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ON THE EXISTENCE OF PROBABILITY MEASURES WITH GIVEN MARGINALS.

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Abstract. As an immediate consequence of a minimax theorem Ky Fan [2] proved a characterization of consistency of a system of convex inequalities. The aim of this paper is to show that by means of Ky Fan's theorem one can obtain simple and unified proofs of some existence theorems on probability measures with given marginals, avoiding in this way approximation and disintegration techniques.

1. Preliminaries. In fact we shall make use of the following special case of Theorem 1 of Ky Fan [2]:

THEOREM 1. Let K be a compact, convex set in a real topological vector space X. Let $(F_s)_{s\in S}$ be a family of continuous, linear, real functions on X and $(a_s)_{s\in S}$ be a family of real numbers. Then the system

(1) $$F_s(x) = a_s,\ s \in S$$

is consistent on K (i.e. there exists an $x_o \in K$ satisfying (1)), if and only if for any finite set of indices $s_1,\dots,s_m \in S$ and

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any real numbers $\alpha_1,\dots,\alpha_m$ it holds

$$(2)\qquad \min_{x\in K} \sum_{j=1}^{m} \alpha_j F_{s_j}(x) \le \sum_{j=1}^{m} \alpha_j a_{s_j} .$$

In what follows, $(\Omega_i,\mathcal{B}_i)$, $1 \le i \le n$, are measurable spaces ($\mathcal{B}_i$ being a σ-field in Ω_i), and $(\Omega,\mathcal{B}) = \bigotimes_{i=1}^{n} (\Omega_i,\mathcal{B}_i)$ is their product space. The projection of Ω onto Ω_i is denoted by π_i. There is given a set $\mathcal{A}$ of nonempty subsets $A \subset \{1,\dots,n\}$. By $(\Omega_A,\mathcal{B}_A)$ we denote the product space of the $(\Omega_i,\mathcal{B}_i)$, $i \in A$, and by π_A the projection of Ω onto Ω_A. When λ is a signed measure on $(\Omega,\mathcal{B})$, then its marginals $\pi_A\lambda$ on $(\Omega_A,\mathcal{B}_A)$ are defined by

$$(\pi_A\lambda)(B_A) = \lambda(\pi_A^{-1}(B_A)),\ B_A \in \mathcal{B}_A.$$

In the first part of Section 2 (Theorem 2, Corollaries 1, 2) topological assumptions are needed. Here the Ω_i are polish spaces and $\mathcal{B}_i$ are the Borel σ-fields. By $\mathcal{R}_i$ and $\mathcal{R}$ we denote the fields generated by the open sets in Ω_i and Ω respectively. Let $C_b(\Omega)$ be the space of all bounded, continuous, real functions on Ω. As it is wellknown, the topological dual $(C_b(\Omega))^*$ can be identified with $rba(\Omega,\mathcal{R})$ - the space of all regular, bounded, additive set functions on $(\Omega,\mathcal{R})$ (cf. Dunford-Schwartz [1], Theorem 2, p. 262). The following lemma ensures countable additivity of certain elements of $rba(\Omega,\mathcal{R})$.

LEMMA 1. Let Ω_i be polish spaces, $1 \le i \le n$, let $\mu \in rba(\Omega,\mathcal{R})$ be nonnegative, and let the marginals $\pi_i\mu$ be countably additive on $\mathcal{R}_i$, $1 \le i \le n$. Then μ is countably additive on $\mathcal{R}$. Consequently there exists a unique measure $\bar{\mu}$ on $\mathcal{B}$ that extends μ and, especially, $\int f d\mu = \int f d\bar{\mu}$, for all $f \in C_b(\Omega)$.

Proof. Let $\mathcal{R}_o$ be the field generated by the cartesian products $\prod_{i=1}^{n} R_i$, where $R_i \in \mathcal{R}_i$, $1 \le i \le n$, and μ_o be the restriction of μ on $\mathcal{R}_o$. As in [9], Lemma 2, inner compact regularity of μ_o can be proved, and hence countable additivity of μ_o. Therefore, μ_o possesses a unique extension to a measure $\bar{\mu}$ on $\mathcal{B}$. Using regularity of μ and $\bar{\mu}$ it is easy to see that $\mu(R) = \bar{\mu}(R)$ for all $R \in \mathcal{R}$.

