Analysis of recursive algorithms by the contraction method

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ABSTRACT Several examples of the asymptotic analysis of recursive algorithms are investigated by the contraction method. The examples concern random permutations and binary search trees. In these examples it is demonstrated that the contraction method can be applied successfully to problems with contraction constants converging to one and with nonregular normalizations as logarithmic normalizations, which are typical in search type algorithms. An advantage of this approach is its generality and the possibility to obtain quantitative approximation results.

1 Introduction

The contraction method is a general method for the study of recursive algorithms. It is based essentially on the application of suitable probability metrics which reflect the structure of the algorithm. The contraction principle was introduced in its general form in Rachev and Rüschendorf [RR91]. A wellknown problem of this method and the application of probability metrics to limit theorems are nonregular normalizations of logarithmic type (which are quite typical in a wide area of algorithms) since this behaviour is not reflected in the regularity structure of probability metrics, while power normalizations n^a can be captured more easily. A second difficulty are contraction factors converging asymptotically to one.

In this paper we deal with several examples which show this problematic behaviour by the use of a modified version of the contraction method. In sections 2 and 3 we consider the number of inversions and the 'MAX'-algorithm for random permutations and in sections 4 and 5 we consider successful and unsuccessful searching in binary random trees. Each of these examples needs some special arguments in order to achieve approximation by a limit distribution; so in general the contraction method can not be considered as an 'automatic' method.

The advantage of the contraction method are its generality which allows e.g. to consider recursions in very general spaces and the fact that it often allows to obtain quantitative approximations.

The examples of this paper which are based on parts of the dissertation of Cramer [Cr95] show that the method can also deal with more difficult situations concerning the contraction factors and the normalizations.

Some of the probability metrics used in this paper are the Kolmogorov-metric ρ for real random variables X, Y defined by

$$\rho(X,Y) \ := \ \sup\left\{|F_X(x) - F_Y(x)| : x \in \mathbb{R}^1\right\}, \tag{1.1}$$

F_X, F_Y the corresponding distribution functions, and the Zolotarev-metric

$$\zeta_r(X,Y) := \sup\left\{|\mathbb{E}[f(X) - f(Y)]| : \left|f^{(m)}(x) - f^{(m)}(y)\right| \le |x-y|^\alpha\right\} \tag{1.2}$$

where $r \ = \ m + \alpha, \ m \in \mathbb{N}_0, \ 0 < \alpha \le 1$. Convergence w.r.t. ζ_r implies weak convergence.

For $\mathbb{E}\,X^j \ = \ \mathbb{E}\,Y^j$, $j = 1, \dots, m$ and $r > 1$

$$\zeta_r(X,Y) \ \le \ \frac{\Gamma(1+1/p)}{\Gamma(1+r)} \ \kappa_r(X,Y). \tag{1.3}$$

where

$$\kappa_r(X,Y) \ := \ r \int |x|^{r-1} \cdot |F_X(x) - F_Y(x)| \, \mathrm{d}x \tag{1.4}$$

is the r-difference pseudomoment. Observe that

$$\kappa_r(X,Y) \ \le \ \mathbb{E}\,|X|^r \ + \ \mathbb{E}\,|Y|^r. \tag{1.5}$$

For the general properties of these and related metrics we refer to Rachev [Ra91].

2 The number of inversions of a random permutation

In a permutation $\sigma = (a_1, \dots, a_n)$ a pair (a_i, a_j), $i < j$ is called inversion if $a_i > a_j$. Denote by I_n the number of inversions in a random permutation of size n. Then one obtains the following recursion

$$I_n \ \stackrel{d}{=} \ I_{n-1} \ + \ X_n, \ I_1 = 0 \tag{2.1}$$

where $X_n \sim \mathrm{U}(\{0, \dots, n-1\})$ is uniformly distributed on $0, \dots, n-1$ and I_{n-1}, X_n are independent. This leads to the moment generating function and first moments

$$G_n(z) \ = \mathbb{E} z^{I_n} \ = \ \frac{1}{n!} \cdot \frac{(1-z^2) \cdot \ldots \cdot (1-z^n)}{(1-z)^{n-1}} \tag{2.2}$$

$$\mathbb{E}\,I_n \ = \ \frac{n\,(n-1)}{4}, \quad \mathrm{Var}\,I_n \ = \ \frac{(n-1)\,n\,(2\,n+5)}{72} \tag{2.3}$$

(cf. Hofri [Ho87, pg. 122-124]). For the normalized version

$$\widehat{I_n} \;:=\; \frac{I_n \;-\; \mathbb{E}\, I_n}{\sqrt{\operatorname{Var} I_n}} \tag{2.4}$$

one obtains the following Berry-Esseen type result (Note that we assume that all the occuring random variables are defined on the same probability space).

Theorem 2.1 *For $n \geq 7$ holds*

$$\rho\left(\widehat{I_n}, \mathcal{N}(0,1)\right) \;\leq\; C \cdot n^{-\frac{1}{2}}, \tag{2.5}$$

with $C = 2.75 \cdot \frac{8^4}{6 \cdot 128} \sqrt{\frac{7}{6}}$.

Proof: W.l.g. we assume that $I_n = \sum_{i=1}^n X_i$ where X_i are independent, $X_i \sim \mathrm{U}(\{0,\ldots,i-1\})$. By the Berry-Esseen theorem (cf. Bhattacharya and Ranga Rao [BR76, Th. 12.4])

$$\rho\left(\widehat{I_n}, \mathcal{N}(0,1)\right) \;\leq\; 2.75\, \frac{S_{n,3}}{(S_{n,2})^{3/2}} \tag{2.6}$$

where

$$S_{n,m} \;:=\; \sum_{k=1}^{n} \mathbb{E}|X_k - \mathbb{E}\, X_k|^m. \tag{2.7}$$

We have $X_1 \equiv 0$ and for $k \geq 2$ (by some calculations)

$$\mathbb{E}|X_k - \mathbb{E}\, X_k|^3 \;\leq\; \frac{k^3}{32}, \quad \operatorname{Var} X_k \;=\; \frac{k^2-1}{12}.$$

