On a comparison result for Markov processes

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Abstract

A comparison theorem is stated for Markov processes in polish state spaces. We consider a general class of stochastic orderings induced by a cone of real functions. The main result states that *stochastic monotonicity* of one process and comparability of the infinitesimal generators imply ordering of the processes. Several applications to convex type and to dependence orderings are given. In particular Liggett's theorem on the association of Markov processes is a consequence of this comparison result.

1 Introduction

This paper is motivated by Liggett's (1985) characterization of (positive) association in Markov processes, which is a main tool to establish this strong dependence notion. This result is the basis of many important applications and it has been modified and extended in various ways. For its role in connection with several interesting models in applied probability see in particular Szekli (1995).

Liggett's theorem is based on the notion of stochastic monotonicity and on the infinitesimal generator A of the Markov process X. The main result in our paper is on the comparison of two Markov processes X, Y with respect to a general class of stochastic orderings $\prec_{\mathcal{F}}$, induced by some cone $\mathcal{F}$ of real functions on the state space E. Stochastic monotonicity and ordering of the infinitesimal generators A, B are the basic ingredients of the comparison result.

Positive dependence of a random vector $Z = (Z_1, \dots, Y_d)$ is typically defined by a comparison with its copy $Z^\perp$ with independent components with respect to some class of (positive) dependence functions. Therefore, as a consequence of the comparison result we obtain also several results on positive dependence orderings. In particular Liggett's association theorem is a consequence of this comparison result.

Ordering conditions for Markov processes in terms of infinitesimal generators have been given in several papers. Massey (1987), Herbst and Pitt (1991), Chen and Wang (1993), Chen (2004), and Daduna and Szekli (2006) describe stochastic ordering for discrete state spaces, for diffusions and for diffusions with jumps in terms of generators. For bounded generators and in the case of discrete state spaces Daduna and Szekli (2006) give a comparison result for the stochastic ordering in terms of comparison of the generators. For an infinite dimensional system of interacting diffusions a comparison result for the directionally convex ordering has been established in Cox, Fleischmann, and Greven (1996) and Greven, Klenke, and Wakolbinger (2002) under the condition that the diffusion coefficients are comparable. For Lévy processes in Bäuerle, Müller, and Blatter (2006) as well as in Bergenthum and Rüschendorf (2007) comparison of the supermodular as well as of further orderings has been derived in terms of the corresponding ordering of the infinitesimal generator.

The proof of the main comparison result in the present paper is given in the same framework as in Liggett's result and uses a similar idea as in Liggett's proof of the characterization of association (see Szekli (1995, chapter 3.7)). The same idea of proof has also been used before in the paper of Cox et al. (1996) and Greven et al. (2002) mentioned above, for the directionally convex ordering of interacting diffusions. The author of this paper is grateful to a reviewer for a hint to these papers.

Motivated by comparison results for option prices there has been developed an alternative approach to comparison theorems based on stochastic analysis (Itô's formula and Kolmogorov's backward equation) which allows even to go beyond the frame of Markov processes to semimartingales (see Bergenthum and Rüschendorf (2007) for recent developments on this approach). For the case of Markov processes the results of Bergenthum and Rüschendorf (2007) are comparable to the results in this paper. In comparison the approach via generators in this paper is however more direct and simple.

2 The comparison result

For a homogeneous Markov process $X = (X_t)_{t\geq 0}$ with values in a compact partially ordered set E Liggett (1985) established an important criterion for the positive dependence notion of *association* of X_t, $t \geq 0$. Let X be a strongly continuous Feller process with corresponding semigroup $S = (S_t)_{t\geq 0}$ of transition operators on $C_b(E)$. Let A denote the infinitesimal generator of S with domain D_A. Then $D_A \cap \mathcal{F}_i^+$ is dense in $\mathcal{F}_i^+$ the class of bounded non-decreasing nonnegative functions on E (see Szekli (1995)).

X is called *associated in time* if for all $0 \leq t_1 < \cdots < t_k$ the vector $(X_{t_1}, \ldots, X_{t_k})$ is associated, i.e.

$$E \prod_{i=1}^k f_i(X_{t_i}) \geq \prod_{i=1}^k E f_i(X_{t_i}) \tag{2.1}$$

for all $f_i \in \mathcal{F}_i^+$. If (2.1) holds for all $t_1 \leq \cdots \leq t_k$ then we call X associated in time and space which combine association in time with the association of X_t in space.

