Stochastic ordering of risks, influence of dependence and a.s. constructions

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Abstract

In this paper we review and extend some key results on the stochastic ordering of risks and on bounding the influence of stochastic dependence on risk functionals. The first part of the paper is concerned with a.s. constructions of random vectors and with diffusion kernel type comparisons which are of importance for various comparison results. In the second part we consider generalizations of the classical Fréchet-bounds, in particular for the distribution of sums and maxima and for more general monotonic functionals of the risk vector. In the final part we discuss three important orderings of risks which arise from Δ-monotone, supermodular, and directionally convex functions. We give some new criteria for these orderings. For the basic results we also take care to give references to "original sources" of these results.

1 Introduction

It has been recognized in recent years that the methods and tools of stochastic ordering and construction of probabilities with given marginals are of essential relevance for the problem of modeling multivariate portfolios and bounding functions of dependent risks like the value at risk, the expected excess of loss and other financial derivatives and risk measures Even if many results on stochastic ordering and dependent risks have been developed in early years, a new impetus on reconsidering this field came recently from financial modelling and risk management and a lot of papers in economics and insurance journals is devoted to this subject (see e.g. the recent article of Embrechts, Höing and Juri (2003) and the references therein). Stochastic ordering and marginal modelling has a long history and several books and conference proceedings to this subject have appeared. To mention are in particular proceedings of conferences on marginal modelling and stochastic ordering ([11], [41], [71], [3], [9]) as well as the comprehensive volumes of Stoyan (1977), Marshall and Olkin (1979), Tong (1980), Mosler (1982), Shaked and Shantikumar (1994), Joe (1997), Nelsen (1999), and Müller and Stoyan (2002).

The main purpose of this paper is to point out and partially extend some of the orderings and results on orderings which seem to be of particular importance for bounding risks and the influence of dependence on functionals. For several of the key results we also want to give some of the early and original references. The field of stochastic orders is very diversified. But some of the recent work and results has already been stated and established in early papers on stochastic ordering.

We essentially restrict to "integral orders $\prec_{\mathcal{F}}$" on the probability measures induced by some function class $\mathcal{F}$ and defined by

$$P \prec_{\mathcal{F}} Q \quad \text{if} \quad \int f\, dP \le \int f\, dQ \quad \text{for all} \quad f \in \mathcal{F}, \tag{1.1}$$

such that f is integrable w.r.t. P and Q. Some natural questions for the analysis of a stochastic order $\prec$ are to find simple and maximal generators $\mathcal{F}$ of $\prec$ such that $\prec$ and $\prec_{\mathcal{F}}$ are equivalent orderings (or to find at least large classes $\mathcal{F}$ such that $P \prec Q$ implies $P \prec_{\mathcal{F}} Q$). This aspect is discussed in most of the books mentioned above. Additional particular references to the subject of integral stochastic orders are Rüschendorf (1979), Reuter and Riedrich (1981), Mosler and Scarsini (1991b) Marshall (1991), Müller (1997), and Denuit and Müller (2001).

The plan of this paper is to discuss at first a.s. construction of random vectors which lie at the core of several ordering results. Related are kernel representation results which give 'pointwise' characterization of stochastic orders by diffusion kernels. Each ordering generates a notion of positive resp. negative dependence by comparing a probability measure

$$P \in M(P_1, \dots, P_n) \tag{1.2}$$

– the class of all probability measures with marginals $P_1, \dots, P_n$– to the product $\otimes_{i=1}^n P_i$ of its marginals. If $\otimes_{i=1}^n P_i \prec P$, then we speak of positive dependence of P, if $P \prec \otimes_{i=1}^n P_i$, then of negative dependence. Related is the problem to describe the maximal influence of dependence on a function f (or class of functionals $\mathcal{F}$),

$$\begin{aligned} M(f) &= \sup\left\{\int f\, dP;\ P \in M(P_1, \dots, P_n)\right\} \qquad (1.3)\\ \text{resp.} \quad m(f) &= \inf\left\{\int f\, dP;\ P \in M(P_1, \dots, P_n)\right\} \end{aligned}$$

which we call the problem of *(generalized) Fréchet-bounds.* One of the most prominent results in stochastic ordering are the classical Fréchet-bounds due to Hoeffding (1940) and Fréchet (1951),

a) For a n-dimensional *df* F holds: $F \in \mathcal{F}(F_1, \dots, F_n)$ – the Fréchet class of n-dimensional *df*'s with marginals $F_1, \dots, F_n$ – if and only if

$$F_- \le F \le F_+, \tag{1.4}$$

where $F_+(x) := \min_{1\le i\le n}\{F_i(x_i)\}$ and $F_-(x) := \max\{0, \sum_{i=1}^n F_i(x_i) - (n-1)\}$ are the upper and lower Fréchet-bound.

b) Moreover, $F_+ \in \mathcal{F}_n$ is a n-dimensional *df*, while $F_- \in \mathcal{F}_n$ if and only if $n = 2$ or

$$\begin{aligned} \text{for n} > 2 \text{ either} \quad \sum_{i=1}^n F_i(x_i) &\le 1 \quad \text{for all } x \text{ with } F_j(x_j) < 1,\ \forall j \qquad (1.5)\\ \text{or} \quad \sum_{i=1}^n F_i(x_i) &\ge n-1 \quad \text{for all } x \text{ with } F_j(x_j) > 0,\ \forall j. \end{aligned}$$

The important characterization in b) of the cases where $F_- \in \mathcal{F}_n$ is due to Dall'Aglio (1972). For a review on various aspects of Fréchet-bounds see Rüschendorf (1991b) (in the following abbreviated by Ru (1991b)). The most important general technique to determine generalized Fréchet-bounds is duality theory. A comprehensive survey of this theory and its applications is given in Rachev and Ru (1998, Vol. I/II). We will describe some interesting aspects of the problem of Fréchet-bounds on the influence of dependence in chapter 3.

For the application to the comparison of risks it has turned out (see the interesting recent book of Müller and Stoyan (2002)) that of particular importance are the classes of supermodular (quasi-monotone), directionally convex and Δ-monotone functions $\mathcal{F}^{sm}$, $\mathcal{F}^{dcx}$, $\mathcal{F}^{\Delta}$ together with the induced orderings and some variants like $\mathcal{F}^{idcx}$, the increasing directionally convex functions. In section four we describe and extend some of the basic comparison criteria for these functions.

We use some standard notation throughout. $X \sim P$ means that the random variable X has distribution P. We write for some ordering $\prec$, $X \prec Y$ synonymously for $P \prec Q$ or $F \prec G$, where F, G are the *dfs* of X, Y and P, Q are the distributions. $\leq_{st}$ denotes the usual stochastic order w.r.t. nondecreasing functions.

2 Stochastic Ordering and A.S. Construction of Random Variables

For the comparison of distributions P, Q w.r.t. some stochastic order it is in some cases useful to compare explicit a.s. constructions of $rv's$ X, Y where $X \sim P$ and $Y \sim Q$. A general useful construction for $P, Q \in M^1(\mathbb{R}^n)$, the set of probability measures on $\mathbb{R}^n$ is the following *'standard construction'*:

Let $F \in \mathcal{F}_n$ be a n-dimensional *df* and let $V_1, \dots, V_n$ be independent $rv's$ uniformly distributed on $[0, 1]$, independent of $X \sim F$. Let $V = (V_1, \dots, V_n)$ and let $F_{i|1,\dots,i-1}(x_i|x_1, \dots, x_{i-1})$ denote the conditional $df's$ of X_i given $X_j = x_j$, $j \leq i-1$. We define $\tau_F : \mathbb{R}^n \times [0,1]^n \to \mathbb{R}^n$ by

$$\tau_F(x, \lambda) = (F_1(x_1, \lambda_1), F_{2|1}(x_2, \lambda_2|x_1), \dots, F_{n|1,\dots,n-1}(x_n, \lambda_n|x_1, \dots, x_{n-1})) \tag{2.1}$$

where $F_{i|1,\dots,i-1}(x_i|x_1, \dots, x_{i-1}) = P(X_i < x_i|X_j = x_j,\ j \leq i-1) + \lambda_i P(X_i = x_i|X_j = x_j,\ j \leq i-1)$. We define the 'inverse' transformation τ_F^{-1} recursivly

$$\tau_F^{-1}(u) = z = (z_1, \dots, z_n), \tag{2.2}$$

with $z_1 = F_1^{-1}(u_1)$, $z_2 = \inf\{y : F_{2|1}(y|z_1) \geq u_2\} = F_{2|1}^{-1}(u_2|z_1)$, ..., $z_n = F_{n|1,\dots,n-1}^{-1}(u_n|z_1, \dots, z_{n-1})$.

