed finitely additive measures absolutely continuous w.r.t.* P.

The restriction to subspaces M_i does in general not imply positivity of the representing measures – as in the complete market. In consequence we obtain as in section 2.1 an extension of the characterization of Pareto equilibria in incomplete models which for finite models Ω was given in Heath and Ku (2004).

Theorem 2.11 *In the incomplete market case the equilibrium condition (E_M) is equivalent to the following condition:*

$$\exists Q_i \in \mathcal{P}_i \textit{ and } \exists Q \in \overset{\mathrm{s}}{\mathrm{ba}}(P) \textit{ such that } Q_{/M_i} = Q_{i/M_i},\ \ 1 \le i \le n, \tag{2.22}$$

i.e., there exists a common scenario Q *on the trading spaces* M_i. Q *is a signed measure which is positive on* M_i.

3 The optimal risk allocation problem

The allocation results for the infimal convolution risk measure $\widehat{\varrho}$ in section 2.1 leave open the question how to allocate optimally risk when the equilibrium condition does not hold. It is of interest to note that in many situations of practical relevance the equilibrium condition (E) does not hold. The most simple example of this type is the case where $\varrho_i(X) = E_{Q_i}(-X)$, $1 \le i \le n$, where Q_i are P continuous probability measures, which represent the view towards risk of the i-th trader. If not all views Q_i are identical, then the equilibrium condition does not hold. Generally one can say that (E) does not hold if the views of the traders towards risk are in some sense too different. In the other direction the equilibrium condition does hold if

$$\varrho_i(X_i) \ge E(-X_i), \quad 1 \le i \le n, \tag{3.1}$$

the expectation w.r.t. P, because, then for any decomposition (X_i) of zero $\sum_{i=1}^n X_i = 0$ holds

$$\sum_{i=1}^n \varrho_i(X_i) \ge E \sum_{i=1}^n (-X_i) = 0$$

and thus $\hat{\varrho}(0) = 0$. In particular, for all law invariant convex risk measures ϱ_i, i.e. where $\varrho_i(X)$ only depends on the law of X w.r.t. P, condition (3.1) holds. Thus the equilibrium conditon holds, if $\varrho_1, \dots, \varrho_n$ are all law invariant.

Thus it is a problem of interest, to modify the risk allocation problem so that it makes sense also in case the equilibrium condition does not hold. The main idea to deal with this situation is, to restrict the class of allowed allocations in order to admit no 'pathological' allocations. This is similar to the restriction to admissible strategies in portfolio theory in order to avoid that effects like doubling strategies may occur in risk allocation. Some types of restrictions have been introduced in the insurance literature (see the introduction of this paper). Our aim is to introduce restrictions as weak as possible which still yield a senseful version of the optimal allocation problem and thus allow to determine w.r.t. this class Pareto optimal allocations resp. risk sharing strategies. We introduce at first some stricter boundedness assumptions on the allocations and show later (in Remark 3.6) that a weakened form of these restrictions leads to the same optimal allocations.

Define a decomposition $X = \sum_{i=1}^n X_i$ to be **admissible** if

$$X(\omega) \ge 0 \text{ implies that } 0 \le X_i(\omega) \le X(\omega) \text{ and } X(\omega) \le 0 \text{ implies that } X(\omega) \le X_i(\omega) \le 0.$$

Thus for $X(\omega) \ge 0$ any trader has to take some nonnegative share of the risk, while for $X(\omega) \le 0$ only non positive shares are allowed. This restriction does not permit uncontrolled 'borrowing' as e.g. unrestricted puts and calls and as a consequence prevents risk arbitrage. We will see in Theorem 3.5 that these restrictions are mathematically well justified. Economically the introduced restrictions can be motivated by a game theoretic argument considering two kind of players, the traders on one side and some regulatory agent on the other side, whose intension is to prevent 'artificial' risk reduction on the side of the traders, which one might call 'risk arbitrage'. The means to do this is to put suitable restrictions on the class of allocations. In the following we will show that the restrictions introduced have some kind of minimax character. They are the 'mildest' form of restrictions which prevent risk arbitarge. From an applied point of view restrictions as introduced here can be seen as introducing a protection against artificial risk reduction in the group of traders. This artificial risk reduction is possible if the risk measures used by the traders are too different which is described mathematically by the property that the equilibrium condition does not hold. The idea of this restriction is thus similar to the corresponding restrictions on strategies in portfolio theory.