Theorem 2.1 (Liggett (1985)) *Assume that*

1.) X *is stochastically monotone, i.e.* $f \in \mathcal{F}_i^+$ *implies* $T_t f \in \mathcal{F}_i^+$

2.) $Afg \geq fAg + gAf$ *for all* $f \in D_A \cap \mathcal{F}_i^+$ (2.2)

3.) $\mu = P^{X_0}$ *is associated,*

then X *is associated in time and space; in particular* $\mu_t = P^{X_t}$ *is associated for all* $t \geq 0$.

Theorem 2.1 was stated in Liggett (1985) for compact partially ordered metric spaces and in Szekli (1995) for products of normally ordered polish spaces. Stochastic monotonicity in the finite discrete case has been characterized by Harris (1977) and Cox (1984). The proof of Liggett's result is essentially based on a representation of a solution of a Cauchy problem for $F : [0, \infty) \to C(E)$ with $F(t) \in D_A$, $\forall t \geq 0$ and

$$F'(t) = AF(t), F(0) = f \in \mathcal{D}_A. \tag{2.3}$$

In the following we derive in a similar framework as in Liggett's theorem a comparison theorem between two Markov processes with values in a polish space E. The ordering on the set of probability measures $M^1(E)$ on E is defined by a cone $\mathcal{F}$ of real valued functions on E by

$$\mu \leq_{\mathcal{F}} \nu \quad \text{if} \int f d\mu \leq \int f d\nu, \quad \forall f \in \mathcal{F}.$$

Similarly we define $X \leq_{\mathcal{F}} Y$ for random variables X, Y in E. The order generating class is not uniquely defined and typically there are many bounded or smooth and bounded order generating classes. Typical examples of orderings described in this way are the usual stochastic ordering, various convex orderings, and dependence orderings like the concordance, the supermodular and the directionally convex ordering (for definitions and properties see Müller and Stoyan (2002)).

Let X, Y be homogeneous strongly continuous Markov processes with values in a polish space E which have the Feller property. Denote the corresponding semigroups by $S = (S_t)$, $T = (T_t)$, and the infinitesimal generators by A, B with domains D_A, D_B. Let $\mathcal{F} \subset C_b(E)$ be a cone of bounded, continuous, real functions on E and denote by $<_{\mathcal{F}}$ the corresponding 'stochastic' order on $M^1(E)$. We assume that

$$\mathcal{F} \subset D_A \cap D_B. \tag{2.4}$$

Theorem 2.2 (Conditional ordering result) *Let X, Y be homogeneous Markov processes such that*

1.) X *is* stochastically monotone, *i.e.* $S_t f \in \mathcal{F}$ *for all* $f \in \mathcal{F}$ *and*

2.) $Af \leq Bf \quad [P^{X_0}]$ *for all* $f \in \mathcal{F}$ (2.5)

then

$$S_t f \leq T_t f \quad [P^{X_o}],\ f \in \mathcal{F}. \tag{2.6}$$

Proof: Define for $f \in \mathcal{F}$, $F : [0,\infty) \to C_b(E)$ by $F(t) := T_t f - S_t f$. Then $F(t)$ satisfies the differential equation

$$\begin{aligned} F'(t) &= BT_t f - AS_t f \\ &= B(T_t f - S_t f) + (B - A)(S_t f). \end{aligned} \tag{2.7}$$

Note that by assumption $S_t f \in \mathcal{F}$ and thus $H(t) := (B - A)(S_t f)$ is well defined and $H(t) \geq 0$ by assumption (2.5). Thus F solves the Cauchy problem

$$F'(t) = BF(t) + H(t), \quad F(0) = 0 \tag{2.8}$$

The solution of (2.8) is uniquely determined and is given by (see Liggett (1985, Th. 2.15) and Szekli (1995, pg. 157))

$$\begin{aligned} F(t) &= T_t F(0) + \int_0^t T_{t-s} H(s)\, ds \\ &= \int_0^t T_{t-s} H(s)\, ds \quad \text{as } F(0) = 0. \end{aligned} \tag{2.9}$$

$H(s) \geq 0$ implies that $F(t) \geq 0$, for all t and thus the statement in (2.6). □

Remark 2.3 *a) As mentioned in the introduction the same idea of proof was used before for the case of directionally convex ordering of certain interacting diffusions in Cox, Fleischmann, and Greven (1996) and Greven, Klenke, and Wakolbinger (2002). Theorem 2.2 can be considered as a general formulation of this comparison argument.*

b) The notion of generator can be generalized to extended generators *allowing for a larger class of not necessarily bounded continuous functions in their domain* $\mathbb{D}_A$*. This is defined by the property that* $f \in \mathbb{D}_A$ *if*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Af(X_s)\, ds \quad \in \mathcal{M}, \tag{2.10}$$

where $\mathcal{M}$ *is the class of martingales (see Jacod (1979, Chapter 13)). This property is closely connected with the strong Markov property of* X*. It leads naturally to considering similar ordering properties for the more general class of semimartingales (see Bergenthum and Rüschendorf (2007)).*

c) *For several classes of examples in particular for Lévy processes, diffusion processes, and jump processes the propagation of ordering condition has been studied (see Bergenthum and Rüschendorf (2007)).*