Theorem 2.1 (Regression construction) *Let X be a n-dimensional random vector with df F, then:*

a) $U := \tau_F(X, V)$ has independent components, uniformly distributed on $[0, 1]$.

b) $Z = \tau_F^{-1}(V)$ is a rv with df F; Z is called the "regression construction" of F.

c)
$$X = \tau_F^{-1}(\tau_F(X, U)) \quad a.s. \tag{2.3}$$

Remark 2.2 *1) Part a) of* (2.1) *is due in the case of absolutely continuous conditional df's to Rosenblatt (1952). a) and b) were stated in this form in Ru (1981b). b) had been given before in an equivalent form in O'Brien (1975), while c) is from Ru and de Valk (1993). The one dimensional case was used since long time for the simulation of rv's.*

2) By the recursive definition in (2.2) *one obtains Z also as a function of* $(V_1, \dots, V_n)$ *which we denote by τ_F^**

$$Z = \tau_F^*(V) \sim F \tag{2.4}$$

where $\tau_F^*(V) = (h_1(V_1), h_2(V_1, V_2), \dots, h_n(V_1, \dots, V_n))$. In this functional form the construction is called "standard representation" of F. It gives a construction of a random vector with df F as function of independent uniforms. The functions h_i represent conditional df's.

3) A "copula" of X (resp. F) is any df C with uniform marginals such that

$$C(F_1, \dots, F_n) = F \tag{2.5}$$

where F_i are the marginal df's. If U is a random vector with $U \sim C$ then

$$(F_1^{-1}(U_1), \dots, F_n^{-1}(U_n)) \sim F. \tag{2.6}$$

U represents some aspects of the dependence structure of $F(resp. X)$. To obtain a copula one can apply Theorem 2.1 in the one-dimensional case and consider $\overline{U} := (\tau_{F_1}(X_1, V_1), \dots, \tau_{F_n}(X_n, V_n))$. Then the df C of $\overline{U}$ is a copula and $X = (F_i^{-1}(\overline{U}_i))$ a.s.

We next give some applications of the standard resp. regression construction.

Corollary 2.3 (Stochastic ordering) *If $F, G \in \mathcal{F}_n$ and $V = (V_1, \dots, V_n)$ is an iid uniform sequence. Then*

$$\tau_F^{-1}(U) \le \tau_G^{-1}(V) \text{ implies } F \le_{st} G, \tag{2.7}$$

where $\le_{st}$ denotes the usual stochastic ordering w.r.t. $\mathcal{F}^m$ the class of monotonically nondeceasing functions.

Remark 2.4 *Condition* (2.7) *is stated in Ru (1981b). It implies various sufficient conditions for stochastic ordering going back to classical results of Veinott (1965), Kalmykov (1962) and Stoyan (1972) in the context of markov chains. The regression and standard construction is used essentially in various papers on stochastic ordering. The positive dependence ordering 'conditional increasing in sequence CIS' just says that the components h_i of τ_F^* are monotonically nondecreasing. This is used essentially in many papers e.g. in Müller and Scarsini (2001) to state sufficient conditions for the supermodular ordering of positive dependent* *sequences (see also section 4). An application of the standard construction to convex ordering analogously to* (2.7) *is given in Shaked and Shantikumar (1994). An alternative application to positive regression dependence ordering of rank statistics as well as to further statistical ordering results is given in Ru (1986).*

As second application we consider the following optimal coupling problem: Determine for some $P, Q \in M^1(\mathbb{R}^n)$ and with $S_n(X) = \sum_{i=1}^n X_i$:

$$\inf \left\{ E|S_n(X) - S_n(Y)|^2 \;:\; X \sim P,\; Y \sim Q \right\}, \tag{2.8}$$

i.e. the problem is to construct two n-dimensional random vectors X, Y with distributions P, Q such that the sums $S_n(X) = \sum_{i=1}^n X_i$, $S_n(Y) = \sum_{i=1}^n Y_i$ are as close as possible in L^2-distance. The answer to this problem is :

Corollary 2.5 (Optimal coupling of sums, Ru (1986))
For $P, Q \in M^1(\mathbb{R}^n)$ let P_1, P_2 denote the distributions of the sums $\sum X_i$, $\sum Y_i$ resp., then

$$\inf \left\{ E \left| \sum_{i=1}^n X_i - \sum_{i=1}^n Y_i \right|^2 \;:\; Y \sim P,\; Y \sim Q \right\} = \ell_2^2(P_1, P_2) \tag{2.9}$$

where $\ell_2^2(P_1, P_2) = \int_0^1 (F_1^{-1}(u) - F_2^{-1}(u))^2\, du$ is the squared minimal ℓ_2-metric.

For the proof one applies the regression construction to the extended random vectors $(\sum_{i=1}^n X_i, X)$ resp. $(\sum_{i=1}^n Y_i, Y)$ and obtains directly (2.9). Of course similar results hold for the coupling of other functionals like
$(\sum_{i=1}^n X_i, \max_{i \le n} X_i)$ simultaneously to $(\sum_{i=1}^n Y_i, \max_{i \le n} Y_i)$ etc.

The standard construction does not in general give a pointwise a.s. construction of random vectors $X \sim P$, $Y \sim Q$ such that $X \le Y$ a.s. if $P \le_{st} Q$. But Strassen's comparison theorem implies the existence of such a.s. representations. This result was extended to closed partial order's $\prec$ on a polish space S. The order $\prec$ on S induces the stochastic order $\prec_{st}$ on $M^1(S)$ the set of probability measures on S for the corresponding class $\mathcal{F}^m$ of monotonically nondecreasing functions w.r.t. $\prec$.

Theorem 2.6 (Strassen's theorem) *Let $\prec$ be a closed partial order on a polish space S and P, Q probability measures on S. Then: $P \prec_{st} Q$ if and only if there exist rv's $X \sim P$, $Y \sim Q$ (on some space $(\Omega, \mathcal{A}, R)$) such that*

$$X \prec Y \quad a.s. \tag{2.10}$$

Remark 2.7 (2.10) *was introduced in Strassen (1965) and extended in various ways in Kamae, Krengel and O'Brien (1978), Kellerer (1984) and Ramachandran and Ru (1995). A proof by means of Strassen's abstract set representation theorem is given in Ru (1980b, Theorem 1) in the case of the Schur-ordering $\prec_S$ on $\mathbb{R}^n$ defined by:*

$$\begin{aligned} a \prec_S b \quad \text{if} \quad & \sum_{i=1}^k a_{(i)} \le \sum_{i=1}^k b_{(i)}, \quad i \le k \le n-1 \\ \text{and} \quad & \sum_{i=1}^n a_i = \sum_{i=1}^n b_i, \end{aligned} \tag{2.11}$$

where $a_{(1)} \geq \ldots \geq a_{(n)}$ *are the conponents arranged in decreasing order. The monotone functions w.r.t.* $\prec_S$ *are the Schur-convex functions and* (2.10) *implies:*

Corollary 2.8 (Schur-convex ordering) *Let* $P, Q \in M^1(\mathbb{R}^n)$*, then:*

$$\begin{aligned} P \prec_S Q \iff & \text{ There exist } X \sim P,\ Y \sim Q \text{ with } X \prec_S Y \qquad (2.12) \\ \iff & \text{ There exist } X \sim P \text{ and a random doubly stochastic} \\ & \text{matrix } \Pi, \text{ such that } \Pi X \sim Q \end{aligned}$$

A more general comparison result for integral stochastic orders $\prec_{\mathcal{F}}$ is based on $\prec_{\mathcal{F}}$-diffusions.