2. Probability measures with given marginals. The problem we want to consider first is the following. Let Ω_i be polish spaces, $1 \le i \le n$, and λ_A be given probability measures on $(\Omega_A,\mathcal{B}_A)$, $A \in \mathcal{A}$. Furthermore, let Λ be a subset of $M_1(\Omega,\mathcal{B})$ - the set of all probability measures on $(\Omega,\mathcal{B})$. The problem is to characterize the existence of a $\lambda^* \in \Lambda$ with marginals $\pi_A\lambda^* = \lambda_A$, $A \in \mathcal{A}$.

Considering $M_1(\Omega,\mathcal{B})$ as a subset of $(C_b(\Omega))^*$ we assume that

(3) Λ is a convex and relatively closed subset of $M_1(\Omega,\mathcal{B})$ w.r.t. the weak*-topology.

Another assumption is more technical:

$$(4)\qquad \bigcup_{A\in\mathcal{A}} A = \{1,\dots,n\}.$$

THEOREM 2. Under assumptions (3) and (4) there exists a $\lambda^* \in \Lambda$ with marginals λ_A, $A \in \mathcal{A}$, if and only if

$$(5)\qquad \inf\{\int(\sum_{A\in\mathcal{A}} f_A \circ \pi_A)d\lambda;\ \lambda \in \Lambda\} \le \sum_{A\in\mathcal{A}} \int f_A d\lambda_A,$$

for all $f_A \in C_b(\Omega_A)$, $A \in \mathcal{A}$.

Proof. The necessity of (5) is obvious. To prove sufficiency, apply Theorem 1 with $X = rba(\Omega,\mathcal{R})$, $K = \bar{\Lambda}$ - the weak *-closure of Λ in X - (K is by Alaoglu's Theorem compact) - and with the system of linear equalities

$$(1')\qquad \int f_A \circ \pi_A d\lambda = \int f_A d\lambda_A,\ f_A \in C_b(\Omega_A),\ A \in \mathcal{A}.$$

Then condition (2) becomes

(2') $$\min \{\int(\sum_{A\in\mathcal{A}} f_A\circ\pi_A)d\lambda;\ \lambda \in \bar{\Lambda}\} \le \sum_{A\in\mathcal{A}} \int f_A d\lambda_A,$$

for all $f_A \in C_b(\Omega_A)$, $A \in \mathcal{A}$, which is fulfilled by (5).

Therefore, Theorem 1 yields the existence of $\lambda^* \in \bar{\Lambda}$ satisfying (1'). Assumption (4) implies that $\pi_i\lambda^*$, $1 \le i \le n$, are countably additive on $\mathcal{R}_i$ and from Lemma 1 and assumption (3) it follows that λ^* can be chosen in $\bar{\Lambda} \cap M_1(\Omega,\mathcal{B}) = \Lambda$. ┘

REMARK 1.

a) For $\mathcal{A}$ being the set of singletons $\{i\}$, $1 \le i \le n$, Theorem 2 has been proved by Strassen [10], Theorem 7, under a somewhat weaker topological assumption on Λ.

b) For $\Lambda = M_1(\Omega,\mathcal{B})$ the left hand side of (5) equals $\inf\{\sum_{A\in\mathcal{A}} f_A\circ\pi_A(\omega);\ \omega \in \Omega\}$, and in this case Theorem 2 has been proved by Kellerer [8], Satz 2.2, for $\Omega_i = \mathbb{R}$, $1 \le i \le n$. ┘

COROLLARY 1. Let (4) be satisfied and let $F \subset \Omega$ be closed and $0 \le \varepsilon \le 1$. Then there exists a probability measure λ^* on $(\Omega,\mathcal{B})$ with marginals λ_A, $A \in \mathcal{A}$, and $\lambda^*(F) \ge 1-\varepsilon$ if and only if

(6) $$(1-\varepsilon)\inf\{\sum_{A\in\mathcal{A}} f_A\circ\pi_A(\omega);\ \omega \in F\} + \varepsilon \inf\{\sum_{A\in\mathcal{A}} f_A\circ\pi_A(\omega);\ \omega\in\Omega\}$$

$$\le \sum_{A\in\mathcal{A}} \int f_A d\lambda_A, \text{ for all } f_A \in C_b(\Omega_A),\ A \in \mathcal{A}.$$