Define the componentwise (resp. product) ordering of processes X, Y by

$$(X) \leq_{\mathcal{F}} (Y) \quad \text{if} \quad Eh(X_{t_1}, \dots, X_{t_k}) \leq Eh(Y_{t_1}, \dots, Y_{t_k}) \tag{2.11}$$

for all functions h that are componentwise in $\mathcal{F}$. In particular $(X) \leq_{\mathcal{F}} (Y)$ implies that

$$X_t \leq_{\mathcal{F}} Y_t \quad \text{for all } t \geq 0. \tag{2.12}$$

As consequence of the conditional ordering result in Theorem 2.2 and the separation theorem for the ordering Markov processes (see Bergenthum and Rüschendorf (2007, Proposition 3.1)) we obtain the following ordering result for the processes:

Corollary 2.4 (Comparison result) *If the conditions of Theorem 2.2 hold true and if additionally $X_0 \leq_{\mathcal{F}} Y_0$, then the componentwise ordering $(X) \leq_{\mathcal{F}} (Y)$ of the processes X, Y holds.*

3 Association and applications

We next derive Liggett's association result (2.5) as consequence of Theorem 2.2 and Corollary 2.4 in Section 2. Let $E = \mathbb{R}^d$ and $X = (X_t)$ be a Markov process with values in E as in the introduction. Then $\mu_t = P^{X_t}$ is associated if and only if

$$(X_t, Y_t) \leq_{\mathcal{F}} (X_t, X_t), \tag{3.1}$$

where Y is a conditionally independent copy of X, i.e. $Y_0 = X_0$, Y, X are conditionally independent given X_0 and $Y|Y_0 = x$, $X|X_0 = x$ are identically distributed. Further, $\mathcal{F}$ is defined by $\mathcal{F} = \left\{f \otimes g; f, g \in \mathcal{F}_i^+\right\}$, $f \otimes g(x,y) = f(x)g(y)$. Let $(S_t), A$ denote the semigroup resp. infinitesimal generator of X (denoted by $X \sim ((S_t), A)$. Then $(X_t, X_t) \sim ((\widetilde{S}_t), \widetilde{A})$, $(X_t, Y_t) \sim ((\widetilde{T}_t), \widetilde{B})$, where

$$\widetilde{T}_t f \otimes g(x,y) = S_t f(x) S_t g(y), \tag{3.2}$$

$$\widetilde{S}_t f \otimes g(x,y) = S_t f g(x), \tag{3.3}$$

$$\widetilde{B} f \otimes g(x,y) = Af(x)g(y) + f(x)Ag(y) \tag{3.4}$$

and

$$\widetilde{A} f \otimes g(x,y) = Afg(x). \tag{3.5}$$

For (3.4), (3.5) we use the assumption that $X_0 = Y_0$.

Corollary 3.1 (Association, Liggett (1985)) *Under the conditions 1)–3) of Theorem 2.1 holds that X is associated in time and space.*

Proof: By condition 1) X is stochastically monotone w.r.t. $\mathcal{F}_i^+$ and therefore for $f \otimes g \in \mathcal{F}$ we obtain from (3.2), (3.3) that (X_t, X_t) and (X_t, Y_t) both are stochastically monotone w.r.t. $\mathcal{F} = \mathcal{F}_i^+ \otimes \mathcal{F}_i^+$. Further Liggett's condition $Afg \geq fAg + gAf$ for $f, g \in \mathcal{F}_i^+$ implies that

$$\widetilde{A} f \otimes g \geq \widetilde{B} f \otimes g \left[P^{(X_0,Y_0)}\right] \tag{3.6}$$

Thus by the conditional comparison Theorem 2.2, we obtain $\widetilde{S}_t f \otimes g \geq \widetilde{T}_t f \otimes g \left[P^{(X_0,Y_0)}\right]$, which is equivalent to

$$S_t fg \geq S_t f S_t g \left[P^{X_0}\right] \tag{3.7}$$

Thus X_t is conditionally associated given X_0. Assumption 3 and Corollary 2.4 imply that X is associated in time and space. □