Definition 2.9 ($\prec_{\mathcal{F}}$-diffusions) *Let* $\mathcal{F}$ *be a class of functions on some space* $(E, \mathcal{A})$*. A markov kernel* K *on* E *is called a* $\mathcal{F}$*-diffusion if*

$$\varepsilon_x \prec_{\mathcal{F}} K(x, .) \quad \text{for all } x \in E. \tag{2.13}$$

A $\prec_{\mathcal{F}}$-diffusion kernel K 'diffuses' locally in any point x mass w.r.t. $\prec_{\mathcal{F}}$. The composition KP is defined by $KP(A) = \int K(x, A)\, P(dx)$.

Proposition 2.10 *Let* K *be a* $\prec_{\mathcal{F}}$*-diffusion then*

$$P \prec_{\mathcal{F}} KP \quad \text{for all } P \in M^1(E). \tag{2.14}$$

Proof: The proof is obvious from the idea of diffusions. For $f \in \mathcal{F}$ holds

$$\begin{aligned} \int f\, dKP &= \int \left(\int f(y)\, K(x, dy) \right) dP(x) \\ &\geq \int f(x)\, d(P(x) \end{aligned}$$

□

From a general kernel representation result in Strassen (1965) one can obtain a converse of (2.14) and characterize several stochastic integral orders by corresponding $\mathcal{F}$-diffusions.A result of this type was stated in Ru (1980b) for several examples including the stochastic order, the convex and convex increasing order which were well established before and are related to famous results of Blackwell, Stein, Sherman, Cartier, Meyer and Strassen. It also included the class of symmetric convex functions and the class of norm increasing functions. The list of examples was further extended in Mosler and Scarsini (1991b). The proof in Ru (1980b) uses an idea from the theory of balagage (Meyer (1966, Theorem 53)) and Strassen's kernel representation theorem (Strassen (1965, Theorem 3)). For a general formulation of this result we define

$$h_f(x) = \sup \left\{ \int f\, dP;\ \varepsilon_x \prec_{\mathcal{F}} P \right\} \tag{2.15}$$

for $f \in C_b(E)$ and assume that $\mathcal{F} \subset C_b(E)$ is some order generating class. Let $\mathcal{F}^\circ$ denote the convex maximal generator of the order dual to $\prec_{\mathcal{F}}$ such that $P \prec_{\mathcal{F}} Q$ is equivalent to $Q \prec_{\mathcal{F}^\circ} P$. Let $\overline{\mathcal{F}^\circ}^{P,Q}$ be the set of all pointwise limits of sequences in $\mathcal{F}^\circ$ which are uniformly integrable w.r.t. P, Q.

Theorem 2.11 ($\mathcal{F}$-diffusions) *Let E be a polish space, $\mathcal{F} \subset C_b(E)$ with dual convex cone $\mathcal{F}^\circ$ and $P, Q \in M^1(E)$. Assume that for $f \in C_b(E)$, $h_f \in \overline{\mathcal{F}^\circ}^{P,Q}$. Then $P \prec_\mathcal{F} Q$ if and only if there exists a $\mathcal{F}$-diffusion K such that*

$$Q = KP. \tag{2.16}$$

Proof: Define $\Pi_x := \{P \in M^1(E) : \varepsilon_x \prec_\mathcal{F} P\}$. Then Π_x is convex, weakly closed and $h_f(x) = \sup\{\int f\,dP;\ P \in \Pi_x\}$. For $f \in C_b(E)$ holds $f(x) \le h_f(x)$ and thus since $h_f \in \overline{\mathcal{F}^\circ}^{P,Q}$

$$\int f\,dQ \le \int h_f\,dQ \le \int h_f\,dP \tag{2.17}$$

This implies by Strassens kernel representation theorem (Strassen (1965, Theorem 3)) the existence of a kernel K on E with $Q = KP$ and $K(x, .) \in \Pi_x$ for all $x \in E$, i.e. K is a $\mathcal{F}$-diffusion. □

Remark 2.12 *a) The first general formulation of the diffusion characterization theorem* (2.16) *is due to Meyer (1966, Theorem 53). It is formulated in the context of integral stochastic orders $\prec_\mathcal{F}$ in Müller and Stoyan (2002, Theorem 2.6.1): Suppose that for any $f, g \in \mathcal{R}_\mathcal{F}$ – the maximal generator of $\prec_\mathcal{F}$ – holds*

$$\max(f, g) \in \mathcal{R}_\mathcal{F} \tag{2.18}$$

then the equivalence in (2.16) *holds true.*

b) We consider the following examples of applications of (2.16)

1) *If $\mathcal{F} = \mathcal{F}^{sym,cx}$ is the set of symmetric convex functions on $\mathbb{R}^n$. Then h_f is symmetric and concave; so it lies in the closure of the dual cone $\mathcal{F}^\circ$. From* (2.16) *we obtain: $P \prec_{sym,cx} Q$ if and only if*

$$\exists\, X \sim P,\ Y \sim Q \text{ such that } X \prec_S E(Y_{(\ )}|X), \tag{2.19}$$

where $\prec_S$ is the Schur order, $Y_{(\ )}$ is the ordered vector (see Ru (1981)). It is interesting that there is a difference to the condition for stochastic Schur-ordering in (2.12)

2) *If $\mathcal{F}^{||\ ||}$ is the class of norm increasing functions $f(x) = g(||x||)$ in $C_b(\mathbb{R}^n)$, then $\varepsilon_x \prec_\mathcal{F} P$ iff P has support in $\{y : ||y|| \ge ||x||\}$. Further for any $f \in C_b$ holds $h_f(x) = \sup\{\int f\,dP : \varepsilon_x \prec_{\mathcal{F}^{||\ ||}} P\}$ is norm decreasing, $||x|| \le ||y||$ implies $h_f(y) \le h_f(x)$. So $h_f \in \overline{\mathcal{F}^\circ}^{P,Q}$ and we obtain:*

$$P \prec_{\mathcal{F}^{||\ ||}} Q \text{ iff there exist } X \sim P,\ Y \sim Q \tag{2.20}$$

such that $||X|| \le ||Y||$ a.s. (see Ru (1980b).

3 Fréchet-Bounds – Extremal Risk

As mentioned in the introduction Fréchet-bounds deal with the basic problem in risk theory to describe the maximal influence of stochastic dependence on the expectation of a functional $\varphi(x_1, \dots, x_n)$. Examples of interest are e.g. convex functionals of the joint position $x_1 + \dots + x_n$, where x_i are risks with distributions P_i. A typical case is $\varphi(x) = (\sum_{i=1}^n x_i - k)^+$, the excess of loss function. A nonconvex function of interest is $\varphi_t(x) = 1_{[t,\infty)}(\sum_{i=1}^n x_i)$ which yields just the inverse of the value at risk functional. Of interest is also the maximal risk of the components $\max_{i \le n} x_i$ and variants hereof. For a detailed introduction to this kind of questions we refer to Embrechts et al. (2003). An extension of the classical Fréchet-bounds in (1.4) is the following result which in particular implies sharpness of the classical Fréchet-bounds.