This implies $\sum_{k=1}^n \operatorname{Var} X_k \geq \frac{(n-1)^3}{36}$ and $\sum_{k=1}^n \mathbb{E}|X_k - \mathbb{E}\, X_k|^3 \leq \frac{(n+1)^4}{128}$. So from (2.6) we obtain for $n \geq 7$

$$\rho\left(\widehat{I_n}, \mathcal{N}(0,1)\right) \;\leq\; 2.75\, \frac{6^3}{128} \left(\frac{n+1}{n-1}\right)^4 \sqrt{\frac{n}{n-1}}\, n^{-\frac{1}{2}} \;\leq\; C\, n^{-\frac{1}{2}}.$$

□

Recursion (2.1) leads to a sum of independent variables and, therefore, allows the application of the classical tools of the central limit theorem. On the other hand it is an interesting test example for the contraction method, since the contraction factors of the normalized recursion converge to one. Also the approximation result in terms of the ζ_3-metric is of independent interest. It gives the same convergence rate as in Theorem 2.1 uniformly for integrals of functions $f(I_n)$ with $\|f^{(3)}\|_\infty \leq 1$, where $\widetilde{I_n}$ is the normalization defined by

$$\widetilde{I_n} \;:=\; \frac{I_n - \mathbb{E}\, I_n}{n^{3/2}}. \tag{2.8}$$

Theorem 2.2 *Let* $\sigma_n^2 := \mathrm{Var}(\widetilde{I_n})$ *and* $Z_n \sim \mathcal{N}(0, \sigma_n^2)$, *then for some* $C > 0$ *and all* $n \in \mathbb{N}$

$$\zeta_3(\widetilde{I_n}, Z_n) \;\leq\; C\, n^{-\frac{1}{2}}. \tag{2.9}$$

Proof: $\widetilde{I_n}$ satisfies the modified recursion

$$\widetilde{I_n} \;\stackrel{d}{=}\; \left(\frac{n-1}{n}\right)^{3/2} \widetilde{I_{n-1}} \;+\; \widetilde{X_n} \tag{2.10}$$

where $\widetilde{X_n} \;:=\; \frac{X_n - \mathbb{E}\, X_n}{n^{3/2}}$. Let (Z_n) be independent $Z_n \sim \mathcal{N}(0, \sigma_n^2)$ and define the 'adapted' normal copy of (2.10)

$$Z_n^\star \;:=\; \left(\frac{n-1}{n}\right)^{3/2} Z_{n-1} \;+\; \widetilde{X_n}. \tag{2.11}$$

Let $Y_i \sim \mathcal{N}(0, \tau_i^2)$ be independent of $(\widetilde{X_i}, Z_{i-1})$ where
$\tau_i^2 := \sigma_i^2 \;-\; \left(\frac{i-1}{i}\right)^3 \sigma_{i-1}^2 = \frac{\mathrm{Var}\, X_i}{i^3} \geq 0$, then

$$Z_i \;\stackrel{d}{=}\; \left(\frac{i-1}{i}\right)^{3/2} Z_{i-1} \;+\; Y_i. \tag{2.12}$$

Using the homogeneity of order three of ζ_3 we obtain

$$\begin{aligned}
\zeta_3(\widetilde{I_n}, Z_n) \;&\leq\; \zeta_3\left(\left(\frac{n-1}{n}\right)^{3/2} \widetilde{I_{n-1}} + \widetilde{X_n}, \left(\frac{n-1}{n}\right)^{3/2} Z_{n-1} + \widetilde{X_n}\right) \\
&\quad + \zeta_3(Z_n^\star, Z_n) \\
&\leq\; \left(\frac{n-1}{n}\right)^{9/2} \zeta_3(\widetilde{I_{n-1}}, Z_{n-1}) \;+\; \zeta_3(Z_n^\star, Z_n).
\end{aligned}$$

and by iteration using $Z_1 = \widetilde{I_1} = 0$ one obtains the 'basic estimate':

$$\zeta_3(\widetilde{I_n}, Z_n) \;\leq\; \sum_{i=2}^{n} \left(\frac{i}{n}\right)^{9/2} \zeta_3(Z_i^\star, Z_i). \tag{2.13}$$

Note that $\mathbb{E}\, Z_i = \mathbb{E}\, Z_i^\star = 0$ and $\mathbb{E}\, Z_i^2 = \mathbb{E}(Z_i^\star)^2$. Therefore, by (1.3), (1.5) and (2.12) and some calculations (cf. Cramer [Cr95])

$$\begin{aligned}
\zeta_3(Z_i^\star, Z_i) \;&=\; \zeta_3\left(\widetilde{X_i} + \left(\frac{i-1}{i}\right)^{3/2} Z_{i-1}, Y_i + \left(\frac{i-1}{i}\right)^{3/2} Z_{i-1}\right) \\
&\leq\; \zeta_3\left(\widetilde{X_i}, Y_i\right) \;\leq\; \frac{\Gamma(2)}{\Gamma(4)}\, \kappa_3\left(\widetilde{X_i}, Y_i\right) \\
&=\; \int_{-\infty}^{0} x^2\, |F_{\widetilde{X}_i}(x) - F_{Y_i}(x)|\, \mathrm{d}x \\
&\leq\; \frac{7}{2^6 \cdot 3^2}\, i^{-3/2} \;+\; \frac{1}{2^5}\, i^{-5/2}.
\end{aligned}$$

Therefore, by some simple calculations

$$
\begin{aligned}
\zeta_3(\widetilde{I_n}, Z_n) &\le \sum_{i=2}^{n} \left(\frac{i}{n}\right)^{9/2} \zeta_3(Z_i^\star, Z_i) \\
&\le \sum_{i=2}^{n} \left(\frac{i}{n}\right)^{9/2} \left(\frac{1}{2^5}\, i^{-5/2} \;+\; \frac{7}{2^6\cdot 3^2}\, i^{-3/2}\right) \\
&\le \frac{1}{n^{9/2}} \left[\frac{1}{2^5}\left(n^2+\frac{1}{3}\,n^3\right) \;+\; \frac{7}{2^6\cdot 3^2}\left(n^3+\frac{1}{4}\,n^4\right)\right] \\
&= \frac{7}{2^8\cdot 3^2}\cdot\frac{1}{\sqrt{n}} \;+\; \mathcal{O}\left(n^{-\frac{3}{2}}\right).
\end{aligned}
$$

□

Note that the contraction factor in this example is only of order $\left(\frac{n-1}{n}\right)^{\frac{3}{2}}$ and so we can not obtain a uniform bound but have to estimate carefully the individual terms.