Let $A(X)$ denote the set of admissible decompositions (X_i) of X. Our definition puts constraints on the compensation structure with respect to the losses. In comparison to

Gerber's constraints (see section 1) our constraints depend on the risk X and are not fixed upper or lower bounds on the X_i. We will show that this class of restrictions avoids the unwanted pathological effects and leads to an allocation problem which can not be improved without obtaining pathologies. It will turn out also that we can weaken boundedness of the restrictions somewhat without changing the allocation problem (see Remark 3.6).

We define the **admissible infimal convolution** ϱ_* by

$$\varrho_*(X) := \inf\left\{\sum_{i=1}^n \varrho_i(X_i);\ \ (X_i) \in A(X)\right\}. \tag{3.2}$$

ϱ_* describes the optimal total risk w.r.t. all admissible allocations. Obviously, we have $\varrho_* \le \min \varrho_i$. ϱ_* is a subadditive, homogeneous monotone risk functional – in particular $\varrho_*(0) = 0$ – but ϱ_* is not translation invariant in general. From the definition of ϱ_* we only obtain

$$\begin{aligned}\varrho_*(1) &= \inf\left\{\sum_{i=1}^n \varrho_i(X_i);\ \ 0 \le X_i, \sum_{i=1}^n X_i = 1\right\} \\ &\le \varrho_i(1) = -1.\end{aligned} \tag{3.3}$$

In the following theorem we derive an essential simplified dual representation of ϱ_* in terms of the scenario measures $\mathcal{P}_i$ of ϱ_i. To obtain this representation we make use of an alternative description of the admissible decompositions like in multiple decision problems:

$$A(X) = \left\{(\varphi_i X);\ \ 0 \le \varphi_i \le 1, \sum_{i=1}^n \varphi_i = 1\right\}. \tag{3.4}$$

For $P_i \in \mathcal{P}_i$ let $P_1 \wedge \cdots \wedge P_n$ denote the lattice infimum of (P_i) in the lattice $\mathrm{ba}(P)$ and let $P_1 \vee \cdots \vee P_n$ denote the lattice supremum of (P_i) in $\mathrm{ba}(P)$. A careful introduction to finitely additive measures and their lattice structure is given in Rao and Rao (1983), see in particular Theorem 2.2.1. In the case that P_i are probability measures with densities f_i w.r.t. μ then $P_1 \wedge \cdots \wedge P_n$ resp. $P_1 \vee \cdots \vee P_n$ have densities $\min\{f_i\}$ resp. $\max\{f_i\}$ w.r.t. μ. The admissible infimal convolution ϱ_* admits the following useful dual representation of ϱ_* using the lattice infima and suprema. This representation simplifies essentially the calculation of ϱ_* and is useful also in the following.

Theorem 3.1 *Let $\varrho_j = \varrho_{\mathcal{P}_j}$ be coherent risk measures.*

1) The admissible infimal convolution ϱ_* has the dual representation

$$\varrho_*(X) = \sup\left\{\int X_- d\bigwedge_j P_j - \int X_+ d\bigvee_j P_j;\ \ P_j \in \mathcal{P}_j, 1 \le j \le n\right\}. \tag{3.5}$$

2) $$\mathcal{A}_{\varrho_*} = \left\{X \in L^\infty(P);\ \text{such that} \int X_- d\bigwedge P_j \le \int X_+ d\bigvee P_j \ \text{for all}\ P_j \in \mathcal{P}_j\right\}. \tag{3.6}$$

Proof:

1) For $P_i \in \mathcal{P}_i$, $1 \le i \le n$ and $Y := -X \in L^\infty(P)$ holds

$$a_{P_1,\dots,P_n}(Y) := \inf\left\{\sum_{i=1}^n \int \varphi_i Y dP_i;\ \ 0 \le \varphi_i, \sum_{i=1}^n \varphi_i = 1\right\}$$

has a solution (φ_i^*) and if $Y(\omega) > 0$, then $\{\varphi_i^* > 0\} \subset \{P_i = \bigwedge_{j=1}^n P_j\}$ and if $Y(\omega) < 0$, then $\{\varphi_i^* > 0\} \subset \{P_i = \bigvee_{j=1}^n P_j\}$. Thus

$$\begin{aligned} a_{P_1,\dots,P_n}(Y) &= \int_{Y\geq 0} Y d \bigwedge_{j=1}^n P_j + \int_{Y<0} Y d \bigvee_{j=1}^n P_j \\ &= \int Y_+ d \bigwedge P_j - \int Y_- d \bigvee P_j \\ &= \int X_- d \bigwedge P_j - \int X_+ d \bigvee P_j. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \varrho_*(X) &= \inf_{(\varphi_i)} \sum_i \varrho_i(\varphi_i X) = \inf_{(\varphi_i)} \sum_i \sup_{P_i\in\mathcal{P}_i} \int (-\varphi_i X) dP_i \\ &= \inf_{(\varphi_i)} \left[-\sum_{i=1}^n \inf_{P_i\in\mathcal{P}_i} \int \varphi_i X dP_i \right] \\ &= -\sup_{(\varphi_i)} \sum_i \inf_{P_i\in\mathcal{P}_i} \int \varphi_i X dP_i \end{aligned}$$

We now apply a useful and general version of the minimax theorem which can be found in Müller (1971).