Theorem 3.1 (Sharpness of Fréchet-bounds, Ru (1981a))
Let $(E_i, \mathcal{A}_i)$ *be polish spaces,* $P_i \in M^1(E_i, \mathcal{A}_i)$ *and* $A_i \in \mathcal{A}_i$, $1 \le i \le n$, *then for any* $P \in M(P_1, \dots, P_n)$ *holds*

$$\left(\sum_{i=1}^n P_i(A_i) - (n-1) \right)_+ \le P\left(A_1 \times \dots \times A_n\right) \le \min\left\{P_i(A_i)\right\} \tag{3.1}$$

and the upper and lower bounds in (3.1) *are attained.*

As consequence we get sharp bounds for the influence of dependence. As a first example we consider the maximal risk of the components. Let $X = (X_1, \dots, X_n)$ be a random vector, with $X_i \sim P_i$ being real *rv*'s with *df*'s F_i. Then with $A_i = (-\infty, t]$ (3.1) implies sharp bounds for the maxima $M_n = \max_{i \le i \le n} X_i$.

Corollary 3.2 (Maximally dependent *rv*'s)

$$\begin{aligned} H^-(t) = \left(\sum_{i=1}^n F_i(t) - (n-1) \right)_+ &\le P\left(\max_{i \le n} X_i \le t \right) \\ &\le \min_{1 \le i \le n} F_i(t) = H^+(t) \end{aligned} \tag{3.2}$$

Remark 3.3 *a) Corollary 3.2 is due for* $F_1 = \dots = F_n$ *to Lai and Robbins (1976). The general case is from Lai and Robbins (1978) and by different methods from Meilijson and Nadas (1979), Tchen (1980) and Ru (1980a). Also a random vector* $\widetilde{X} = (\widetilde{X}_1, \dots, \widetilde{X}_n)$ *is constructed with* $\widetilde{X}_i \sim F_i$ *and*

$$M_n(\widetilde{X}) = \max_{i \le n} \widetilde{X}_i \sim H^-. \tag{3.3}$$

$\widetilde{X}$ *yields the lower bound in* (3.2). *It is called* maximally dependent *random vector in Lai and Robbins (1976). The upper bound* $H^+(t)$ *is attained by the comonotonic vector* $X^* = (F_1^{-1}(U), \dots, F_n^{-1}(U))$, *where* U *is uniform on* $[0, 1]$. *In stochastic ordering terms* (3.2), *is equivalent to*

$$M_n(X^*) \le_{st} M_n(X) = \max_{i \le n} X_i \le_{st} M_n(\widetilde{X}) \tag{3.4}$$

Strongly positive dependent rv's have in stochastic order small maxima i.e. they have small maximal risks of the components.

b) *The most simple way to explain the upper bound in* (3.4) *is the following argument due to Lai and Robbins (1976). In fact this is a typical argument for the duality approach to problems of this kind (see Rachev and Ru (1998)). Note that for any real* $v \in \mathbb{R}$ *:*

$$M_n(X) = \max_{i \le n} X_i \le v + \sum_{i=1}^{n} (X_i - v)_+ \tag{3.5}$$

Equality holds in (3.5) *iff for some 'splitting point'* v^* *the sets* $\{X_i \ge v^*\}$ *are pairwise disjoint and* $\bigcup_{i=1}^{n} \{X_i \ge v^*\} = \Omega$. *The maximally dependent random vector* $\widetilde{X}$ *is constructed such that there is a splitting point* v^* *as above. In the case* $F_1 = \cdots = F_n$, *Lai and Robbins (1976) proved the extremely interesting result that the maximally dependent case is close to the independent case in the following asymptotic sense under the usual domain of attraction conditions for maxima (*$\sim$ *denote here asymptotic equivalence).*

$$EM_n\left(X^{\perp}\right) \sim EM_n\left(\widetilde{X}\right) \sim F^{-1}\left(1 - \frac{1}{n}\right), \tag{3.6}$$

where $X^{\perp}$ *is an iid sequence with df* F *and* $a_n = F^{-1}(1 - \frac{1}{n})$ *is the usual normalization for the maximum law.*

There are many alternative applications of (3.1) e.g. to get sharp bounds for the concentration probabilities or to get sharp multivariate Fréchet-bounds (see Ru (2003)).

Corollary 3.4 *a)* ***Maximal concentration.***

$$\begin{aligned} \left(\sum_{i=1}^{n} (F_i(b_i) - F_i(a_i)) - (n-1) \right)_+ &\le P\left(X_i \in [a_i, b_i],\, 1 \le i \le n\right) \\ &\le \min_{i \le i \le n} \left(F_i(b_i) - F_i(a_i)\right) \end{aligned} \tag{3.7}$$

The bounds in (3.7) *are sharp.*

b) ***(Sharp) multivariate Fréchet-bounds.*** *If* X_i *are* k_i*-dimensional random vectors with df's* F_i, $1 \le i \le n$ *and* F *is the df of* $X = (X_1, \ldots, X_n)$ *then for any* $x_i \in \mathbb{R}^{k_i}$, $1 \le i \le n$*:*

$$\left(\sum_{i=1}^{n} F_i(x_i) - (n-1) \right)_+ \le F(x_1, \ldots, x_n) \le \min_{1 \le i \le n} (F_i(x_i)) \tag{3.8}$$

and the multivariate Fréchet-bounds are sharp.

Of particular interest in risk theory is the distribution and risk of the combined portfolio given by the sum $S_n(X) = \sum_{i=1}^{n} X_i$.

The following basic ordering result for the ordering of sums has been stated first in Meilijson and Nadas (1979) for the convex increasing order and in Ru (1983) for the convex order.

Theorem 3.5 (Maximal sums w.r.t. convex order) *Let X be a random vector with marginal df's $F_1, \dots, F_n$, then:*

a) ***Convex increasing order***

$$E\left(\sum_{i=1}^n X_i - t\right)_+ \leq$$

$$\psi_+(t) := \inf_{v=(v_1,\dots,v_n)} \left\{ \left(\sum v_i - t\right)_+ + \sum_{i=1}^n E(X_i - v_i)_+ \right\} \tag{3.9}$$

The bound in (3.9) *is sharp.*

b) ***Convex order***

$$\sum_{i=1}^n X_i \prec_{cx} \sum_{i=1}^n F_i^{-1}(U) \tag{3.10}$$

and $E(\sum_{i=1}^n F_i^{-1}(U) - t)_+ = \psi_+(t)$

Remark 3.6 *1) The proof of a) was given by Meilijson and Nadas (1979) by a duality argument similar to that in* (3.5)*. Also a construction of a rv attaining the upper bound is given there (for an even more general situation). The result of Meilijson and Nadas (1979) describes a sharp upper bound for the ordering* $\prec_{icx}$ *w.r.t. increasing convex functions which is also called stopp-loss ordering,* $\prec_{sl}$ *(in particular in the economics and insurance literature). That the comonotone case yields the maximum w.r.t. the convex order in* (3.10) *was stated in Ru (1983) as consequence of a more general result for supermodular functions and based on the rearrangement method which in the discrete case goes back to inequalities of Lorentz (1953). Implicitly this result is also contained in the 'Lorentz Theorem' of Tchen (1980, Theorem 5), observing that for* φ *convex,* $\varphi(x_1 + \cdots + x_n)$ *is quasimonotone (in Tchens terminology) or supermodular in the now more common terminology. This convex ordering result for sums of random variables and also the simple duality proof have been detected and rederived again several times in the literature.*

2) The sharpness of the bound in (3.9) *resp. in* (3.10) *also implies that*

$$\begin{aligned} E\left(\sum_{i=1}^n F_i^{-1}(U) - t\right)_+ &= \psi_+(t) \\ &= \inf\left\{\sum_{i=1}^n E(X_i - v_i)_+;\ \sum_{i=1}^n v_i = t\right\}. \end{aligned} \tag{3.11}$$