3 The number of records

The number of records in a sequence $x_1, \ldots, x_n$ is relevant for the behaviour of the 'MAX'-algorithm which determines the maximum element of the sequence (cf. Hofri [Ho87, pg. 112/113]). Let M_n denote the number of maxima of a random permutation read from the left to the right. Then one obtains the recursion

$$M_n \overset{d}{=} M_{n-1} \;+\; X_n, \tag{3.1}$$

where $X_n \sim \mathfrak{B}\left(1, \frac{1}{n}\right)$ and X_n, M_{n-1} are independent. Define $M_1 = 0$, then

$$M_n \overset{d}{=} \sum_{i=2}^{n} X_i \tag{3.2}$$

(X_i) independent, and

$$\mathbb{E}\, M_n = \mathrm{H}_n \;-\; 1, \quad \operatorname{Var} M_n \;=\; \mathrm{H}_n \;-\; \mathrm{H}_n^{(2)} \tag{3.3}$$

where $\mathrm{H}_n^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k}$, $\quad \mathrm{H}_n = \mathrm{H}_n^{(1)} = \ln n + \gamma + \mathcal{O}\left(n^{-1}\right)$,
$\mathrm{H}_n^{(2)} \underset{n\to\infty}{\longrightarrow} \zeta(2) \;=\; \frac{\pi^2}{6}$ (cf. Hofri [Ho87]).

Define the normalized version

$$\widehat{M_n} \;:=\; \frac{M_n - \mathbb{E}\, M_n}{\sqrt{\operatorname{Var} M_n}}, \tag{3.4}$$

then as in section 2 we obtain the normal approximation but with logarithmic rate only.

Theorem 3.1 *For all $n \in \mathbb{N}$ and some constant $C > 0$ holds*

$$\rho\left(\widehat{M_n}, \mathcal{N}(0,1)\right) \;\leq\; \frac{C}{\sqrt{\ln n}}. \tag{3.5}$$

Proof: Follows from the Berry-Esseen bound (2.6), where $\mathbb{E}\, X_k = \frac{1}{k}$, $\operatorname{Var} X_k = \frac{k-1}{k^2}$, $\mathbb{E}|X_k - \mathbb{E}\, X_k|^3 = \frac{k^3-3k^2+4k-2}{k^4}$.

Therefore, $\sum_{k=2}^n \mathbb{E}|X_k - \mathbb{E}X_k|^3 \sim \ln n$ and $\sum_{k=2}^n \operatorname{Var} X_k \sim \ln n$ and so we obtain (3.5) (C can easily be calculated in explicit form). □

The normalization of M_n is logarithmic in n. For a convergence result as in (3.5) by the contraction method we shall make use of the ζ_3-metric. It turns out that in this example one obtains contraction factors of order $\sqrt{\frac{\ln(n-1)}{\ln n}}$ which converge fast to one. Nevertheless by careful estimation of the individual terms the method as expounded in the proof of Theorem 2.2 also can be applied to this case. Define

$$\widetilde{M_n} \;:=\; \frac{M_n - \mathbb{E}\, M_n}{\sqrt{\ln n}}. \tag{3.6}$$

Theorem 3.2 *For $\sigma_n^2 := \operatorname{Var} \widetilde{M_n}$, $Z_n \sim \mathcal{N}(0, \sigma_n^2)$ holds*

$$\zeta_3(\widetilde{M_n}, Z) \;=\; \mathcal{O}\left(\frac{1}{\sqrt{\ln n}}\right). \tag{3.7}$$

Proof: $\widetilde{M_n}$ satisfies the recursion

$$\widetilde{M_n} \;\stackrel{d}{=}\; \sqrt{\frac{\ln(n-1)}{\ln n}}\, \widetilde{M_{n-1}} \;+\; \widetilde{X_n} \tag{3.8}$$

where $\widetilde{X_n} := \frac{X_n - \mathbb{E}\, X_n}{\sqrt{\ln n}}$. Let (Z_n) be independent, $Z_n \sim \mathcal{N}(0, \sigma_n^2)$,

$$Z_n^\star \;:=\; \sqrt{\frac{\ln(n-1)}{\ln n}}\, Z_{n-1} \;+\; \widetilde{X_n} \tag{3.9}$$

the normal 'copy' and

$$Y_n \;\sim\; \mathcal{N}(0, \tau_n^2), \quad \tau_n^2 \;:=\; \sigma_n^2 \;-\; \frac{\ln(n-1)}{\ln n}\, \sigma_{n-1}^2 \;=\; \operatorname{Var} \widetilde{X_n} \tag{3.10}$$

then

$$Z_n \;\stackrel{d}{=}\; \sqrt{\frac{\ln(n-1)}{\ln n}}\, Z_{n-1} \;+\; Y_n. \tag{3.11}$$

and as in section 2

$$\begin{aligned} \zeta_3(\widetilde{M_n}, Z_n) \;&\leq\; \zeta_3(\widetilde{M_n}, Z_n^\star) + \zeta_3(Z_n^\star, Z_n) \\ &\leq\; \left(\frac{\ln(n-1)}{\ln n}\right)^{3/2} \zeta_3(\widetilde{M_{n-1}}, Z_{n-1}) \;+\; \zeta_3(Y_n, \widetilde{X_n}) \end{aligned}$$

which yields by iteration

$$\zeta_3(\widetilde{M_n}, Z_n) \;\le\; \left(\frac{\ln 2}{\ln n}\right)^{3/2} \zeta_3(\widetilde{M_2}, Z_2) \;+\; \sum_{i=3}^{n} \left(\frac{\ln i}{\ln n}\right)^{3/2} \zeta_3(Y_i, \widetilde{X_i}). \quad (3.12)$$