Proof. The set $\Lambda = \{\lambda \in M_1(\Omega,\mathcal{B});\ \lambda(F) \ge 1-\varepsilon\}$ is convex and relatively closed in $M_1(\Omega,\mathcal{B})$ (by Portmanteau's Theorem). With $f_{\mathcal{A}} := \sum_{A\in\mathcal{A}} f_A\circ\pi_A$ and $\inf_B f_{\mathcal{A}} := \inf_{\omega\in B} f_{\mathcal{A}}(\omega)$ the left hand side of (5) equals

$$\inf_{\lambda\in\Lambda}\{(\inf_F f_{\mathcal{A}})\lambda(F) + (\inf_{F^c} f_{\mathcal{A}})(1-\lambda(F))$$

$$= \begin{cases} (1-\varepsilon)\inf_F f_{\mathcal{A}} + \varepsilon \inf_{F^c} f_{\mathcal{A}}, & \text{if } \inf_F f_{\mathcal{A}} \ge \inf_{F^c} f_{\mathcal{A}} \\ \inf_F f_{\mathcal{A}}, & \text{if } \inf_F f_{\mathcal{A}} < \inf_{F^c} f_{\mathcal{A}} \end{cases}$$

$$= (1-\varepsilon)\inf_F f_{\mathcal{A}} + \varepsilon \inf_\Omega f_{\mathcal{A}}.$$ ┘

Let now $(\Omega_i,\mathcal{B}_i)$, $1 \le i \le n$, be arbitrary measurable spaces. By $B(\Omega,\mathcal{B})$ we denote the space of all bounded, $\mathcal{B}$-measurable, real functions on Ω. The topological dual $(B(\Omega,\mathcal{B}))^*$ can be identified with $ba(\Omega,\mathcal{B})$ - the space of all bounded, additive set functions on $(\Omega,\mathcal{B})$ (cf. Dunford-Schwartz [1], Theorem 1, p. 258).

The proof of the following lemma can be omitted.

LEMMA 2. Let $\nu,\mu \in ba(\Omega,\mathcal{B})$, $\nu \le \mu$, and let $f \in B(\Omega,\mathcal{B})$. Then

(7) $$\max\{\int f d\lambda;\ \lambda \in ba(\Omega,\mathcal{B}),\ \nu \le \lambda \le \mu\} = \int f_+ d\mu - \int f_- d\nu,$$

where f_+ and f_- are the positive and negative parts of f. ┘

The following theorem is due to Kellerer [7], Satz (4.2), Satz (4.3).

THEOREM 3. Let ν and μ be bounded signed measures on $(\Omega,\mathcal{B})$ with $\nu \le \mu$ and λ_A be bounded signed measures on $(\Omega_A,\mathcal{B}_A)$, $A \in \mathcal{A}$. Then there exists a signed measure λ^* on $(\Omega,\mathcal{B})$ with marginals λ_A, $A \in \mathcal{A}$, and $\nu \le \lambda^* \le \mu$, if and only if

$$(8)\qquad \sum_{A\in\mathcal{A}} \int f_A d\lambda_A \le \int (\sum_{A\in\mathcal{A}} f_A \circ \pi_A)_+ d\mu - \int (\sum_{A\in\mathcal{A}} f_A \circ \pi_A)_- d\nu$$

for all $f_A \in B(\Omega_A, \mathfrak{B}_A)$, $A \in \mathcal{A}$.

<u>Proof.</u> Let $X = ba(\Omega, \mathfrak{B})$ and $K = \{\lambda \in X;\ \nu \le \lambda \le \mu\}$. Clearly, K is convex and closed (w.r.t. weak $*$-topology), and it is bounded, and hence compact by Alaoglu's Theorem. Moreover, each element of K is countably additive. By Theorem 1 the system

$$(1'')\qquad \int 1_{B_A} \circ \pi_A d\lambda = \int 1_{B_A} d\lambda_A,\ B_A \in \mathfrak{B}_A,\ A \in \mathcal{A},$$

is consistent on K, if and only if

$$(2'')\qquad \max_{\lambda\in K} \int (\sum_{A\in\mathcal{A}} f_A \circ \pi_A) d\lambda \ge \sum_{A\in\mathcal{A}} \int f_A d\lambda_A$$

for all $\mathfrak{B}_A$-measurable step-functions f_A, $A \in \mathcal{A}$. By Lemma 2 condition (2") equals condition (8) (the extension of (8) from step-functions f_A to elemtents of $B(\Omega_A, \mathfrak{B}_A)$ is obvious). ┘

