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On a Class of Extremal Problems in Statistics

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Summary: Let m denote the infimum of the integral of a function φ w. r. t. all probability measures with given marginals. The determination of m is of interest for a series of stochastic problems. In the present paper we prove a duality theorem for the determination of m and give some examples for its application. We consider especially the problem of extremal variance of sums of random variables and prove a theorem for the existence of random variables with given marginal distributions, such that their sum has variance zero.

1. Introduction

A basic problem of dealing with dependent random variables is the following one. Let φ be a function of n variables and let $P_1, \ldots, P_n$ be n one-dimensional probability measures; then determine the minimum and the maximum of the integral of φ w.r.t. all probability measures with marginals $P_1, \ldots, P_n$. By means of results of this type one can describe the influence of dependence on a stochastic problem which is defined by the function φ.

Solutions for this stochastic optimization problem are known only in very few special cases. A very nice solution in the case $\varphi(x_1, \ldots, x_n) = \max \{x_i \mid 1 \leqq i \leqq n\}$ has been given in recent papers by Lai and Robbins [10], [11]. Their result is that $\max \{x_i \mid 1 \leqq i \leqq n\}$ is for arbitrary dependent random variables not much larger than for independent random variables. So they are able to prove limit theorems for the maximum of a sequence of arbitrary dependent random variables.

In section two of this paper we prove a duality theorem for the general optimization problem and give some examples where solutions are found by an application of this theorem. A useful aspect of this duality theorem is that on one hand it allows to give bounds for the minimum and on the other hand it describes the support of an 'optimal measure'.

In section three we consider the problem of extremal variance of sums of random variables which is of the type described above. For some cases we are able to give a solution of this problem. We isolate the more special question for

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the existence of n random variables with constant sum and with given marginals, prove a duality theorem for this question, and get solutions in some special cases.

2. A Duality Theorem

Let $\varphi: [0, 1]^n \to R^1$ be a continuous function and let $P_1, \ldots, P_n$ be n probability measures on $[0, 1]\,\mathfrak{B}^1$, where $\mathfrak{B}^1$ is the Borel σ-Algebra on R^1. Define

$$\mathfrak{H}(P_1, \ldots, P_n) := \{P \mid P \text{ is a probability measure on } [0, 1]^n\, \mathfrak{B}^n \text{ with marginals } P_1, \ldots, P_n\}.$$

$\mathfrak{H}(P_1, \ldots, P_n)$ is a convex set of probability measures which is compact w.r.t. the vague topology.

The '*primal*' problem we want to consider is to determine

$$m := \inf \{\textstyle\int \varphi \mathrm{d}P \mid P \in \mathfrak{H}(P_1, \ldots, P_n)\}. \tag{1}$$

It is clear from the note above that there exists *a* $P^* \in \mathfrak{H}(P_1, \ldots, P_n)$ with $m = \int \varphi \mathrm{d}P^*$. The problem of determining $\sup \{\int \varphi \mathrm{d}P \mid P \in \mathfrak{H}(P_1, \ldots, P_n)\}$ is included in this formulation of the optimization problem by taking $-\varphi$ instead of φ.

For a topological space X we denote by $C(X)$ the set of all continuous, real functions on X. In the following duality theorem we give an optimization problem which is 'dual' to the primal optimization problem.

Theorem 1:

$$m = \sup \left\{ \sum_{i=1}^{n} \int f_i \mathrm{d}P_i \mid f_i \in C[0, 1], \quad 1 \leqq i \leqq n, \right. \tag{2}$$

$$\left. \sum_{i=1}^{n} f_i(x_i) \leqq \varphi(x_1, \ldots, x_n), \quad \forall x = (x_1, \ldots, x_n) \in [0, 1]^n \right\}.$$

Proof: The proof of Theorem 1 is based on the following duality theorem of ISII [8], [9], Th. 2.3, in topological vectorspaces (cf. also DIETER [2], GOLSTEIN [5], Kapitel 2, 3). Let X be a convex cone with vertex 0 in a real vectorspace, let Z be a topological vectorspace, $z_0 \in Z$ and let a pseudoorder be induced on Z by a convex cone $\mathfrak{C}$ with vertex 0. Let $F: X \to R^1$, $\psi: X \to Z$ be linear functions such that $\mathring{\mathfrak{C}} \neq \emptyset$ and $0 \in \overline{\psi(X) - \mathring{\mathfrak{C}} + z_0}$, where $\mathring{A}$ denotes the interior of A. Then,

$$\sup \{F(x) \mid x \in X, \psi(x) + z_0 \geqq 0\} = \inf \{z^*(z_0);\ z^* \in Z^*, z^* \geqq 0, z^*(\psi(x)) + F(x) \leqq 0, \forall x \in X\} \tag{3}$$

where Z^* is the topological dual of Z.