By the moment estimate $\zeta_3(\widetilde{M_2}, Z_2) < \infty$
and since $\operatorname{Var}\widetilde{X_i} = \operatorname{Var} Y_i = \tau_i^2 = \frac{1}{\ln i} \cdot \frac{i-1}{i^2}$

$$\begin{aligned}
\zeta_3(Y_i, \widetilde{X_i}) \;&\le\; \frac{1}{6}\left(\mathbb{E}|Y_i|^3 \;+\; \mathbb{E}|\widetilde{X_i}|^3\right) \\
&\le\; \frac{1}{(\ln i)^{3/2}} \cdot \frac{1}{6}\left(\frac{1}{i} + \frac{\sqrt{8}}{\sqrt{\pi}} \cdot \frac{1}{i\sqrt{i}}\right) \qquad (3.13)
\end{aligned}$$

using that $\mathbb{E}\left|X_i - \frac{1}{i}\right|^3 \le \frac{1}{i}$.

Therefore, we finally obtain from (3.12)

$$\begin{aligned}
\zeta_3(\widetilde{M_n}, Z_n) \;&\le\; \frac{1}{6} \cdot \left(\frac{\ln 2}{\ln n}\right)^{3/2} \\
&\quad + \sum_{i=3}^{n} \left(\frac{\ln i}{\ln n}\right)^{3/2} \cdot \frac{1}{(\ln i)^{3/2}} \cdot \frac{1}{6}\left(\frac{1}{i} + \frac{\sqrt{8}}{\sqrt{\pi}} \cdot \frac{1}{i\sqrt{i}}\right) \\
&=\; \frac{1}{6\,(\ln n)^{3/2}}\left[(\ln 2)^{3/2} + \sum_{i=3}^{n} \frac{1}{i} + \sum_{i=3}^{n} \frac{1}{i\sqrt{i}} \cdot \frac{\sqrt{8}}{\sqrt{\pi}}\right] \\
&\le\; \frac{1}{6\,(\ln n)^{3/2}}\left[(\ln 2)^{3/2} + 2\ln n\right] \\
&=\; \frac{1}{3} \cdot \frac{1}{\sqrt{\ln n}} \;+\; \mathcal{O}\left((\ln n)^{-3/2}\right).
\end{aligned}$$

□

4 Unsuccessful searching in binary search trees

An introduction to random search trees is given in Mahmoud [Ma92]. In this and the following section we deal with the analysis of inserting and retrieving randomly ordered data in binary search trees by the contraction method.

Let U_n denote the number of comparisons necessary to insert a new random element in a random search tree. A search tree is called random if it arises from a random permutation. An element to be inserted in a tree is called random if any of the $n+1$ free leaves of the tree has probability $\frac{1}{n+1}$ to be attained.

U_n satisfies the recursion

$$U_n \stackrel{d}{=} U_{n-1} + Y_n, \quad U_0 \;=\; 0, \qquad (4.1)$$

where U_{n-1}, Y_n are independent, $Y_n \sim \mathfrak{B}(1, \frac{2}{n+1})$. For $n = 1$ one comparison with the root is necessary. For $n \geq 2$ insertion of the $(n+1)$-th element needs as many comparisons in the n-tree as in the $(n-1)$-tree except in the case that one comparison with the n-th element is necessary. The probability that no comparison with this element is necessary is $\frac{n-1}{n+1}$.

From (4.1) one obtains

$$\mathbb{E}\, U_n = 2\,(\mathrm{H}_{n+1} - 1), \quad \mathrm{Var}\, U_n = 2\,\mathrm{H}_{n+1} - 4\,\mathrm{H}_{n+1}^{(2)} + 2\,. \tag{4.2}$$

Brown and Shubert (1984) (cf. Mahmoud [Ma92, pg. 76]) proved a central limit theorem for U_n based on Lyapunov's theorem and generating functions. Since by (4.1)

$$U_n \stackrel{d}{=} \sum_{i=1}^{n} Y_i\,, \quad Y_i \sim \mathfrak{B}(1, \frac{2}{i+1}), \quad (Y_i) \text{ independent}, \tag{4.3}$$

this argument can be simplified to yield:

Theorem 4.1 *Define* $\widehat{U_n} := \frac{U_n - \mathbb{E}\, U_n}{\sqrt{\mathrm{Var}\, U_n}}$, *then for some constant* $C > 0$ *and all* n

$$\rho\left(\widehat{U_n}, \mathcal{N}(0,1)\right) \leq \frac{C}{\sqrt{\ln n}}. \tag{4.4}$$

Proof: Observe that $\frac{S_{n,3}}{S_{n,2}^{3/2}} \sim \frac{1}{\sqrt{2\,\ln n}}$ (cf. Mahmoud [Ma92, pg. 77]). Therefore, (4.4) is a consequence of (2.6). □

From Deheuvels and Pfeifer [DP86] one obtains that $\frac{1}{\ln n}$ is the exact order of approximation of $\widehat{U_n}$ by a Poisson distribution. This indicates that the logarithmic rate in the Berry-Esseen bound (4.4) should give essentially the right order of approximation. The following approximation by the contraction method supports that the logarithmic order is sharp.

The contraction method can be applied in much the same way as in section 3. We, therefore, only give a sketch of this application. For more details we refer to Cramer [Cr95].