<u>REMARK 2.</u> The assumption that ν and μ are bounded guarantees compactness of K and countable additivity of a solution $\lambda^* \in K$. Boundedness of μ can be weakened (cf. Kellerer [7], Satz 4.5): Assume that $\pi_{A_0}\mu$ is σ-finite for some $A_0 \in \mathcal{A}$, or - more generally - that μ is "rechtecksnormal" as defined in Kellerer [7], p. 171. Then Theorem 3 can be proved applying Ky Fan's theorem in [2] on the system of linear equalities $\pi_A\lambda = \lambda_A$, $A \in \mathcal{A}$, <u>and</u> inequalities $\lambda \le \mu$, and with $K = \{\lambda \in ba(\Omega, \mathfrak{B}): \lambda' := (\lambda-\nu)/\gamma \ge 0,\ \|\lambda'\| \le 1\}$, where $\gamma := \|\lambda_A - \pi_A\nu\|$ (note that (8) implies that $\lambda_A \ge \pi_A\nu$, $A \in \mathcal{A}$, and that γ does not depend on $A \in \mathcal{A}$; the transformations $\lambda \to \lambda'$, $\lambda_A \to \lambda'_A = (\lambda_A - \pi_A\nu)/\gamma$, $\mu \to \mu' = (\mu-\nu)/\gamma$ will be convenient). The countable additivity of a solution λ^* then follows as in 3.4.2, p. 542, of Jacobs [5]. ┘

We now consider the problem treated in Theorem 3 in more detail, when $\mathcal{A}$ is the set of singletons $\{i\}$, $1 \le i \le n$, $\lambda_{\{i\}} = \lambda_i$ are probability measures and the lower bound ν is zero. This problem has a very interesting history. It has been solved for n = 2 by Fréchet [3], Kellerer [7], Satz 4.4, Strassen [10], Theorem 6, Hansel-Troallic [4].

<u>COROLLARY 2.</u> Let λ_i be probability measures on $(\Omega_i, \mathfrak{B}_i)$, $1 \le i \le n$, and μ be a bounded measure on $(\Omega, \mathfrak{B})$. Then there exists a probability measure λ^* on $(\Omega, \mathfrak{B})$ with marginals λ_i, $1 \le i \le n$, and $\lambda^* \le \mu$, if and only if

$$(9)\qquad \int (\sum_{i=1}^n f_i \circ \pi_i - 1)_+ d\mu - \sum_{i=1}^n \int f_i d\lambda_i \ge -1$$

for all nonnegative $f_i \in B(\Omega_i, \mathfrak{B}_i)$, $1 \le i \le n$, or, equivalently, for all $f_i \in B(\Omega_i, \mathfrak{B}_i)$ with values in $[0,1]$, $1 \le i \le n$.

<u>Proof.</u> By Theorem 3 there exists a λ^* satisfying the conditions of the corollary, if and only if

$$(10)\qquad \sum_{i=1}^n \int f_i d\lambda_i \le \int (\sum_{i=1}^n f_i \circ \pi_i)_+ d\mu,$$

for all $f_i \in B(\Omega_i, \mathfrak{B}_i)$, $1 \le i \le n$.

Clearly (10) implies (9). For the converse implication let $c_i = \inf_{\Omega_i} f_i$, $1 \le i \le n$, and $c = \sum_{i=1}^n c_i$. With $g_i = f_i - c_i$, $1 \le i \le n$, (10) is equivalent to:

$$(11)\qquad \sum_{i=1}^n \int g_i d\lambda_i + c \le \int (\sum_{i=1}^n g_i \circ \pi_i + c)_+ d\mu$$

for all nonnegative $g_i \in B(\Omega_i, \mathfrak{B}_i)$, $1 \le i \le n$, and $c \in \mathbb{R}$.

But (9) implies $\pi_i\mu \ge \lambda_i$, $1 \le i \le n$, and hence (11) is satisfied if $c \ge 0$. For $c < 0$ multiplication of both sides of (9) by $|c|$

yields (11).