We define: $X := C^n[0, 1]$, $Z := C[0, 1]^n$,

$$F(f_1, \ldots, f_n) := \sum_{i=1}^{n} \int f_i \mathrm{d}P_i, \quad \psi(f_1, \ldots, f_n)(x) := -\sum_{i=1}^{n} f_i(x_i)$$

for $x=(x_1, \ldots, x_n)$, $z_0:=\varphi$ and the cone $\mathfrak{C}:=\{f\in C[0, 1]^n \mid f\geqq 0\}$. The topology on $Z=C[0, 1]^n$ is given by the norm $\|f\|:=\sup\{|f(x)| \mid x\in[0, 1]^n\}$. The left side of (3) is identical to our dual problem

$$M_1:=\sup\left\{\sum_{i=1}^n \int f_i \mathrm{d}P_i \mid f_i\in C[0, 1], 1\leqq i\leqq n, \quad \sum_{i=1}^n f_i\circ\pi_i\leqq\varphi\right\},$$

where π_i denotes the projection on the i-th component, $1\leqq i\leqq n$.

By RIESZ' representation theorem (cf. DUNFORD, SCHWARTZ [3], Th. 3, pg. 265) the dual space of $C[0, 1]^n$ is the space of signed measures on $[0, 1]^n$ $\mathfrak{B}^n$ and hence $\{z^*\in Z^* \mid z^*\geqq 0\}$ equals the set of measures on $[0, 1]^n$. Therefore, the right hand side of (3) is identical to

$$M_2:=\inf\left\{\int\varphi\mathrm{d}\mu \mid \mu \text{ is a measure on } [0, 1]^n\ \mathfrak{B}^n \text{ and}\right.$$

$$\left.-\int\sum_{i=1}^n f_i\circ\pi_i\mathrm{d}\mu\leqq-\sum_{i=1}^n\int f_i\mathrm{d}P_i, \forall f_i\in C[0, 1], 1\leqq i\leqq n\right\}.$$

Since $f_i\in C[0, 1]$ implies $-f_i\in C[0, 1]$, we get

$$M_2:=\inf\left\{\int\varphi\mathrm{d}\mu \mid \mu \text{ is a measure with } \int\sum_{i=1}^n f_i\circ\pi_i\mathrm{d}\mu=\sum_{i=1}^n\int f_i\mathrm{d}P_i,\right.$$

$$\left.\forall f_i\in C[0, 1], 1\leqq i\leqq n\right\}.$$

Taking $f_i\equiv 1$, $1\leqq i\leqq n$ we get that admissible μ are normalized on 1. With $f_i\equiv 0$ for $i\neq j$ we obtain further that admissible μ have as j-th marginal P_j, since by Theorem 1.3 of BILLINGSLEY [1] continuous functions are measure determining. Therefore, the right hand side of (3) is equal to our primal problem

$$M_2=\inf\{\int\varphi\mathrm{d}\mu \mid \mu\in\mathfrak{H}(P_1, \ldots, P_n)\}.$$

We have to check the regularity conditions. Obviously it holds $\mathring{\mathfrak{C}}\neq\emptyset$. It remains to show that 0 is an interior point of $\psi(X)-\mathfrak{C}+z_0$, i.e. for $f\in C[0, 1]^n$ with $\|f\|\leqq\varepsilon$ there exist $g\in C^m[0, 1]$ and $h\in C[0, 1]^n$, $h\geqq 0$ such that $f=\psi(g)-h+\varphi$. Since φ is continuous, it is bounded, $|\varphi|\leqq K$. With $g_i:\equiv-\frac{1}{n}(K+\varepsilon)$, $1\leqq i\leqq n$ and $g:=(g_1, \ldots, g_n)$ we have that

$$h:=-\sum_{i=1}^n g_i\circ\pi_i-f+\varphi\geqq 0.$$

Therefore, by ISII's Theorem $M_1=M_2$ which is identical to (2). ∎

The following proposition shows that there exists a solution of the *dual problem* in (2) if one enlarges the space $C[0, 1]$.