Theorem 4.2 *Define* $\widetilde{U_n} := \frac{U_n - \mathbb{E}\, U_n}{\sqrt{\ln n}}$, $\sigma_n^2 := \mathit{Var}\, \widetilde{U_n}$, $Z_n \sim \mathcal{N}(0, \sigma_n^2)$, *then for some* $C > 0$ *and all* $n \in \mathbb{N}$

$$\zeta_3\left(\widetilde{U_n}, Z_n\right) \leq \frac{C}{\sqrt{\ln n}}. \tag{4.5}$$

Proof: $\widetilde{U_n}$ satifies the recursion

$$\widetilde{U_n} \stackrel{d}{=} \sqrt{\frac{\ln(n-1)}{\ln n}}\, \widetilde{U_{n-1}} + \widetilde{Y_n}, \quad \widetilde{Y_n} := \frac{Y_n - \mathbb{E}\, Y_n}{\sqrt{\ln n}}. \tag{4.6}$$

Define

$$Z_n^\star := \sqrt{\frac{\ln(n-1)}{\ln n}}\, Z_{n-1} + \widetilde{Y_n}, \tag{4.7}$$

$$\tau_n^2 := \sigma_n^2 - \frac{\ln(n-1)}{\ln n}\,\sigma_{n-1}^2 = \operatorname{Var}\widetilde{Y_n}. \tag{4.8}$$

Then for $W_n \sim \mathcal{N}(0,\tau_n^2)$ independent of (Z_n), (Y_n)

$$Z_n \stackrel{d}{=} \sqrt{\frac{\ln(n-1)}{\ln n}}\, Z_{n-1} + W_n. \tag{4.9}$$

And so as in section 3

$$\zeta_3\left(\widetilde{U_n}, Z_n\right) \le \left(\frac{\ln 2}{\ln n}\right)^{3/2} \zeta_3\left(\widetilde{U_2}, Z_2\right) + \sum_{i=3}^{n} \left(\frac{\ln i}{\ln n}\right)^{3/2} \zeta_3\left(W_i, \widetilde{Y_i}\right) \tag{4.10}$$

As $\mathbb{E}\,\widetilde{U_2} = 0 = \mathbb{E}\,Z_2$, $\operatorname{Var}\widetilde{U_2} = \sigma_2^2 = \operatorname{Var} Z_2$ one obtains
$\zeta_3\left(\widetilde{U_2}, Z_2\right) \le \frac{1}{6}\left(\mathbb{E}\left|\widetilde{U_2}\right|^3 + \mathbb{E}\left|Z_2\right|^3\right) < \infty$. Also,

$$\begin{aligned}
\zeta_3\left(W_i, \widetilde{Y_i}\right) &\le \frac{1}{6}\left(\mathbb{E}|W_i|^3 + \mathbb{E}\left|\widetilde{Y_i}\right|^3\right)\\
&= \frac{1}{6}\left[\frac{2\sqrt{2}}{\sqrt{\pi}}\,\tau_i^3 + \frac{1}{(\ln i)^{3/2}}\left(\frac{i-1}{i+1}\left(\frac{2}{i+1}\right)^3 + \frac{2}{i+1}\left(\frac{i-1}{i+1}\right)^3\right)\right]\\
&\le \frac{2}{6\,(\ln i)^{3/2}\,(i+1)}\left[1 + \frac{4}{\sqrt{\pi\,(i+1)}}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\zeta_3\left(\widetilde{U_n}, Z_n\right) &\le \frac{1}{(\ln n)^{3/2}}\frac{1}{6}\left(\frac{10}{81} + \frac{8}{27\sqrt{\pi}}\right)\\
&\quad + \frac{1}{(\ln n)^{3/2}}\sum_{i=3}^{n}\frac{1}{i+1}\cdot\frac{1}{3}\left(1 + \frac{4}{\sqrt{\pi(i+1)}}\right)\\
&\le \frac{1}{\sqrt{\ln n}} \qquad \text{for } n \ge n_0.
\end{aligned} \tag{4.11}$$

□

Remark: Regarding recursion (4.6) one obtains convergence also under alternative distributional assumptions on $\widetilde{Y_n}$ (resp. Y_n). Let μ_r be any $(r,+)$-ideal, simple metric (cf. Rachev [Ra91]) then as in (4.10) one obtains for a recursion of this type the basic estimate

$$\mu_r\left(\widetilde{U_n}, Z_n\right) \le \left(\frac{\ln 2}{\ln n}\right)^{r/2} \mu_r\left(\widetilde{U_2}, Z_2\right) + \sum_{i=3}^{n}\left(\frac{\ln i}{\ln n}\right)^{r/2} \mu_r\left(W_i, \widetilde{Y_i}\right) \tag{4.12}$$

Then

$$\mu_r\left(\widetilde{U_n}, Z_n\right) \underset{n\to\infty}{\longrightarrow} 0 \tag{4.13}$$

if

$$\text{a)} \quad \mu_r\left(\widetilde{U_2}, Z_2\right) < \infty, \quad \mu_r\left(W_i, \widetilde{Y_i}\right) < \infty, \quad i \geq 3$$

and

$$\text{b)} \quad \mu_r\left(W_i, \widetilde{Y_i}\right) = o\left(\frac{1}{i \ln i}\right). \tag{4.14}$$

For the proof of (4.13) let for $\varepsilon > 0$, $k_0 \in \mathbb{N}$ such that $\mu_r(W_k, \widetilde{Y_k}) \leq \dfrac{\varepsilon}{k \ln k}$, then

$$\begin{aligned}
\limsup_{n\to\infty} \mu_r\left(\widetilde{U_n}, Z_n\right) &\leq \limsup_{n\to\infty} \left(\frac{\ln 2}{\ln n}\right)^{r/2} \mu_r\left(\widetilde{U_2}, Z_2\right) \\
&\quad + \limsup_{n\to\infty} \frac{1}{(\ln n)^{r/2}} \sum_{i=3}^{k_0-1} (\ln i)^{r/2} \mu_r\left(W_i, \widetilde{Y_i}\right) \\
&\quad + \limsup_{n\to\infty} \frac{1}{(\ln n)^{r/2}} \sum_{i=k_0}^{n} (\ln i)^{r/2-1} \frac{1}{i}\, \varepsilon \\
&\leq 0 + 0 + \limsup_{n\to\infty} \varepsilon \frac{1}{\ln n} \sum_{i=k_0}^{n} \frac{1}{i} \leq \varepsilon
\end{aligned}$$

□

In the preceding example of unsuccessful searching the estimates of $\mu_r(W_i, \widetilde{Y_i})$ of order $\frac{1}{i(\ln i)^{3/2}}$ even allow to obtain the convergence rate $\frac{1}{\sqrt{\ln n}}$.