Minimax Theorem: *Let $f : A\times B \to \overline{\mathbb{R}}$, $A, B \neq \emptyset$ be a game of concave-convex type, i.e.*

1) $\forall\, b_1, b_2 \in B$, $\alpha \in [0,1]$ there exists a $b \in B$ such that for all $a \in A$ holds

$$f(a,b) \leq (1-\alpha) f(a,b_1) + \alpha f(a,b_2).$$

2) $\forall\, a_1, a_2 \in A$, $\alpha \in [0,1]$ there exists an $a \in A$ such that for all $b \in B$ holds

$$f(a,b) \geq (1-\alpha) f(a_1,b) + \alpha f(a_2,b).$$

If $f < \infty$ and for some topology τ on A holds A is τ-compact and $\forall\, b \in B$, $f(\cdot, b) : A \to \mathbb{R}$ is upper semicontinuous, then

$$\inf_{b\in B} \sup_{a\in A} f(a,b) = \sup_{a\in A} \inf_{b\in B} f(a,b). \tag{3.7}$$

We choose $A = \{(\varphi_i); 0 \leq \varphi_i, \sum \varphi_i = 1\}$, which is compact in weak$*$-topology, $B = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ and $f((\varphi_i),(P_i)) = \sum_{i=1}^n \int \varphi_i X dP_i$. By linearity of f and convexity of $\mathcal{P}_i$ and A the conditions of the minimax theorem are fulfilled and we obtain from the first part of the proof

$$\begin{aligned} \varrho_*(X) &= -\inf_{P_i\in\mathcal{P}_i} \sup_{(\varphi_i)} \sum_i \int \varphi_i X dP_i = -\inf_{P_i\in\mathcal{P}_i} \left(\int X_+ d\bigvee P_i - \int X_- d\bigwedge P_i \right) \\ &= \sup_{P_i\in\mathcal{P}_i} \left(\int X_- d\bigwedge P_i - \int X_+ d\bigvee P_i \right). \end{aligned}$$

2) follows from 1). □

An interesting consequence of Theorem 3.1 is the following characterization of the Pareto equilibrium condition (E) in terms of ϱ_*.

Proposition 3.2 *Let ϱ_i be coherent risk measures and let ϱ_* denote the admissible infimal convolution, then it holds:*

$$\begin{aligned} \varrho_* \text{ is a coherent measure} &\Leftrightarrow \varrho_*(1) = -1 \\ &\Leftrightarrow \text{The Pareto equilibrium condition (E) holds.} \end{aligned}$$

Under (E) we have $\varrho_* = \widehat{\varrho} = \Psi$.

Proof: The first equivalence is obvious since the condition $\varrho_*(1) = -1$ implies translation invariance of ϱ_*. By Theorem 3.1

$$\begin{aligned} \varrho_*(1) &= \sup\Big\{-|\bigvee_j P_j|;\ \ P_j \in \mathcal{P}_j\Big\} \\ &= -\inf\Big\{|\bigvee_j P_j|;\ \ P_j \in \mathcal{P}_j\Big\}. \end{aligned}$$

Thus $\varrho_*(1) = -1$ if and only if there exists a common scenario measure $Q \in \bigcap_{i=1}^n \mathcal{P}_i$ which by Theorem 2.8 is equivalent to the Pareto equilibrium condition (E). □

Remark 3.3 *The condition $\varrho_*(1) = -1$ has the following interpretation. The traders in the market try to allocate their risk in the best possible way which leads to a total risk $\varrho_*(1) \leq -1$ for any risk measures ϱ_i. On the other hand from a regulatory point of view the risk measures should be chosen by the traders in a cautious way in order not to underestimate the whole risk. This game theoretic consideration suggests that in order to obtain that the optimal admissible total risk is reasonable i.e. in our context is a coherent risk measure one might expect that the condition $\varrho_*(1) = -1$ should hold. This idea is confirmed by Proposition 3.2.*

In the general case we modify ϱ_* to obtain a coherent risk measure which we call coherent admissible infimal convolution.