If X^**,* v^* *attain the upper bound in* (3.9) *then*

$$\left(\sum_{i=1}^n X_i^* - t\right)_+ = \sum_{i=1}^n (X_i^* - v_i^*)_+ \quad a.s. \tag{3.12}$$

This equality has a simple geometric meaning and is fulfilled only for the comontonic vector $(F_i^{-1}(U))_{1\le i\le n}$ *(see Meilijson and Nadas (1979) or the recent paper by Kaas, Dhaene, Vyncke, Goovaerts and Denuit (2001))*

3) *Meilijson and Nadas (1979) in fact gave sharp bounds for more general functionals. Let* $I_j \subset \{1,\dots,n\}$, $1 \le j \le k$ *be subsets with* $\bigcup_{j=1}^k I_j = \{1,\dots,n\}$ *and consider* $M = \max_{1\le i\le k} \sum_{j\in I_i} X_j$. *Then for all x:*

$$E(M-x)_+ \le \widetilde{\psi_+}(x) := \inf_v \left\{ \left(\max_{1\le i\le k} \sum_{j\in I_i} v_j - x \right)_+ + \sum_{i=1}^n E\left(X_i - v_i\right)_+ \right\} \tag{3.13}$$

and the upper bound in (3.13) *is pointwise sharp. Furthermore for cyclic directed networks the bound is attained stochastically for some* $P \in M(P_1,\dots,P_n)$.

4) ***Comonotonic vectors, multivariate marginals.*** *While for one dimensional marginals comonotonic vectors maximize the risk for many convex functionals (like in* (3.9), (3.10)*) for* $\varphi(\sum_{i=1}^n x_i)$, φ *convex) this is no longer the case for multivariate marginals, where they can even minimize the risk. For some illustrative examples see Ru (2003). The reason for this is the possible negative dependence in the components of the marginals.*

In the recent paper of Denuit, Dhaene and Ribas (2001) the following interesting result related to (3.9), (3.10) was proved by a simple induction argument:

Theorem 3.7 (Positive dependence increases risk) *If* X *is associated then*

$$\sum_{i=1}^n X_i^\perp \le_{sl} \sum_{i=1}^n X_i. \tag{3.14}$$

Here $X^\perp = (X_1^\perp,\dots,X_n^\perp)$ *has independent components and* $X_i^\perp \sim X_i$.

So positive dependence (association) leads to riskier portfolios. In Christofides and Vaggelatou (2004) and Ru (2003) a general version of this result has been given stating that positive dependence leads to higher risk for a general class of proper risk functions $f(X_1,\dots,X_n)$.

Of particular interest in risk theory is to describe the influence of dependence on the value at risk functional of the combined portfolio $\mathrm{VaR}_\alpha(X_1+\cdots+X_n)$ which is defined as the α-quantile of the combined portfolio $X_1+\cdots+X_n$. For the description of the maximal influence the following functionals are of interest: Given n *df*'s $F_1,\dots,F_n$ consider:

$$\begin{aligned} M_n(t) &= \sup\left\{ P\left(\sum_{i=1}^n X_i \le t\right);\ X_i \sim F_i,\ 1\le i\le n \right\} \\ m_n(t) &= \inf\left\{ P\left(\sum_{i=1}^n X_i < t\right);\ X_i \sim F_i,\ 1\le i\le n \right\} \end{aligned} \tag{3.15}$$

Then

$$1 - m_n(t) = \sup\{P(\sum X_i \geq t);\ X_i \sim F_i,\ 1 \leq i \leq n\} \tag{3.16}$$

and one obtains the sharp upper bound

$$\mathrm{VaR}_\alpha(X_1 + \cdots + X_n) \leq (1 - m_n)^{-1}(\alpha). \tag{3.17}$$

For the case $n = 2$ the following bounds were first established in Sklar (1973) and in more general form in Moynihan, Schweizer and Sklar (1978). The bounds and also their sharpness were independently established in Makarov (1981) and Ru (1982) (For the history of this result see also Schweizer (1991)).

Theorem 3.8 (Maximal sum risks, *n=2*, Makarov (1981), Ru (1982), Sklar (1973)) *Let X be a random vector with marginal df's $F_1, \ldots, F_n$, then for $n = 2$ holds:*

$$\begin{aligned} P(X_1 + X_2 \leq t) &\leq M_2(t) = F_1 \wedge F_2(t) \\ P(X_1 + X_2 < t) &\geq m_2(t) = F_1 \vee F_2(t) - 1, \end{aligned} \tag{3.18}$$

where $F_1 \wedge F_2(t) = \inf_x(F_1(x_-) + F_2(t - x))$ is the infimal convolution function and $F_1 \vee F_2(t) = \sup_x(F_1(x_-) + F_2(t - x))$ is the supremal convolution function.

(3.16) is derived in Ru (1982) as consequence of the following general representation of the upper Fréchet-bounds for $\varphi = 1_A$, $P_1, P_2 \in M^1(\mathbb{R}^n)$ and $A \subset \mathbb{R}^{2n}$ closed:

$$\begin{aligned} M(A) &= \sup\{P(A);\ P \in M(P_1, P_2)\} \\ &= 1 - \sup\{P_2(O) - P_1(\pi_1(A \cap (\mathbb{R}^n \times O)));\ O \subset \mathbb{R}^n \text{ open}\}, \end{aligned} \tag{3.19}$$

where π_1 is the projection on the first component. (3.19) is a consequence of Strassen (1965, Theorem 11) (see Ru (1982,1986)).

The type of bounds in (3.18) extends easily to $n \geq 3$ (see Frank, Nelsen and Schweizer (1987) and Denuit, Genest and Marceau (1999)).

Proposition 3.9 *Let X be a random vector with marginal df's $F_1, \ldots, F_n$. Then for any $t \in \mathbb{R}^1$ holds:*

$$\left(\bigvee_{i=1}^n F_i(t) - (n-1)\right)_+ \leq P\left(\sum_{i=1}^n X_i \leq t\right) \leq \min\left(\bigwedge_{i=1}^n F_i(t), 1\right) \tag{3.20}$$

where $$\bigwedge_{i=1}^n F_i(t) := \inf\left\{\sum_{i=1}^n F_i(u_i);\ \sum_{i=1}^n u_i = t\right\}$$

and $$\bigvee_{i=1}^n F_i(t) := \sup\left\{\sum_{i=1}^n F_i(u_i);\ \sum_{i=1}^n u_i = t\right\}$$

Proof: The proof of (3.20) follows by induction from the case $n = 2$; alternatively by the following simple argument. For any $u_1, \dots, u_n$ with $\sum_{i=1}^n u_i = t$ holds:

$$P\left(\sum_{i=1}^n X_i \le t\right) \le P\left(\bigcup_{i=1}^n \{X_i \le u_i\}\right) \le \sum_{i=1}^n F_i(u_i), \tag{3.21}$$

which gives the upper bound. Similarly, using the Fréchet lower bound in (1.4) we obtain

$$P\left(\sum_{i=1}^n X_i \le t\right) \ge P\left(X_1 \le u_1, \dots, X_n \le u_n\right) \ge \left(\sum_{i=1}^n F_i(u_i) - (n-1)\right)_+. \tag{3.22}$$

□

Remark 3.10 *The bounds in* (3.17) *are however in contrast to the case* $n = 2$ *not sharp. If* $n = 3$, $F_1 = F_2 = F_3$ *are the df of the uniform distribution on* $[0, 1]$ *then*

$$M_3(t) = \begin{cases} \frac{2}{3}t & , 0 \le t \le \frac{3}{2} \\ 1 & , t > \frac{3}{2} \end{cases}, \quad m_3(t) = \begin{cases} \frac{2}{3}t - 1 & , 0 \le t \le 3 \\ 1 & , t \ge 3 \end{cases} \tag{3.23}$$