5 Successful searching in binary search trees

Let for a random binary search tree as in section 4 S_n denote the number of comparisons to retrieve a randomly chosen element in the tree. Based on a formula for $\mathcal{P}(S_n = k)$ by Brown and Shubert (1984) a derivation of $\mathbb{E}\, S_n$, $\operatorname{Var} S_n$ and a central limit theorem for S_n due to Louchard (1987) is derived via generating functions in Mahmoud [Ma92, pg. 78-82]. We next derive a quantitative version of the central limit theorem by the contraction method and moment formulas based on the following (new) recursion for S_n:

$$S_n \stackrel{d}{=} 1 + S_{I_n}, \quad S_0 = 0, \quad S_1 = 1, \tag{5.1}$$

where I_n is independent of (S_i) and $\mathcal{P}(I_n = 0) = \frac{1}{n}$, $\mathcal{P}(I_n = j) = \frac{2j}{n^2}$, $1 \leq j \leq n-1$.

It can be shown that this recursion does not allow the form of a sum of independent random variables as considered in the random search algorithm in Rachev and Rüschendorf [RR91]. Therefore, it does not allow the application of the Berry-Esseen type or Poisson type approximation result. (5.1) arises from the recursion

$$\begin{aligned}
\mathcal{P}(S_n = k) &= \sum_{j=1}^{n} \mathcal{P}(S_n = k,\ j \ \text{ chosen}) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{1}{n^2} \left(1\!\!1_{\{i=j\}}\, \delta_{1k} + 1\!\!1_{\{i<j\}}\, \mathcal{P}(S_{n-i} = k-1) \right. \\
&\quad \left. + \ 1\!\!1_{\{i>j\}}\, \mathcal{P}(S_{i-1} = k-1) \right) \\
&= \frac{\delta_{1k}}{n} + \sum_{i=1}^{n} \frac{n-i}{n^2}\, \mathcal{P}(S_{n-i} = k-1) \\
&\quad + \sum_{i=1}^{n} \frac{i-1}{n^2}\, \mathcal{P}(S_{i-1} = k-1) \\
&= \frac{\delta_{1k}}{n} + \sum_{j=1}^{n-1} \mathcal{P}(S_j = k-1) \cdot \frac{2j}{n^2}. \qquad (5.2)
\end{aligned}$$

An explicit formula for $\mathcal{P}(S_n = k)$ due to Brown and Shubert (1984) (cf. Mahmoud [Ma92, pg. 79]) is relatively involved. Based on this formula the first two moments are derived in Mahmoud [Ma92, pg. 80]. The recursion (5.1) allows a direct calculation.

Proposition 5.1

$$\text{a)} \quad \mathbb{E}\, S_n = 2\left(1 + \frac{1}{n}\right) \mathrm{H}_n - 3 \qquad (5.3)$$

$$\text{b)} \quad \mathrm{Var}\, S_n = \left(2 + \frac{10}{n}\right) \mathrm{H}_n - 4\left(1 + \frac{1}{n}\right) \left[\frac{\mathrm{H}_n^2}{n} + \mathrm{H}_n^{(2)}\right] + 4 \qquad (5.4)$$

Proof:

$$\begin{aligned}
\text{a)} \quad \mathbb{E}\, S_n &= 1 + \mathbb{E}\left(\mathbb{E}(S_{I_n} | I_n)\right) = 1 + \sum_{k=0}^{n-1} \mathcal{P}(I_n = k)\ \mathbb{E}\, S_k \\
&= 1 + \sum_{k=0}^{n-1} \frac{2\,k}{n^2}\ \mathbb{E}\, S_k. \qquad (5.5)
\end{aligned}$$

With $Q_n := n \cdot \mathbb{E}\, S_n$, (5.5) leads to $Q_n = n + \frac{2}{n} \sum_{k=1}^{n-1} Q_k$, $Q_1 = 1$, which implies $Q_{n+1} = \frac{2\,n+1}{n+1} + \frac{n+2}{n+1}\, Q_n$. By iteration one obtains

$$Q_n = \frac{2\,n-1}{n} + \sum_{k=1}^{n-1} \frac{2\,k-1}{k} \cdot \frac{n+1}{k+1}$$

$$= (n+1)\left[\sum_{k=1}^{n} \frac{2}{k+1} - \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)\right]$$
$$= (n+1)\left[2\,(\mathrm{H}_{n+1} - 1) - 1 + \frac{1}{n+1}\right]$$
$$= 2(n+1)\,\mathrm{H}_n - 3\,n.$$

b) $$\mathbb{E}\, S_n^2 = 1 + 2\,\mathbb{E}\, S_{I_n} + \mathbb{E}\, S_{I_n}^2$$
$$= 1 + 2\,(\mathbb{E}\, S_n - 1) + \sum_{j=1}^{n-1} \frac{2\,j}{n^2}\,\mathbb{E}\, S_j^2.$$

With $P_n := n \cdot \mathbb{E}\, S_n^2$, $P_n = -n + 2\,Q_n + \frac{2}{n}\sum_{j=1}^{n-1} P_j$.

This yields $\frac{n+1}{2}\,P_{n+1} - \frac{n}{2}\,P_n = \frac{2\,n+1}{2} + 2\,Q_n + P_n$. By a), therefore, one obtains

$$P_{n+1} = 8\,\mathrm{H}_n - \frac{10\,n - 1}{n+1} + \frac{n+2}{n+1}\,P_n$$

and by iteration

$$P_n = \sum_{j=1}^{n}\left(8\,\mathrm{H}_j - \frac{10\,j - 3}{j}\right)\frac{n+1}{j+1}.$$

Using the relation

$$\sum_{j=1}^{n} \frac{\mathrm{H}_j}{j} = \frac{\mathrm{H}_n^{(2)} + \mathrm{H}_n^2}{2}$$

one can explicitly calculate P_n and obtain (5.4). □

Define the normalizations

$$\widetilde{S_n} := \frac{S_n - \mathbb{E}\, S_n}{\sqrt{2\,\ln n}}, \quad \widetilde{S_0} = \widetilde{S_1} = 0 \tag{5.6}$$

and

$$a(k,n) := 1 - \mathbb{E}\, S_n + \mathbb{E}\, S_k,\; b(k) := \operatorname{Var} S_k,\; \sigma_n^2 := \operatorname{Var} \widetilde{S_n}. \tag{5.7}$$

For our derivation we need the following (still unsettled) conjecture:

C) $$\limsup_{n\to\infty} \int y^2 \left|\sum_{k=2}^{n-1} \frac{2\,k}{n^2}\left[\phi\left(\frac{y}{\sqrt{b(n)}}\right) - \phi\left(\frac{y - a(k,n)}{\sqrt{b(k)}}\right)\right]\right| \mathrm{d}y < \infty \tag{5.8}$$

ϕ the distribution function of the standard normal distribution.