Definition 3.4 (Coherent admissible infimal convolution) *We define the coherent admissible infimal convolution risk measure $\widehat{\varrho}_*$ by*

$$\begin{aligned} \widehat{\varrho}_*(X) &:= \inf\{m \in \mathbb{R};\ \ X + m \in \mathcal{A}\} \\ &= \inf\{m \in \mathbb{R};\ \ \varrho_*(X+m) \leq 0\} \end{aligned} \tag{3.8}$$

where $\mathcal{A} = \mathcal{A}_{\varrho_*} = \{X;\ \ \varrho_*(X) \leq 0\}$.

From the definition $\widehat{\varrho}_*$ is a coherent risk measure with acceptance set $\mathcal{A}_{\widehat{\varrho}_*} \supset \mathcal{A}_{\varrho_*} = \mathcal{A}$ and

$$\widehat{\varrho}_* \leq \varrho_* \leq \min \varrho_i. \tag{3.9}$$

The following theorem says that $\widehat{\varrho}_*$ is the largest coherent risk measure $\varrho \leq \min \varrho_i$. This is a justification for our choice of restrictions on the class of decompositions. Essentially less severe restrictions do not lead to a coherent allocation problem and thus admit pathological decompositions.

Theorem 3.5 *Let $\varrho_1, \dots, \varrho_n$ be coherent risk measures, then:*

1) Under condition (E) holds

$$\widehat{\varrho}_* = \widehat{\varrho} = \Psi$$

2) $\widehat{\varrho}_*$ is the largest coherent risk measure ϱ such that $\varrho \leq \min_i \varrho_i$.

Proof:

1) $\widehat{\varrho}_*$ is the largest coherent risk measure ϱ with $\varrho \le \varrho_*$. Under condition (E) ϱ_* is a coherent risk measure by Proposition 3.2. Therefore $\widehat{\varrho}_* = \varrho_* = \widehat{\varrho} = \Psi$.

2) If ϱ is a coherent risk measure $\varrho \le \min \varrho_i$ and if $X \in L^\infty(P)$ has a decomposition $X = \sum_{i=1}^n \varphi_i X, 0 \le \varphi_i, \sum \varphi_i = 1$, then

$$\varrho(X) \le \sum_i \varrho(\varphi_i X) \le \sum_i \varrho_i(\varphi_i X).$$

This implies that $\varrho(X) \le \varrho_*(X)$ and thus

$$\begin{aligned}\mathcal{A}_\varrho &= \{X \in L^\infty(P);\ \ \varrho(X) \le 0\}\\ &\supset \{X \in L^\infty(P);\ \ \varrho_*(X) \le 0\}\\ &= \mathcal{A}_{\varrho_*}\end{aligned}$$

As consequence we obtain

$$\begin{aligned}\varrho(X) &= \inf\{m \in \mathbb{R};\ \ X + m \in \mathcal{A}_\varrho\}\\ &\le \inf\{m \in \mathbb{R};\ \ X + m \in \mathcal{A}_{\varrho^*}\} = \widehat{\varrho}_*(X).\end{aligned}$$ □

Remark 3.6 (Enlarged class of admissible decompositions) *It is possible to enlarge the class of admissible decomposition sets without changing the total risk measure. Define*

$$\widetilde{A}(X) = \left\{(X_i);\ \ \sum_{i=1}^n X_i = X \ \text{and}\ |X_i| \le |X|\right\}. \tag{3.10}$$

Then $\widetilde{A}$ has an equivalent description by

$$\widetilde{A}(X) = \left\{(\varphi_i X);\ \ |\varphi_i| \le 1, \sum \varphi_i = 1\right\}. \tag{3.11}$$

Define $\tilde{\varrho}$ as the corresponding infimal convolution

$$\widetilde{\varrho}(X) = \inf\left\{\sum \varrho_i(X_i);\ \ (X_i) \in \widetilde{A}(X)\right\}. \tag{3.12}$$

Then $\widetilde{\varrho}$ is a subadditive, homogeneous, monotone risk functional. $\widetilde{\varrho}$ is a coherent risk measure if and only if $\widetilde{\varrho}(1) = -1$. We obtain a similar explicit representation result as that for ϱ_* in Theorem 3.1. Introducing

$$\widehat{\widetilde{\varrho}}(X) = \inf\{m \in \mathbb{R};\ \ \widetilde{\varrho}(X + m) \le 0\} \tag{3.13}$$

then $\widehat{\widetilde{\varrho}}$ is a coherent risk measure. Further for any coherent risk measure $\varrho \le \min \varrho_i$ and any $X \in L^\infty(P)$ with decomposition $(X_i) \in \widetilde{A}(X)$ holds

$$\varrho(X) \le \sum_i \varrho(X_i) \le \sum_i \varrho_i(X_i) \ \text{and thus}\ \varrho \le \widehat{\widetilde{\varrho}}. \tag{3.14}$$

This implies by Theorem 3.5 that

$$\widehat{\widetilde{\varrho}} = \widehat{\varrho}_*. \tag{3.15}$$

Thus also with the enlarged class $\widetilde{A}(X)$ of admissible decompositions we obtain the same optimal risk decomposition rule.