(see Ru (1982)). The bounds in (3.20),(3.21) *are in this case more crude.*

$$\min\left(1, \bigwedge_{i=1}^3 F_i(t)\right) = \min(1, t) \ \ \textit{and} \ \left(\bigvee_{i=1}^3 F_i(t) - 2\right)_+ = (t-2)_+ \tag{3.24}$$

For some examples sharp bounds for $n \ge 3$ *have been given in Ru (1982) and Rachev and Ru (1998).*

The simple method of bounding the risk probability in (3.20) has been given and extended in Frank, Nelsen and Schweizer (1987) to general monotonically nondecreasing functions $\psi(x_1, \dots, x_n)$. The resulting bounds are of interest and markable relevance if further information on the underlying *df*'s can be used. The following is essentially a reformulation of corresponding results in Moynihan, Schweizer and Sklar (1978), Frank, Nelsen and Schweizer (1987), Denuit, Genest and Marceau (1999) and Embrechts Höing and Juri (2003). For a *df* H let $\overline{H}$ denote the corresponding multivariate survival function $\overline{H}(x) = P_H([x, \infty))$. For $t \in \mathbb{R}$ let $A_\psi^+(t) := \{u = (u_1, \dots, u_n) : u$ a maximal point in $\mathbb{R}^n$ with $\psi(u) \le t\}$

Theorem 3.11 (Bounds for monotonic functionals)
Let $X = (X_1, \dots, X_n)$ *be a random vector with df* $F \in \mathcal{F}(F_1, \dots, F_n)$ *and let* $\psi(x)$ *be monotonically nondecreasing and lower semicontinuous. Then*

a) ***General bounds.***

$$\begin{aligned}\left(\sup_{u\in A_\psi^+(t)}\sum_{i=1}^n F_i(u_i)-(n-1)\right)_+ &\le P\left(\psi(X)\le t\right) \\ &\le \inf_{u\in A_\psi^+(t)}\sum_{i=1}^n F_i(u_i)\end{aligned} \tag{3.25}$$

b) ***Improved bounds.*** *If G, H are df's, then*

1) $F \ge G$ implies

$$P\left(\psi(X)\le t\right)\ge \sup_{u\in A_\psi^+(t)} G(u). \tag{3.26}$$

2) If $\overline{F}\ge\overline{H}$, then

$$P\left(\psi(X)< t\right)\le 1-\sup_{u\in A_\psi^-(t)} \overline{H}(u), \tag{3.27}$$

where $A_\psi^-(t) := \{u\in\mathbb{R}^n : \psi(u)\ge t\}$

Proof:

a) For any $u\in A_\psi^+(t)$ holds, using maximality of u,

$$P\left(\psi(X)\le t\right)\le P\left(\bigcup_{i=1}^n\{X_i\le u_i\}\right)\le\sum_{i=1}^n F_i(u_i).$$

This implies the upper bound in a). Further,

$$\begin{aligned}P\left(\psi(X)\le t\right) &\ge P\left(X_1\le u_1,\dots,X_n\le u_n\right) \\ &\ge \left(\sum_{i=1}^n F_i(u_i)-(n-1)\right)_+\end{aligned} \tag{3.28}$$

by the lower Fréchet-bound.

b) If $F\ge G$, then in (3.28) we get

$$P\left(\psi(X)\le t\right)\ge\left(\sup_{u\in A_\psi^+(t)} G(u)-(n-1)\right)_+.$$

If $\overline{F}\ge\overline{H}$ then for $u\in A_\psi^-(t)$

$$\begin{aligned}P\left(\psi(X)<t\right) &= 1-P\left(\psi(X)\ge t\right) \\ &\le 1-P\left(X_1\ge u_1,\dots,X_n\ge u_n\right) \\ &= 1-\overline{F}(u)\le 1-\overline{H}(u),\end{aligned}$$

which implies 2).

□

Remark 3.12 *a) For the case $n = 2$ one gets sharp upper and lower bounds for $P(\psi(X) \geq t)$ by applying (3.20) to the set $A = \{x = (x_1, x_2) : \psi(x_1, x_2) \geq t\}$ for any function ψ, in particular for monotonically nondecreasing functions.*

b) Theorem 3.2 in Embrechts, Höing and Juri (2003) state sharpness of the bounds in (3.26) and (3.27). In comparison to Embrechts, Höing and Juri (2003) we omit some continuity assumption on ψ and omit the language of copulas which is not necessary here. The corresponding bound for the value at risk functionals are in consequence of the monotonicity of ψ easy to achieve (see Embrechts et al. (2003, Theorem 4.1)). There is still a lot of open problems in this area. In particular how to obtain applicable and good bounds under additional information on the model.

c) An extension of the bounds in (3.14), (3.25) to increasing functions $\psi(X, Y)$ of k-dimensional vectors has been given in Li, Scarsini and Shaked (1996). For $n = 2$ one gets sharpness by Strassens theorem as in (3.17). Also the partial integration argument from Ru (1980a) can be applied to obtain bounds for $Eg(X + Y)$ for increasing differentiable functions g (see Li, Scarsini and Shaked (1996, Theorem 4.2)) and more general to any Δ-monotone functions $f(X_1, \ldots, X_n)$ for k_i-dimensional random vectors X_i (see Ru (2003)). Several further bounds and techniques for obtaining bounds are discussed in Ru (1991a). One rich source of such bounds are Bonferroni inequalities which in many cases can be proved by a general reduction principle to be sharp. Let e.g. $A_1, \ldots, A_n \in \mathcal{A}$ where $(E, \mathcal{A})$ is any measure space and $P_i \in M^1(E, \mathcal{A})$. Let $X = (X_1, \ldots, X_n)$ be a random vector with $X_j \sim P_j$, $1 \leq j \leq n$ and define the set that at least k of the events $\{X_j \in A_j\}$ holde true,

$$L_k := \bigcup_{J \subset \{1, \ldots, n\},\ |J| = k} \{X_j \in A_j,\ j \in J\}. \tag{3.29}$$

Then

$$\begin{aligned} P(L_k) &\leq b_k := \min_{0 \leq r \leq k-1} \left(1, \frac{1}{k - r} \sum_{i=1}^{n-r} p_{(i)}\right) \\ P(L_k) &\geq a_k := \max \left(0, \frac{\sum_{i=r+1}^{n} p_{(i)} - (k-1)}{n - r - (k-1)}\right) \end{aligned} \tag{3.30}$$

where $p_i = P(X_i \in A_i)$ and $p_{(1)} \leq \cdots \leq p_{(n)}$ (see Ru (1991a)).

The bounds in (3.30) are sharp. They are consequences of a Bonferroni type result in Rüger (1979) and a general reduction principle (see Ru (1991a)). In *particular for real random variables and* $A_i = [t, \infty)$ *one gets sharp upper and lower bounds for the tail of the* k*-th order statistic*

$$P\left(X_{(k)} \geq t\right) \begin{cases} \leq b_k, & \text{with } p_i = P(X_i \geq t) \\ \geq a_k \end{cases} . \tag{3.31}$$

Also extensions to higher order Bonferroni bounds are given in Ru (1991) and to improved bounds in the case that one can use some of the higher order joint marginal distributions.

4 Δ-Monotone, Supermodular and Directionally Convex Function Classes

In various applications of comparing risks it has turned out that Δ-monotone, supermodular and directionally convex functions and variants of them play an eminent role (see Müller and Stoyan (2002)). For the definition we introduce for $f : \mathbb{R}^n \to \mathbb{R}$, $\varepsilon > 0$, the difference operator $\Delta_i^\varepsilon f$ by

$$\Delta_i^\varepsilon f(x) = f(x + \varepsilon e_i) - f(x), \ 1 \leq i \leq n \tag{4.1}$$

where e_i is the i-th unit vector.