Finally note that the left hand side of (9) is decreased by replacing $f_i \geq 0$ by $\min\{1, f_i\}$, $1 \leq i \leq n$. ⌋

REMARK 3. As pointed out in Remark 2 the assumption that μ is bounded can be replaced by the weaker assumption: There exists an $i \leq n$ such that $\pi_i\mu$ is σ-finite (or: μ is "rechtecksnormal"). ⌋

REMARK 4. Of course, in (9) we can restrict to step-functions $f_i = \sum_{k=1}^{m_i} \alpha_k^{(i)} 1_{B_k^{(i)}}$, where $\alpha_k^{(i)} \geq 0$ and $B_k^{(i)} \in \mathfrak{B}_i$, $1 \leq i \leq n$. We may assume that for fixed i the $B_k^{(i)}$ are disjoint and that their union is Ω_i. Then, avoiding positive part operation, (9) can be reformulated as

$$(9')\qquad \min\{\sum_{k_1,\dots,k_n} \delta_{k_1,\dots,k_n} \mu(B_{k_1}^{(1)} \times \dots \times B_{k_n}^{(n)}) - \sum_{i=1}^{n} \sum_{k=1}^{m_i} \alpha_k^{(i)} \lambda_i(B_k^{(i)});\ (\delta, \alpha^{(1)}, \dots, \alpha^{(n)}) \in H_n\} \geq -1,$$

where

$$H_n = \{(\delta, \alpha^{(1)}, \dots, \alpha^{(n)});\ \delta = (\delta_{k_1,\dots,k_n}) \geq 0,$$

$$\alpha^{(i)} = (\alpha_k^{(i)}) \geq 0,\ 1 \leq i \leq n,\ \delta_{k_1,\dots,k_n} \geq \sum_{i=1}^{n} \alpha_{k_i}^{(i)} - 1\},$$

for arbitrarily given partitions $\Omega_i = \bigcup_{k=1}^{m_i} B_k^{(i)}$, $B_k^{(i)} \in \mathfrak{B}_i$.

Clearly the minimum of the left hand side of (9') is attained at some extreme point of the polyhedral set H_n. The following lemma gives the extreme points of H_n in the case $n = 2$.

LEMMA 3. $(\delta^*, \alpha^*, \beta^*)$ is an extreme point of

$$H_2 = \{(\delta,\alpha,\beta);\ \delta = (\delta_{jk})_{\substack{1\leq j\leq p\\ 1\leq k\leq q}} \geq 0,\ \alpha = (\alpha_j)_{1\leq j\leq p} \geq 0,$$

$$\beta = (\beta_k)_{1\leq k\leq q} \geq 0,\ \delta_{jk} \geq \alpha_j + \beta_k - 1\},$$

if and only if there are subsets $J \subset N_p := \{1,\dots,p\}$, $K \subset N_q := \{1,\dots,q\}$, where either $J \neq N_p$, $K \neq N_q$, or $J = N_p$, $K = \phi$, or $J = \phi$, $K = N_q$, such that

$$(12)\qquad \delta_{jk}^* = 1 \text{ on } J \times K,\ \alpha_j^* = 1 \text{ on } J,\ \beta_k^* = 1 \text{ on } K,$$

and all other δ_{jk}^*, α_j^*, β_k^* are zero.

Proof. Let (δ,α,β) be an extreme point of H_2. Then clearly $\delta_{jk} = (\alpha_j + \beta_k - 1)_+$ for all j,k. Let $J_o := \{j \in N_p;\ \alpha_j \notin \{0,1\}\}$, $K_o := \{k \in N_q;\ \beta_k \notin \{0,1\}\}$, and suppose that $J_o \neq \phi$ or $K_o \neq \phi$. Then choose an $\varepsilon > 0$ with $\varepsilon < \min\{\alpha_j, |1-\alpha_j|, \beta_k, |1-\beta_k|;\ j \in J_o,\ k \in K_o\}$, and define two distinct points $(\tilde\delta,\tilde\alpha,\tilde\beta)$, $(\tilde{\tilde\delta},\tilde{\tilde\alpha},\tilde{\tilde\beta}) \in H_2$ by

$$\tilde\alpha_j = \begin{cases} \alpha_j + \varepsilon, & j \in J_o \\ \alpha_j, & j \notin J_o \end{cases},\qquad \tilde{\tilde\alpha}_j = \begin{cases} \alpha_j - \varepsilon, & j \in J_o \\ \alpha_j, & j \notin J_o \end{cases}$$

$$\tilde\beta_k = \begin{cases} \beta_k - \varepsilon, & k \in K_o \\ \beta_k, & k \notin K_o \end{cases},\qquad \tilde{\tilde\beta}_k = \begin{cases} \beta_k + \varepsilon, & k \in K_o \\ \beta_k, & k \notin K_o \end{cases}$$

and

$$\tilde\delta_{jk} = (\tilde\alpha_j + \tilde\beta_k - 1)_+,\ \tilde{\tilde\delta}_{jk} = (\tilde{\tilde\alpha}_j + \tilde{\tilde\beta}_k - 1)_+.$$