Remark 3.7 *Theorem 3.5 justifies the restriction to admissible decompositions for a general formulation of the optimal risk allocation problem. The coherent admissible infimal convolution $\widehat{\varrho}_*$ is in the general case (without equilibrium condition) the relevant coherent risk measure for the allocation problem describing the total 'intrinsic' risk of X after an optimal admissible allocation to the traders.*

4 Allocation of risks and cautious risk attitude

In this section we consider in a market with n traders with risk measures $\varrho_1, \dots, \varrho_n$ the allocation problem from a regulatory agent's point of view (cautious risk attitude). The aim is to apply cautious risk measurement methods in order to meet worst case situations.

So in contrast to sections 2, 3 we do not consider the problem of optimal risk sharing but take the view of a regulator and consider non cooperating traders where the total risk X is distributed in some way to the traders. The regulator would like to take care for the worst case of allocation of risk and the corresponding necessary capital to meet this situation. For the cautious risk attitude a first natural choice of risk measure is

$$\varrho_{\max}(X) := \max_i \varrho_i(X) \tag{4.1}$$

(see Delbaen (2000), Föllmer and Schied (2004)). It is easy to check that $\varrho_{\max}$ is a coherent risk measure. The acceptance set is given by

$$\begin{aligned}\mathcal{A}_{\varrho_{\max}} &= \{X \in L^\infty(P);\ \ \varrho_{\max}(X) \le 0\} \\ &= \{X \in L^\infty(P);\ \ \varrho_i(X) \le 0,\ \ \forall\, i\} \\ &= \bigcap_{i=1}^n \mathcal{A}_{\varrho_i}.\end{aligned} \tag{4.2}$$

The representing scenario set is given by

$$\mathcal{P}_{\varrho_{\max}} = \operatorname{conv}\Big(\bigcup_{i=1}^n \mathcal{P}_i\Big) \tag{4.3}$$

since

$$\begin{aligned}X \in \mathcal{A}_{\varrho_{\max}} &\Leftrightarrow X \in \mathcal{A}_{\varrho_i}, 1 \le i \le n \\ &\Leftrightarrow \forall\, Q_i \in \mathcal{P}_i : E_{Q_i} X \ge 0, 1 \le i \le n \\ &\Leftrightarrow \forall\, Q \in \operatorname{conv}\Big(\bigcup_{i=1}^n \mathcal{P}_i\Big) : E_Q X \ge 0.\end{aligned}$$

$\varrho_{\max}$ is obviously the smallest coherent risk measure majorizing ϱ_i, $1 \le i \le n$. No equilibrium condition is connected with the application of $\varrho_{\max}$.

Remark 4.1 (Incomplete markets) *$\varrho_{\max}$ can also be applied in the incomplete market situation $M \subset L^\infty(P)$, M a linear subspace and with the restrictions $\varrho_i/_M$. A result from the theory of convex cones (cf. Fuchssteiner and Lusky (1981)) implies that any $\mu \in \operatorname{ba}(P/_M)$ with $\mu \le \varrho_{\max}/_M$ has a representation of the form $\mu = \sum_{k=1}^n \lambda_k \mu_k$ with $\lambda_k \ge 0$ and $\mu_k \le \varrho_k$, $1 \le k \le n$, $\mu_k \in \operatorname{ba}(P/_M)$. Thus also in the incomplete market case we get a representation of $\varrho_{\max}/_M$ by*

$$\mathcal{P}^M_{\max} = \Big\{\mu \in \overset{\mathrm{s}}{\operatorname{ba}}(P); \mu/_M \in \operatorname{conv}\Big(\bigcup_{i=1}^n \mathcal{P}_i/_M\Big)\Big\}.$$

A more cautious total risk measure arises when the risk is allocated to the traders in the most unfavorable way. This leads to

Definition 4.2 (Supremal convolution) *For coherent risk measures $\varrho_1, \dots, \varrho_n$ define the supremal convolution $\widehat{\tau} := \varrho_1 \vee \cdots \vee \varrho_n$ by*

$$\widehat{\tau}(X) := \sup\Big\{\sum_{i=1}^n \varrho_i(X_i);\ \ \sum_{i=1}^n X_i = X\Big\}. \tag{4.4}$$

$\widehat{\tau}$ satisfies all conditions of a coherent risk measure except possibly the condition $\widehat{\tau}(0) = 0$. $\widehat{\tau}(0)$ can be interpreted as a possible hidden risk position in the market when decomposing the zero position in an unfavorable way to the traders.