Definition 4.1 *Let* $f : \mathbb{R}^n \to \mathbb{R}$*.*

1) f *is* Δ-monotone *if for every subset* $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ *and* $\varepsilon_1, \dots, \varepsilon_k > 0$ *holds*

$$\Delta_{i_1}^{\varepsilon_1} \dots \Delta_{i_k}^{\varepsilon_k} f(x) \geq 0 \quad \text{for all } x. \tag{4.2}$$

2) f *is* supermodular *if for all* $1 \leq i < j \leq n$, $\varepsilon, \delta > 0$ *and all* x

$$\Delta_i^\varepsilon \Delta_j^\delta f(x) \geq 0. \tag{4.3}$$

3) f *is* directionally convex *if* (4.3) *holds for all* $i \leq j$*.*

Denote by $\mathcal{F}^\Delta$, $\mathcal{F}^{sm}$, $\mathcal{F}^{dcx}$ *the set of all* Δ*-monotone resp. supermodular resp. directionally convex functions then* $\mathcal{F}^\Delta \subset \mathcal{F}^{sm}$ *and* $\mathcal{F}^{dcx} \subset \mathcal{F}^{sm}$*.*

Remark 4.2 *The class of supermodular and directionally convex functions were investigated in early papers of Lorentz (1953) and Ky Fan and Lorentz (1954) in the context of functional inequalities (see also the extensive chapter in Marshall and Olkin (1979)). In Cambanis, Simons and Stout (1976) and Tchen (1980) supermodular functions are called quasimonotone.* Δ*-monotone functions were introduced in Ru (1980). Twice differentiable functions* f *are supermodular (directionally convex) if*

$$\frac{\partial^2}{\partial x_i \partial x_j} f(x) \geq 0 \quad \text{for all } x \text{ and } i < j \text{ (resp. for } i \leq j). \tag{4.4}$$

Differentiable functions f are Δ-monotone if for all $i_1 < i_2 < \cdots < i_k$, $1 \le k \le n$

$$\frac{\partial^k}{\partial x_{i_1} \dots \partial x_{i_k}} f(x) \ge 0. \tag{4.5}$$

Definition 4.3 *For $P, Q \in M^1(\mathbb{R}^n)$ define*

a) $P \le_{uo} Q$ 'upper orthant ordering' *if*

$$P([x,\infty]) \le Q([x,\infty]), \quad \forall\, x \in \mathbb{R}^n \tag{4.6}$$

b) $\le_{sm}$, $\le_{dcx}$ *denote the supermodular ordering resp. directionally convex ordering generated by $\mathcal{F}^{sm}$ resp. $\mathcal{F}^{dcx}$.*

There are corresponding positive/negative dependence notions.

Definition 4.4 *1) P is* positive *(negative)* upper orthant dependent *$-P \in PUOD$ (resp. $P \in NUOD$) if*

$$\bigotimes_{i=1}^{n} P_i \le_{uo} P \quad \left(resp.\ P \le_{uo} \bigotimes_{i=1}^{n} P_i \right) \tag{4.7}$$

2) P is weakly associated *if $E \prod_{i=1}^n f_i(X_i) \ge \prod_{i=1}^n E f_i(X_i)$ for all nondecreasing $f_i \ge 0$.*

Remark 4.5 *A similar notion also exists for lower orthants and is denoted by $\le_{lo}$ resp. PLOD and NLOD. This notion was introduced in Lehmann (1966). The following equivalence holds:*

$$P \in PUOD \text{ if and only if } P \text{ is weakly associated.} \tag{4.8}$$

(For $n = 2$ due to Lehmann (1966), for $n \ge 2$ to Ru (1981c).) In fact more generally it was shown in Bergmann (1978) that: $P \le_{uo} Q$ if and only if for $X \sim P$, $Y \sim Q$

$$E \prod_{i=1}^{n} f_i(X_i) \le E \prod_{i=1}^{n} f_i(Y_i), \tag{4.9}$$

for f_i nondecreasing, $f_i \ge 0$.

The maximal generator of the upper orthant order is the set of Δ-monotone functions.

Theorem 4.6 (Δ-monotone functions, Ru (1980a)) *If $P, Q \in M^1(\mathbb{R}^n)$, then: $P \le_{uo} Q$ if and only if*

$$\int f\, dP \le \int f\, dQ \quad \text{for all } f \in \mathcal{F}^{\Delta} \tag{4.10}$$

which are integrable w.r.t. P and Q, i.e. $\le_{uo}$ is equivalent to $\le_{\mathcal{F}^\Delta}$.

A similar result of course also holds for the lower orthant order and can be combined to characterize the 'concordance ordering'; $P \leq_{con} Q$ if $P \leq_{uo} Q$ and $P \leq_{lo} Q$. A random vector X is called WA (weakly associated) if $\mathrm{Cov}(f(X_J), g(X_I)) \geq 0$ for any disjoint subsets J, I of $\{1, \dots, n\}$ and monotonically nondecreasing functions f, g of these components. Concerning the supermodular ordering the analogous result is the following.

Theorem 4.7 (Supermodular functions) *Let $P, Q \in M^1(\mathbb{R}^n)$,*

a) $n = 2$ Cambanis, Simons and Stout (1976). For $P, Q \in M(P_1, P_2)$ holds:

$$P \leq_{uo} Q \Longleftrightarrow P \leq_{sm} Q \tag{4.11}$$

b) $n \geq 2$ 'The Lorentz Theorem', Tchen (1980), Ru (1983). For $P \in M(P_1, \dots, P_n)$ holds

$$P \leq_{sm} P_+, \tag{4.12}$$

where P_+ is the measure corresponding to the upper Fréchet-bound (the comonotonic measure).

c) Christofides and Vaggelatou (2003).
If X is a weakly associated random vector, then X has positive supermodular dependence , i.e.

$$\bigotimes_{i=1}^{n} P^{X_i} \leq_{sm} P^X. \tag{4.13}$$

Remark 4.8 *a) The interesting result in (4.13) can be stated in the form: Positive dependence implies increasing of the risk. This is of essential interest in risk theory.*

b) (4.12) was proved in Tchen (1980) by discrete approximation and reduction to the Lorentz (1953) inequalities. In Ru (1979, 1983) the problem of generalized Fréchet-bounds was identified with a rearrangement problem for functions and then reduced to the Lorentz inequality.

c) For P and Q with identical (n-1)-dim. marginal distributions one obtains

$$P \leq_{uo} Q \Rightarrow P \leq_{sm} Q \tag{4.14}$$

(Tchen (1980), Ru (1980a, Theorem 3b))

d) There are some useful composition rules which allow to use $\leq_{sm}$ for several models of interest (see Müller and Stoyan (2002) for results and references).

e) From Tchens proof of b) it is clear that $P_- \leq_{sm} P$ if the lower Fréchet-bound P_-, is a df (cf. also Müller and Stoyan (2002, p. 120)).

f) *Since for any φ convex the function $\psi(x) = \varphi(x_1 + \cdots + x_n)$ is supermodular one obtains as consequence of* (4.12) *the statement of* (3.9) *that*

$$\sum_{i=1}^n X_i \leq_{cx} \sum_{i=1}^n F_i^{-1}(U) \tag{4.15}$$

i.e. the sums are maximal in convex order for the comonotone case, in other words the comonotone positively dependent portfolio is the riskiest possible.

g) *Comparison of $P \in M(P_1, \dots, P_n)$ and $Q \in M(Q_1, \dots, Q_n)$ w.r.t. the supermodular ordering $\leq_{sm}$ is only possible if the marginals are identical*

$$P \leq_{sm} Q \text{ implies } P_i = Q_i, \quad 1 \leq i \leq n. \tag{4.16}$$

So in comparison to $\leq_{\mathcal{F}^\Delta} = \leq_{uo}$ the $\leq_{sm}$ ordering is restricted to one marginal class while $\leq_{uo}$ allows comparisons between P and Q if the marginals increase stochastically:

$$P \leq_{uo} Q \text{ implies } P_i \leq_{st} Q_i, \quad 1 \leq i \leq n. \tag{4.17}$$

The comparison by $\mathcal{F}^\Delta$ is however for a smaller class of functions $\mathcal{F}^\Delta \subset \mathcal{F}^{sm}$. On the other hand criteria for $\leq_{sm}$ are not as simple as those for $\leq_{uo}$. The directionally convex order $\leq_{dcx}$ is a 'typical' risk order. It allows comparisons in cases where the marginals increase convexly:

$$P \leq_{dcx} Q \text{ implies } P_i \leq_{cx} Q_i, \quad 1 \leq i \leq n. \tag{4.18}$$

From the copula representation (2.7) of distributions with given marginals the following is immediate: If $P \in M(P_1, \dots, P_n)$, $Q \in M(Q_1, \dots, Q_n)$ have the same copula C and $P_i \leq_{st} Q_i$, $1 \leq i \leq n$, then

$$P \leq_{st} Q. \tag{4.19}$$

($\leq_{st}$ is the multivariate stochastic order w.r.t. increasing functions, see Ru (1981b, Proposition 7).