Then $(\delta,\alpha,\beta) = \frac{1}{2}(\tilde\delta,\tilde\alpha,\tilde\beta) + \frac{1}{2}(\tilde{\tilde\delta},\tilde{\tilde\alpha},\tilde{\tilde\beta})$, since $\alpha = \frac{1}{2}(\tilde\alpha + \tilde{\tilde\alpha})$, $\beta = \frac{1}{2}(\tilde\beta + \tilde{\tilde\beta})$, and

$$\delta_{jk} = \tilde\delta_{jk} = \tilde{\tilde\delta}_{jk} \text{ if } j \in J_o,\ k \in K_o \text{ or } j \notin J_o,\ k \notin K_o,$$

and in the case $j \in J_o$, $k \notin K_o$ or $j \notin J_o$, $k \in K_o$ we have by the choice of ε that

$$\tfrac{1}{2}(\alpha_j + \beta_k + \varepsilon - 1)_+ + \tfrac{1}{2}(\alpha_j + \beta_k - \varepsilon - 1)_+ = (\alpha_j + \beta_k - 1)_+.$$

It follows that $J_o = \phi$ and $K_o = \phi$, and, therefore, there are subsets $J \subset N_p$, $K \subset N_q$ such that $\delta_{jk} = 1$ on $J \times K$, $\alpha_j = 1$ on J, $\beta_k = 1$ on K and all other δ_{jk}, α_j, β_k are zero. It can easily be seen that among these points the extreme points of H_2 are found as in the assertion. ⌋

From Remark 4 and Lemma 3 we obtain the following corollary (cf. Fréchet [3], Kellerer [7], Strassen [10], Hansel-Troallic [4]).

<u>COROLLARY 3.</u> Under the assumptions of Corollary 2 suppose $n = 2$. There exists a probability measure λ^* on $(\Omega,\mathcal{B})$ with marginals λ_1, λ_2, and $\lambda^* \le \mu$, if and only if

$$\mu(B_1 \times B_2) \ge \lambda_1(B_1) + \lambda_2(B_2) - 1 \tag{13}$$

for all sets $B_i \in \mathcal{B}_i$, $i = 1,2$.

<u>Proof.</u> By Lemma 3 the inequality (9') becomes

$$\sum_{j\in J}\sum_{k\in K} \mu(B_j^{(1)} \times B_k^{(2)}) - \sum_{j\in J}\lambda_1(B_j^{(1)}) - \sum_{k\in K}\lambda_2(B_k^{(2)}) \ge -1,$$

which is identical to (13) with $B_1 = \bigcup_{j\in J} B_j^{(1)}$, $B_2 = \bigcup_{k\in K} B_k^{(2)}$. ⌋

<u>REMARK 5.</u> The "only if" part of Lemma 3 does not carry over to the case $n \ge 3$. But the points $(\delta,\alpha^{(1)},\dots,\alpha^{(n)}) \in H_n$ with $\alpha_k^{(i)} \in \{0,1\}$ and $\delta_{k_1,\dots,k_n} = (\sum_{i=1}^{n} \alpha_{k_i}^{(i)} - 1)_+$ again are extreme points when $n \ge 3$. So a necessary condition for (9') (resp. (9)) in the case $n = 3$ is

$$2\mu(B_1\times B_2\times B_3) + \mu(B_1\times B_2\times B_3^C) + \mu(B_1\times B_2^C\times B_3) + \mu(B_1^C\times B_2\times B_3) \tag{14}$$
$$\ge \lambda_1(B_1) + \lambda_2(B_2) + \lambda_3(B_3) - 1, \text{ for all } B_i \in \mathcal{B}_i,\ 1 \le i \le 3.$$

This again shows as observed by Kellerer [6], Satz 6.1, that for $n = 3$ it is not sufficient for (9') that $\mu(B_1\times B_2\times B_3)$ is not less than the lower Fréchet-bound:

$$\mu(B_1\times B_2\times B_3) \ge \lambda_1(B_1) + \lambda_2(B_2) + \lambda_3(B_3) - 2. \tag{15}$$

Condition (15) is obtained from (9') by choosing the points $\delta_{j,k,l} = \frac{1}{2}$, $(j,k,l) \in J \times K \times L$, $\alpha_j = \frac{1}{2}$ on J, $\beta_k = \frac{1}{2}$ on K, $\gamma_l = \frac{1}{2}$ on L, and zero otherwise, which can easily be shown to be extreme points of H_3. ⌋

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