Remark 3.2 *a) Bäuerle, Müller, and Blatter (2006) show that the Liggett condition* (2.5) *yields in the case of Lévy processes the characterization of association of Lévy processes by Samorodnitsky (1995), stating that association of a Lévy process is equivalent to the property that the support of the Lévy measure is contained in the union of the positive and negative orthant of $\mathbb{R}^d$, i.e. all jumps are in the same direction.*

b) Condition 2) in Theorem 2.2 is also a necessary condition for stochastic ordering since for $f \in \mathcal{F}$, $S_t f \leq T_t f$, implies that

$$Af(x) = \lim_{t \downarrow 0} \frac{S_t f(x) - f(x)}{t} \leq \lim_{t \downarrow 0} \frac{T_t f(x) - f(x)}{t} = Bf(x).$$

Condition 1) is in general not a necessary condition.

Example 3.3 *In several cases the local comparison condition for the infinitesimal generators is easy to characterize explicitly.*

a) For pure diffusion processes X, Y in $\mathbb{R}^d$ with diffusion matrices $(a_{ij}) = (a_{ij}(x)), (b_{ij}) = (b_{ij}(x))$ the infinitesimal generators are given by

$$\begin{aligned} Af(x) &= \frac{1}{2} \sum_{ij} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} \\ Bf(x) &= \frac{1}{2} \sum_{ij} b_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}. \end{aligned} \tag{3.8}$$

Thus for convex ordering the comparison condition $Af(x) \geq Bf(x)$*,* $f \in \mathcal{F}_{cx} \cap C^2$ *is equivalent to*

$$C := A - B \geq_{psd} 0 \tag{3.9}$$

in the sense of positive semidefiniteness. The stochastic monotonicity *needs some strong conditions in dimension* $d \geq 2$ *while in* $d = 1$ *it is satisfied generally (see Bergenthum and Rüschendorf (2007) for details). Note that the application to convex ordering needs an extension to unbounded functions if the space is not compact. For the directionally convex ordering* $\geq_{dcx}$ *the corresponding ordering of the infinitesimal generator is given by the more simple comparison*

$$a_{ij}(x) \leq b_{ij}(x), \quad \forall i, j, \quad \forall x. \tag{3.10}$$

Cox, Fleischmann, and Greven (1996) and Greven, Klenke, and Wakolbinger (2002) establish for some class of (infinite dimensional) interacting diffusions that the stochastic monotonicity condition (as defined in Theorem 2.2) is fulfilled for the case that $\mathcal{F} = \mathcal{F}_{dcx}$ *the class of directionally convex functions.*

b) *For integrable Lévy processes without drift and diffusion* $X \sim (0, 0, \nu), Y \sim (0, 0, \nu^*)$*, where* ν, ν^* *are the corresponding Lévy measures, the infinitesimal generator is given by*

$$\begin{aligned} Af(x) &= \int_{\mathbb{R}^d} \Lambda f(x, y) d\nu(y) \text{resp.} \\ A^* f(x) &= \int_{\mathbb{R}^d} \Lambda f(x, y) d\nu^*(y), \end{aligned} \tag{3.11}$$

where $\Lambda f(x, y) = f(x+y) - f(x) - y \cdot \nabla f(x)$*. For the convex resp. directionally convex orderings* $\leq_{cx}, \leq_{dcx}$ *with generating functions* $\mathcal{F}_{cx}, \mathcal{F}_{dcx}$ *the stochastic monotonicity condition is satisfied as* $S_t f(x) = \int f(X_t + x) dP$*. Thus we obtain that the conditions*

$$\begin{aligned} X_0 &\leq_{cx} Y_0 \qquad (X_0 \leq_{dcx} Y_0) \\ \nu &\leq_{cx} \nu^* \qquad (\nu \leq_{dcx} \nu^*) \end{aligned} \tag{3.12}$$

imply that $X \leq_{cx} Y (X_{\leq_{dcx}} Y)$*. A similar result holds for the supermodular ordering* $\leq_{sm}$*. For* (3.12) *note that one has to pose some integrability condition on* f*. As consequence of Remark 3.2 this implies that the convex, directionally convex, and the supermodular orderings* $\leq_{cx}$*,* $\leq_{dcx}$*, and* $\leq_{sm}$ *of two Lévy processes* X *and* Y *are equivalent to the corresponding orderings of the Lévy measures* μ *and* ν*.*