To investigate this we introduce as in section 2.1 Pareto equilibrium conditions:

$$(\mathrm{E}_\tau) \qquad \sum_{i=1}^n X_i = 0 \text{ and } \varrho_i(X_i) \geq 0, \ \forall\, i \text{ implies } \varrho_i(X_i) = 0, \ \forall\, i. \tag{4.5}$$

The corresponding strengthened version of (E_τ) is

$$(\mathrm{SE}_\tau) \qquad \sum_{i=1}^n X_i = 0 \text{ implies } \sum_{i=1}^n \varrho_i(X_i) \leq 0. \tag{4.6}$$

As in Proposition 2.2 we find:

Proposition 4.3 *1) The equilibrium conditions (E_τ) and (SE_τ) are equivalent.*

2) $\widehat{\tau}$ is a coherent risk measure $\Leftrightarrow$ $\widehat{\tau}(0) = 0$
$\Leftrightarrow$ *The equilibrium condition (E_τ) holds.*

In the following proposition we establish that – in spite of the formal similarity of condition (E_τ) to (E) – except in some exceptional cases the equilibrium condition (E_τ) does not hold.

Proposition 4.4 *Consider coherent risk measures $\varrho_i = \varrho_{\mathcal{P}_i}$. Then the equilibrium condition (E_τ) holds if and only if $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n$ and $|\mathcal{P}_i| = 1$.*

Proof: We consider at first the case $n = 2$. Then (E_τ) is equivalent to:

$$\forall X \in L^\infty(P) : \sup_{Q \in \mathcal{P}_1} E_Q X - \inf_{Q \in \mathcal{P}_2} E_Q X \leq 0.$$

This is further equivalent to:

$$\forall X \in L^\infty(P) \text{ holds}: \quad \sup_{Q \in \mathcal{P}_1} E_Q X \leq \inf_{Q \in \mathcal{P}_2} E_Q X \quad \text{and} \quad \sup_{Q \in \mathcal{P}_2} E_Q X \leq \inf_{Q \in \mathcal{P}_1} E_Q X$$

considering X as well as $-X$.

Thus we obtain equivalence to

$$\begin{aligned} \sup_{Q \in \mathcal{P}_1} E_Q X &\leq \inf_{Q \in \mathcal{P}_2} E_Q X \\ &\leq \sup_{Q \in \mathcal{P}_2} E_Q X \leq \inf_{Q \in \mathcal{P}_1} E_Q X, \quad \forall X \in L^\infty(P) \end{aligned}$$

or equivalently

$$\begin{aligned} \sup_{Q \in \mathcal{P}_1} E_Q X &= \inf_{Q \in \mathcal{P}_1} E_Q X \\ &= \inf_{Q \in \mathcal{P}_2} E_Q X = \sup_{Q \in \mathcal{P}_2} E_Q X, \quad \forall X \in L^\infty(P). \end{aligned} \tag{4.7}$$

(4.7) again is equivalent to

$$\mathcal{P}_1 = \mathcal{P}_2 \text{ and } |\mathcal{P}_1| = |\mathcal{P}_2| = 1.$$

In the case $n \geq 2$ we obtain from the recursive structure of the supremal convolution

$$\widehat{\tau} = (\varrho_1 \vee \cdots \vee \varrho_{n-1}) \vee \varrho_n. \tag{4.8}$$

Thus the result for the case $n = 2$ implies:

$$\mathcal{P}_{\varrho_n} = \mathcal{P}_{\varrho_1 \vee \cdots \vee \varrho_{n-1}} = \mathcal{P}_{\varrho_{n-1}} = \cdots = \mathcal{P}_{\varrho_1} \text{ and } |\mathcal{P}_{\varrho_i}| = 1 \tag{4.9}$$

is equivalent to the equilibrium condition (E_τ). $\square$

Remark 4.5 (Pathological risk decompositions) *As consequence of Proposition 4.4 we find that the equilibrium condition (E_τ) is only satisfied in the exceptional case that $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \{P_1\}$. In all other situations it follows from the homogeneity of $\widehat{\tau}$ that*

$$\widehat{\tau}(0) = \infty. \tag{4.10}$$

Thus for any $K > 0$ there exist pathological risk decompositions (X_i) of 0

$$0 = \sum_{i=1}^{n} X_i \text{ with } \sum_{i=1}^{n} \varrho_i(X_i) \geq K. \tag{4.11}$$

Thus in the market there may be hidden allocations of the risk 0 to the traders with arbitrary large sum of risks $\sum \varrho_i(X_i)$ if any decomposition of X is taken into consideration.