Let (Z_n) be independent of (S_n), $Z_n \sim \mathcal{N}(0, \sigma_n^2)$, then under C) we obtain the following quantitative result.

Theorem 5.2 *Under* C) *exists a constant* $K < \infty$, *such that*

$$\zeta_3\left(\widetilde{S_n}, Z_n\right) \leq \frac{K}{\sqrt{\ln n}}. \tag{5.9}$$

Proof: $(\widetilde{S_n})$ satisfies the recursion

$$\widetilde{S_n} \ \stackrel{d}{=} \ \sqrt{\frac{\ln I_n}{\ln n}}\, \widetilde{S_{I_n}} \ + \ c_n(I_n), \tag{5.10}$$

where $c_n(k) := \frac{1-\mathbb{E}\, S_n + \mathbb{E}\, S_k}{\sqrt{2\,\ln n}}$.

Define the normal copy

$$Z_n^\star \ \stackrel{d}{=} \ \sqrt{\frac{\ln I_n}{\ln n}}\, Z_{I_n} \ + \ c_n(I_n). \tag{5.11}$$

Then we obtain from the wellknown properties of ζ_3, using the independence assumptions, the basic recursive estimate

$$\begin{aligned}
\zeta_3\left(\widetilde{S_n}, Z_n\right) \ &\leq \ \zeta_3\left(\widetilde{S_n}, Z_n^\star\right) \ + \ \zeta_3\left(Z_n^\star, Z_n\right) \\
&\leq \ \sum_{k=0}^{n-1} \mathcal{P}(I_n = k)\, \zeta_3\left(\sqrt{\frac{\ln k}{\ln n}}\, \widetilde{S_k} + c_n(k), \sqrt{\frac{\ln k}{\ln n}}\, Z_k + c_n(k)\right) \\
&\quad + \zeta_3(Z_n^\star, Z_n) \\
&\leq \ \sum_{k=2}^{n-1} \frac{2\,k}{n^2} \left(\frac{\ln k}{\ln n}\right)^{3/2} \zeta_3\left(\widetilde{S_k}, Z_k\right) \ + \ \zeta_3\left(Z_n^\star, Z_n\right). \qquad (5.12)
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}\, Z_n^\star \ &= \ \sum_{k=0}^{n-1} \mathcal{P}(I_n = k)\ \mathbb{E}\left(\sqrt{\frac{\ln k}{\ln n}}\, Z_k + c_n(k)\right) \\
&= \ \sum_{k=0}^{n-1} \mathcal{P}(I_n = k)\ \frac{1 - \mathbb{E}\, S_n + \mathbb{E}\, S_k}{\sqrt{2\,\ln n}} \\
&= \ (2\,\ln n)^{-1/2}\, \left[1 - \mathbb{E}\, S_n + \mathbb{E}\, S_{I_n}\right] \ = \ 0 \ = \ \mathbb{E}\, Z_n
\end{aligned}$$

and, similarly,

$$\mathbb{E}\, (Z_n^\star)^2 \ = \ \frac{1}{2\,\ln n}\, \mathrm{Var}\ S_n \ = \ \mathrm{Var}\ \widetilde{S_n},$$

we obtain

$$\zeta_3\left(Z_n^\star, Z_n\right) \ \leq \ \frac{1}{6}\, \kappa_3\left(Z_n^\star, Z_n\right) \ = \ \frac{1}{2} \int x^2 \left|F_{Z_n^\star}(x) - F_{Z_n}(x)\right| \mathrm{d}x.$$

Furthermore,

$$F_{Z_n}(x) \ = \ \phi(x/\sigma_n) \ = \ \phi\left(x\,\sqrt{2\,\ln n}/\sqrt{b(n)}\right)$$

and

$$\begin{aligned} F_{Z_n^\star}(x) &= \sum_{k=0}^{n-1} \mathcal{P}(Z_n^\star \le x \mid I_n = k)\cdot \mathcal{P}(I_n = k) \\ &= \sum_{k=2}^{n-1} \frac{2\,k}{n^2}\,\phi\left(\frac{x\,\sqrt{2\ln n} - a(k,n)}{\sqrt{b(k)}}\right) + \frac{1}{n}\,\mathbb{1}_{[1-\mathbb{E}\,S_n,\infty)}(x\,\sqrt{2\ln n}) \\ &\quad + \frac{2}{n^2}\,\mathbb{1}_{[2-\mathbb{E}\,S_n,\infty)}(x\,\sqrt{2\ln n}). \end{aligned}$$

This implies using the substitution $y = x\cdot\sqrt{2\ln n}$

$$\zeta_3\left(Z_n^\star, Z_n\right) \le \frac{1}{2\cdot(2\ln n)^{3/2}}\,\left[A_n + B_n + C_n\right] \tag{5.13}$$

where

$$\begin{aligned} A_n &:= \frac{1}{n}\int y^2\left|\mathbb{1}_{[1-\mathbb{E}\,S_n,\infty)}(y) - \phi\left(\frac{y}{\sqrt{b(n)}}\right)\right|\mathrm{d}y, \\ B_n &:= \frac{2}{n^2}\int y^2\left|\mathbb{1}_{[2-\mathbb{E}\,S_n,\infty)}(y) - \phi\left(\frac{y}{\sqrt{b(n)}}\right)\right|\mathrm{d}y, \\ C_n &:= \int y^2\left|\sum_{k=2}^{n-1}\frac{2\,k}{n^2}\left[\phi\left(\frac{y-a(k,n)}{\sqrt{b(k)}}\right) - \phi\left(\frac{y}{\sqrt{b(n)}}\right)\right]\right|\mathrm{d}y. \end{aligned}$$