A similar conclusion holds true also for the stochastic order $\leq_{st}$ *and the upper orthant resp. lower orthant orderings* $\leq_{uo}$*,* $\leq_{\ell o}$*. Note that the upper* *orthant ordering is generated by the class $\mathcal{F}_\Delta$ of Δ-monotone functions (see Rüschendorf (1980)) and thus the stochastic monotonicity condition is satisfied for Lévy processes. This is similarly true for the lower orthant ordering $\le_{\ell o}$ and thus for the combination of both orderings, the concordance ordering $\le_c$. For the case of supermodular and concordance orderings this result is stated in Bäuerle, Müller, and Blatter (2006), as well as Bergenthum and Rüschendorf (2007).*

The proof of Corollary 3.1 extends to further positive dependence orderings. Let $\mathcal{F}^+_{ism} \subset \mathcal{F}^+_i$ denote the class of increasing, nonnegative supermodular functions on $\mathbb{R}^d$. Define a random vector $Z = (Z_1, \dots, Z_d)$ to be positive supermodular associated (PSA) if

$$Ef(Z)Eg(Z) \le Ef(Z)g(Z) \tag{3.13}$$

for all $f, g \in \mathcal{F}^+_{ism}$. PSA is a weakening of the notion of association. By Christofides and Vaggelatou (2004) association of Z implies positive supermodular dependence (PSMD) i.e.

$$Z^\perp \le_{sm} Z \tag{3.14}$$

where $Z^\perp$ is a copy of Z with independent components $Z_i^\perp$, such that $Z_i^\perp \stackrel{d}{=} Z_i$. Obviously PSA of Z implies positive upper orthant dependence (PUOD) and positive concordance dependence (PCD), the combination of positive upper and lower orthant dependence.

Let $\mathcal{F}^s$ denote the cone

$$\mathcal{F}^s = \{f \otimes g; \quad f, g \in \mathcal{F}^+_{ism}\} \tag{3.15}$$

Then $f, g \in \mathcal{F}^+_{ism}$ implies that $fg \in \mathcal{F}^+_{ism}$. Thus by the representation of the semigroups and generators as in the case of association (see (3.2)–(3.5)) we get the following variant of Corollary 3.1.

Corollary 3.4 (Positive supermodular association) *Let X be a Markov process as in Section 2 and assume that*

1. *X is stochastically monotone w.r.t. $\mathcal{F}^+_{ism}$* (3.16)

2. *$Afg \ge fAg + gAf$ for all $f \in D_A \cap \mathcal{F}^+_{ism}$* (3.17)

3. *$\mu = P^{X_0}$ is PSA,*

then X is PSA in time and space; in particular $\mu_t = P^{X_t}$ is PSA for all $t \ge 0$.

Remark 3.5 *a) A similar result as in Corollary 3.4 also holds if we replace the class $\mathcal{F}_{ism}^+$ by the classes F_{idcx}^+ of nonnegative, increasing directionally convex functions or by $\mathcal{F}_\Delta^+$ the class of (increasing) Δ-monotone functions. As consequence we obtain sufficient conditions for positive increasing directionally convex dependence (PDCD) and for positive upper orthant dependence (PUOD). For this conclusion note that $f, g \in \mathcal{F}_\Delta^+$ implies that $fg \in \mathcal{F}_\Delta^+$ as can be seen for differentiable f, g by considering k-th derivatives. In a similar way we get sufficient conditions for positive lower orthant dependence (PLOD) in space. As consequence we get sufficient conditions for positive concordance dependence (PCD). For a discussion of these dependence and ordering concepts we refer to Müller and Stoyan (2002, Chapter 3.8).*

b) As particular consequence of Liggett's theorem and Corollary 3.4 consider a Lévy process X_t with $X_1 \stackrel{d}{=} N(0, \Sigma)$ starting in $X_0 = 0$. Then $Af(x) = \frac{1}{2}\Sigma_{i,j}\sigma_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}$, where $\Sigma = (\sigma_{ij})$. Liggett's condition (2.5)
$Afg \geq (Af)g + fAg$ for $f, g \in \mathcal{F}_i^+$ is equivalent to

$$\sum_{i,j} \sigma_{ij} \left[\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} + \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right] \geq 0 \text{ for all } f, g \in \mathcal{F}_i^+ \cap C_b^2 \tag{3.18}$$

and thus to $\sigma_{ij} \geq 0$ for all i, j, which is the well-known characterization of association of normal vectors (due to Pitt (1982)). The same condition also holds for the PSA condition (3.17)*. Thus for a normally distributed random vector $X \sim N(0, \Sigma)$,*

$$\begin{aligned} X \text{ is associated} &\Leftrightarrow X \text{ is PSA} \\ &\Leftrightarrow \sigma_{ij} \geq 0 \text{ for all } i, j. \end{aligned} \tag{3.19}$$

Since the PSA dependence is between the association concept and the PLOD this conclusion os however obvious.

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