As consequence of Proposition 4.4 it is necessary to prevent the regulatory agent to be too strict since this would lead to 'artificial' worst case scenarios with too high risk. It seems natural to restrict the class of admissible decompositions as in section 3. Let

$$A(X) := \left\{ (X_i) = (\varphi_i X);\ \ 0 \leq \varphi_i \leq 1, \sum_{i=1}^{n} \varphi_i = 1 \right\} \tag{4.12}$$

denote the class of admissible decompositions and define the **admissible supremal convolution**

$$\tau^*(X) := \sup \left\{ \sum_{i=1}^{n} \varrho_i(X_i);\ \ (X_i) \in A(X) \right\}. \tag{4.13}$$

Then τ^* is a subadditive homogeneous monotone risk functional (with $\tau^*(0) = 0$) but τ^* is not translation invariant in general. Obviously

$$\varrho_{\max} \leq \tau^* \leq \widehat{\tau}. \tag{4.14}$$

As in Theorem 3.1, τ^* can be calculated explicitly in terms of the representation scenarios $\mathcal{P}_i$ of ϱ_i.

Theorem 4.6 *Let $\varrho_i = \varrho_{\mathcal{P}_i}$ be coherent risk measures and τ^* the corresponding admissible supremal convolution. Then*

1) $$\tau^*(X) = \sup \left\{ \int X_- d \bigvee_i P_i - \int X_+ d \bigwedge_i P_i;\ \ P_i \in \mathcal{P}_i, 1 \leq i \leq n \right\}$$

2) $$\mathcal{A}_{\tau^*} = \left\{ X \in L^\infty(P);\ \text{such that} \int X_- d \bigvee_i P_i \leq \int X_+ d \bigwedge_i P_i,\ \ \forall P_i \in \mathcal{P}_i \right\}.$$

Proof:

1) As in the proof of Theorem 3.1 we obtain

$$\begin{aligned} \tau^*(X) &= \sup_{(\varphi_i)} \sum_i \varrho_i(\varphi_i X) \\ &= \sup_{(\varphi_i)} \sum_i \sup_{P_i \in \mathcal{P}_i} E_{P_i} \varphi_i Y, \text{ with } Y := -X \\ &= \sup_{(P_i) \in (\mathcal{P}_i)} \sup_{(\varphi_i)} \sum_i \int \varphi_i Y dP_i \\ &= \sup_{(P_i)} \left(\int Y_+ d \bigvee_i P_i - \int Y_- d \bigwedge_i P_i \right) \\ &= \sup_{(P_i)} \left(\int X_- d \bigvee_i P_i - \int X_+ d \bigwedge_i P_i \right). \end{aligned}$$

2) follows from 1). $\square$

Remark 4.7 *a) As consequence of Theorem 4.6 we see that*

$$\tau^*(1) = -\inf\Big\{\big|\bigwedge P_i\big|;\;\; P_i \in \mathcal{P}_i\Big\} \tag{4.15}$$

and thus

$$\begin{aligned}\tau^*(1) = -1 &\Leftrightarrow \big|\bigwedge P_i\big| = 1, \quad \forall P_i \in \mathcal{P}_i \\ &\Leftrightarrow \mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n \text{ and } |\mathcal{P}_i| = 1 \\ &\Leftrightarrow \text{The Pareto equilibrium condition } (E_\tau) \text{ holds.}\end{aligned} \tag{4.16}$$

b) ***Enlarged class of admissible strategies.*** *As in section 3 we can enlarge the class of admissible decompositions to* $\widetilde{A}(X) = \{(X_i);\; \sum_{i=1}^n X_i = X \text{ and } |X_i| \le |X|\}$. *We then get a corresponding risk functional*

$$\widetilde{\tau}(X) := \sup\Big\{\sum \varrho_i(X_i);\;\; (X_i) \in \widetilde{A}(X)\Big\}. \tag{4.17}$$

Similarly to Theorem 4.6 we obtain an explicit representation of $\widetilde{\tau}$ *by*

$$\begin{aligned}\widetilde{\tau}(X) = \sup\Big\{&\int X_- d\bigvee P_i - \int X_+ d\bigwedge P_i \\ &+\frac{n-1}{2}\Big(\int X_+ d\bigvee P_i - \int X_+ d\bigwedge P_i\Big);\;\; P_i \in \mathcal{P}_i, 1 \le i \le n\Big\}.\end{aligned} \tag{4.18}$$

As consequence we obtain that in contrast to the infimum case in section 2.2 the enlargement of admissible strategies leads to a different risk measure

$$\tau^*(X) \le \widetilde{\tau}(X) \tag{4.19}$$

and equality holds only under the Pareto equilibrium condition (E_τ).