Under C) we obtain $C_n \le M_C$, $\forall\ n\in\mathbb{N}$ and some constant M_C. Also for $n \ge n_0$, $\mathbb{E}\,S_n \ge 1$ and

$$\begin{aligned} A_n &\le \frac{1}{n}\int y^2\left|\phi\left(\frac{y}{\sqrt{b(n)}} - \mathbb{1}_{[0,\infty)}(y)\right)\right|\mathrm{d}y + \frac{1}{n}\int y^2\,\mathbb{1}_{[1-\mathbb{E}\,S_n,0)}(y)\,\mathrm{d}y \\ &\le \frac{1}{n}\cdot\frac{1}{3}\,2\,\frac{\sqrt{2}}{\sqrt{\pi}}\,\sqrt{b(n)}^3 + \frac{1}{n}\cdot\frac{1}{3}\,(\mathbb{E}\,S_n - 1)^3 \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

as $b(n) = \operatorname{Var} S_n \sim 2\ln n$ and $\mathbb{E}\,S_n \sim 2\ln n$. Therefore, $A_n \le M_A$, $\forall\ n$ and similarly $B_n \le M_B$, $\forall n$ and, together, for a constant M we obtain

$$\zeta_3\left(Z_n, Z_n^\star\right) \le \frac{M}{(\ln n)^{3/2}}. \tag{5.14}$$

Application of Euler's summation formula (cf. Hofri [Ho87, pg. 19])

$$\sum_{j=1}^{n-1} f(j) = \int_1^n f(x)\,\mathrm{d}x + \sum_{k=1}^{m}\frac{B_k}{k!}\left[f^{(k-1)}(n) - f^{(k-1)}(1)\right] + R_m, \tag{5.15}$$

where

$$R_m = \frac{(-1)^{m+1}}{m!}\int_1^n B_m(\{x\})\,f^{(m)}(x)\,\mathrm{d}x, \quad \{x\} = x - \lfloor x\rfloor,$$

and (B_k) the Bernoulli numbers, $B_m(x) = \sum_{k\geq 0} \binom{m}{k} B_k\, x^{m-k}$ the m-th Bernoulli-polynomial, to the function $f(x) = x \ln x,\ x \geq 1$ yields for $m = 2$ after some calculations

$$\sum_{j=2}^{n-1} j \ln j \;=\; \frac{1}{2} n^2 \ln n - \frac{1}{4} n^2 - \frac{1}{2} n \ln n + \mathcal{O}(\ln n). \tag{5.16}$$

Consider a sufficiently large n_0 such that for $n \geq n_0$, $\sum_{j=2}^{n-1} j \ln j \leq \frac{1}{2} n^2 \ln n - \frac{1}{4} n^2$. Choose $\widetilde{M}$ large enough that (5.9) with $\widetilde{M}$ instead of K holds for $n < n_0$ and define $K := \max\left(\widetilde{M}, 2M\right)$. Then we finally obtain from (5.12), (5.14) by induction, assuming (5.9) for all $k < n$, that

$$\begin{aligned}
\zeta_3\left(\widetilde{S_n}, Z_n\right) &\leq \sum_{k=2}^{n-1} \frac{2k}{n^2} \left(\frac{\ln k}{\ln n}\right)^{3/2} \frac{K}{\sqrt{\ln k}} + \frac{M}{(\ln n)^{3/2}} \\
&= \frac{1}{(\ln n)^{3/2}} \cdot \frac{2}{n^2} \cdot K \sum_{k=2}^{n-1} k \ln k + \frac{M}{(\ln n)^{3/2}} \\
&\leq \frac{1}{(\ln n)^{3/2}} \left[\frac{2K}{n^2}\left(\frac{1}{2} n^2 \ln n - \frac{1}{4} n^2\right) + M\right] \\
&\leq \frac{1}{(\ln n)^{3/2}} \left[K \ln n - \frac{K}{2} + \frac{K}{2}\right] = \frac{K}{\sqrt{\ln n}}.
\end{aligned}$$

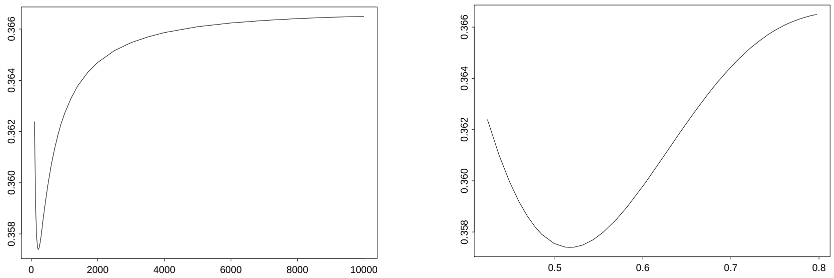
□

Remarks:

a) In the preceding example one can not give a direct proof for the convergence of $\widetilde{S_n}$ based on estimates for the terms in the 'basic estimate' but one obtains convergence by an induction argument which uses the Euler summation formula in a crucial way. This extension of the contraction technique seems to be potentially useful also for further examples.

b) To verify the conjecture C) used in our proof it seems not possible to put the absolute values under the sum. Numerical calculations with 'Maple' for $n \leq 10\,000$ indicate that C) is correct. Denote the integral in (5.8) for $n \in \mathbb{N}$ by $f(n)$, then calculation on the range -25 to 25 with a Newton-Cote algorithm with exactness of order 10^{-5} one obtains the following curves of $f(n)$ against n resp. against $\ln(\ln(\ln n))$, which indicate boundedness of f.

Figure 1:



$f(n)$ against n $\qquad$ $f(n)$ against $\ln(\ln(\ln(n)))$

$\square$

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