The situation is more complicated if the marginals increase in convex order. Here the analog of (4.19) is wrong, see Müller and Scarsini (2001). The reason is that negative dependence can destroy this conclusion, as the following simple example of that paper shows.

Example 4.9 *Consider $n = 2$ and rv's $X=(W, -W)$, $Y = (W, -EW)$ for some integrable random variable W. Then $Y_i \leq_{cx} X_i$, $i = 1, 2$, but $X_1 + X_2 = W - W = 0$ while $Y_1 + Y_2 = W - EW$ i.e. $X_1 + X_2 \leq_{cx} Y_1 + Y_2$.*

However in the positive direction Müller and Scarsini (2001) proved the interesting result, that under a strong positive dependence assumption the analog of (4.19) is true. Let F_+ denote as usual the upper Fréchet-bound of $\mathcal{F}(F_1, \dots, F_n)$.

Theorem 4.10 (Directionally convex ordering) *Let F_i, G_i be one dimensional df's, $1 \leq i \leq n$.*

a) 'The Ky-Fan-Lorentz Theorem'
Ru (1983). If $F_i \leq_{cx} G_i$, $1 \leq i \leq n$ then

$$F_+ \leq_{dcx} G_+. \tag{4.20}$$

b) Müller and Scarsini (2001). If $F \in \mathcal{F}(F_1, \dots, F_n)$ and $G \in \mathcal{F}(G_1, \dots, G_n)$ have the same conditionally increasing (CI) copula C and if $F_i \leq_{cx} G_i$, $1 \leq i \leq n$, then

$$F \leq_{dcx} G. \tag{4.21}$$

Remark 4.11 *Müller and Scarsini (2001) give a proof of a) using mean preserving spread (see Theorem 3.12.13 of their paper) while the proof in Ru (1983) is based on the Ky Fan and Lorentz Theorem. The second main ingredient of the proof of b) is the a.s. standard construction of random vectors in* (2.4)*: $X = \tau_F^*(V) = (h_1(V_1), \dots, h_n(V_1, \dots, V_n))$, where the functions h_i are monotonically nondecreasing for CI distribution functions.* (4.21) *is not valid any more under the weaker dependence assumption of association or of conditional increasing in sequence CIS (see Müller and Scasini (2001)).*

The following weakening of the WA-notion was introduced in Ru (2003):
X is smaller than Y in the weakly conditional in sequence order – $X \leq_{WCS} Y$ – if for all $t, 1 \leq i \leq n-1$ and f monotonically nondecreasing

$$\mathrm{Cov}(1(X_i > t), f(X_{(i+1)})) \leq \mathrm{Cov}(1(Y_i > t), f(Y_{(i+1)})) \tag{4.22}$$

where $X_{(i+1)} = (X_{i+1}, \dots, X_n)$. X is called weakly associated in sequence (WAS) if $X^* \leq_{WCS} X$, where X^* is the corresponding version of X with independent components; equivalently for all t

$$P^{X_{(i+1)}|X_i > t} \geq_{st} P^{X_{(i+1)}}. \tag{4.23}$$

The following result extends and unifies Theorem 4.7 and 4.10.

Theorem 4.12 (WCS-Theorem, Ru (2003)) *Let X, Y be random vectors with marginals P_i, Q_i.*

a) If $P_i = Q_i, 1 \leq i \leq n$ and $X \leq_{WCS} Y$ then $X \leq_{sm} Y$

b) If $P_i \leq_{cx} Q_i, 1 \leq i \leq n$ and $X \leq_{WCS} Y$ then $X \leq_{dcx} Y$.

The ordering $\leq_{WCS}$ combines an increase in positive dependence with a convex increase of the marginals. Some examples for this ordering are given in Ru (2003). In particular one obtains as corollary:

Corollary 4.13 *If $F \in \mathcal{F}(F_1, \dots, F_n)$ and $F_i \leq_{cx} G_i$, $1 \leq i \leq n$, then*

a)

$$F \leq_{dcx} G_+ \tag{4.24}$$

b) If $X \sim F$*, then*

$$\sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} G_i^{-1}(U), \tag{4.25}$$

where U *is uniformly distributed on* $(0,1)$*.*

Finally we state some new criteria for the $\leq_{sm}$ and $\leq_{dcx}$ ordering in functional dependence models which are related to Bäuerle (1997, Theorem 3.1) and Bäuerle and Müller (1998). For the proofs see Ru (2003). Let (U_i) be independent rv's and (V_i), V any random variables independent of (U_i). Further let

$$X_i = g_i(U_i, V_i),\ Y_i = g_i(U_i, V),\ Z_i = \widetilde{g}_i(U_i, V_i),\ W_i = \widetilde{g}_i(U_i, V) \tag{4.26}$$

where $V_i \sim V$ and $g_i(u, \cdot)$, $\widetilde{g}_i(u, \cdot)$ are monotonically nondecreasing. Let $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n)$, Z, W denote the corresponding vectors and let $\leq_{ccx}$ denote the componentwise convex order. X, Y, Z, W describe functional models where the dependence is obtained in functional form from some inner and outer factors U_i resp. V_i. These type of models are of particular relevance in various applications in insurance and in economics.

Theorem 4.14 *For the* X*,* Y*,* Z*,* W *specified as in* (4.26) *holds:*

a) Bäuerle (1997). $X \leq_{sm} Y$*,* $Z \leq_{sm} W$

b) If for all v*,* $\widetilde{g}_i(U_i, v) \leq_{cx} g_i(U_i, v)$ *then* $Z \leq_{ccx} X$*,* $W \leq_{ccx} Y$ *and* $Z \leq_{dcx} Y$*.*

c) If $g_i(U_i, v) \leq_{cx} \widetilde{g}_i(U_i, v)$ *then* $X \leq_{ccx} Z$*,* $Y \leq_{ccx} W$*, and* $X \leq_{dcx} W$*.*

For the proofs of b) and c) see Ru (2003).

Remark 4.15 *The random vectors* Z*,* Y *and* X*,* W *which are compared w.r.t.* $\leq_{dcx}$ *in Theorem 4.14 do not have the same dependence structure (copula), which was a basic assumption for the proof of the* $\leq_{dcx}$ *ordering result in Theorem 4.10 b). Also the* X *and* W *vectors are not necessarily positive dependent. Since we do not postulate any independence for the* (V_i)*, we can describe any multivariate df* F *by a random vector of the form as for* X*. Thus this comparison result applies to many models. Similarly as in Bäuerle (1997) one could add in Theorem 4.14 a further random influence component* W *and consider models of the form* $X_i = g_i(U_i, V_i, W)$*,* $Y_i = g_i(U_i, V_i, W)$ *etc.*

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