Definition 4.8 (Coherent admissible supremal convolution) *The coherent admissible supremal convolution risk measure* $\widehat{\tau}^*$ *is defined by*

$$\widehat{\tau}^*(X) = \inf\{m \in \mathbb{R};\;\; X + m \in \mathcal{A}_{\tau^*}\} = \inf\{m \in \mathbb{R};\;\; \tau^*(X+m) \le 0\}, \tag{4.20}$$

where $\mathcal{A}_{\tau^*} = \{X \in L^\infty(P);\;\; \tau^*(X) \le 0\}$.

In the following proposition we characterize $\hat{\tau}^*$ by a maximality property.

Proposition 4.9 *1) The coherent admissible supremal convolution risk measure* $\widehat{\tau}^*$ *is the largest coherent risk measure* ϱ *such that*

$$\tau^* \ge \varrho \ge \varrho_{\max}.$$

2) If the ϱ_i *have the Fatou-property, then* $\mathcal{A}_{\widehat{\tau}^*} = \mathcal{A}_{\tau^*}$ *and the scenarios representation set* $\mathcal{P}^*$ *of* $\widehat{\tau}^*$ *is given by*

$$\mathcal{P}^* = \overline{\operatorname{conv}\mathcal{Q}}^{L^1(P)},$$

where $\mathcal{Q} = \Big\{\frac{Q}{|Q|} \in M^1(P);\; \exists P_i \in \mathcal{P}_i, 1 \le i \le n, \text{ with } \bigwedge P_i \le Q \le \bigvee P_i\Big\}$ *and* $M^1(P)$ *denotes the* P*-continuous probability measures.*

Proof:

1) $\widehat{\tau}^*$ is the largest coherent risk measure ϱ with $\varrho \le \tau^*$.

2) By the bipolar theorem and the representation of τ^* in Theorem 4.6 it is to show that

$$\int X_- d \bigvee P_i \leq \int X_+ d \bigwedge P_i, \quad \forall P_i \in \mathcal{P}_i$$

is equivalent to

$$\int X_- dQ \leq \int X_+ dQ, \quad \forall Q \in \mathcal{P}^*.$$

If $Q \in \mathcal{P}^*$ and $\bigwedge P_i \leq \alpha Q \leq \bigvee P_i$ with some positive constant α then

$$\alpha \int X_- dQ \leq \int X_- d \bigvee P_i \leq \int X_+ d \bigwedge P_i \leq \alpha \int X_+ dQ.$$

For the other direction let $P_i \in \mathcal{P}_i, 1 \leq i \leq n$ and define $Q := 1_{\{X \geq 0\}} \bigwedge P_i + 1_{\{X<0\}} \bigvee P_i$. Then $\frac{Q}{|Q|} \in \mathcal{P}^*$ and

$$\int X_- d \bigvee P_i = \int X_- dQ \leq \int X_+ dQ = \int X_+ d \bigwedge P_i.$$

Thus $\mathcal{P}^*$ is the representation set of $\widehat{\tau}^*$ since $\mathcal{A}_{\widehat{\tau}^*} = \mathcal{A}_{\tau^*} = (\mathcal{P}^*)^0$ and thus

$$\operatorname{conv}(\{0\} \cup \mathcal{P}^*) = (\mathcal{A}_{\widehat{\tau}^*})^0. \qquad \square$$

Conclusion: The classical risk allocation (risk sharing) problem is only well defined if the view towards risks of the traders is not too different, in mathematical terms if the Pareto equilibrium condition holds. Thus it is of interest to extend the allocation problem in a senseful way to the general case. An effective way for the formulation of a general allocation problem is to introduce restrictions on the class of admissible allocations in a suitable way. This is done in this paper for the risk sharing problem (under ‘optimistic risk attitude’) and also under a ‘cautious risk attitude’ (regulatory agents view). As consequence we obtain new relevant coherent risk measures for allocation problens. A simplified dual representation for these intrinsic risk measures is derived. The choice of the restriction conditions is justified by the fact that essentially less restrictive conditions would allow ‘pathological’ allocations (‘risk arbitrage’) and in particular would not allow to obtain Pareto optimal allocations.

Acknowledgement: The authors thank a reviewer for several useful hints and remarks on this paper.

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