Let $B[0, 1]$ be the set of all BOREL-measurable, real, bounded functions on $[0, 1]$.

Proposition 2:

$$m=\sup\left\{\sum_{i=1}^n\int f_i\mathrm{d}P_i \mid f_i\in B[0, 1], 1\leqq i\leqq n, \sum_{i=1}^n f_i\circ\pi_i\leqq\varphi\right\},$$

and the supremum is attained.

Proof. For $f_i \in B[0, 1]$, $1 \leqq i \leqq n$, with $\sum_{i=1}^{n} f_i \circ \pi_i \leqq \varphi$ and for $P \in \mathfrak{H}(P_1, \ldots, P_n)$ we obtain

$$\sum_{i=1}^{n} \int f_i \mathrm{d}P_i = \sum_{i=1}^{n} \int f_i \circ \pi_i \mathrm{d}P = \int \left(\sum_{i=1}^{n} f_i \circ \pi_i \right) \mathrm{d}P \leqq \int \varphi \mathrm{d}P ,$$

and hence

$$m \geqq \sup \left\{ \sum_{i=1}^{n} \int f_i \mathrm{d}P_i \mid f_i \in B[0, 1],\ 1 \leqq i \leqq n,\ \sum_{i=1}^{n} f_i \circ \pi_i \leqq \varphi \right\}.$$

Since $C[0, 1] \subset B[0, 1]$ Theorem 1 implies equality.

For the existence of functions $f_1^*, \ldots, f_n^*$, for which the supremum is attained, we first note that there exists a $K \in R_+^1$ such that

$$m = \sup \left\{ \sum_{i=1}^{n} \int f_i \mathrm{d}P_i \mid f_i \in B[0, 1],\ |f_i| \leqq K,\ 1 \leqq i \leqq n,\ \sum_{i=1}^{n} f_i \circ \pi_i \leqq \varphi \right\}. \quad (4)$$

Let $f_1, \ldots, f_n \in B[0, 1]$ with $\sum_{i=1}^{n} f_i \circ \pi_i \leqq \varphi$ be given and let $b_i := \sup f_i$, $1 \leqq i \leqq n$, and let $a := \inf \varphi$, $A := \sup \varphi$. Defining

$$g_i := f_i + \frac{1}{n} \sum_{j=1}^{n} b_j - b_i, \quad 1 \leqq i \leqq n ,$$

we obtain

$$\sum_{i=1}^{n} g_i \circ \pi_i \leqq \varphi, \quad \sum_{i=1}^{n} \int g_i \mathrm{d}P_i = \sum_{i=1}^{n} \int f_i \mathrm{d}P_i$$

and

$$\sup g_i = \frac{1}{n} \sum_{i=1}^{n} b_j = \bar{b}, \quad 1 \leqq i \leqq n .$$

It follows that $\bar{b} \leqq \frac{A}{n}$. In the next step define $h_i(x) := \max\{g_i(x), c\}$ with

$$c := \min_{1 \leqq r \leqq n} \left(\frac{a}{r} - \frac{n-r}{nr} A \right) = a - \frac{n-1}{n} A .$$

Then

$$\sum_{i=1}^{n} h_i(x_i) = \sum_{i=1}^{n} g_i(x_i) \leqq \varphi(x_1, \ldots, x_n) ,$$

if $g_i(x_i) \geqq c$, $1 \leqq i \leqq n$. Let now $g_i(x_i) < c$ for exactly r indices i, $1 \leqq r \leqq n$, then

$$\sum_{i=1}^{n} h_i(x_i) \leqq rc + (n-r)\, \bar{b} \leqq rc + \frac{n-r}{n} A \leqq a \leqq \varphi(x_1, \ldots, x_n) ,$$

and $g_i \leqq h_i$, $1 \leqq i \leqq n$, implies

$$\sum_{i=1}^{n} \int g_i \mathrm{d}P_i \leqq \sum_{i=1}^{n} \int h_i \mathrm{d}P_i .$$

With $K:=\max\left\{|c|, \frac{|A|}{n}\right\}$ we obtain (4). Now let $(f_1^{(k)}, \ldots, f_n^{(k)})$, $k \in \boldsymbol{N}$, be a sequence such that

$$f_i^{(k)} \in B_K[0,1] := \{f \in B[0,1] \mid |f| \leqq K\}, \quad 1 \leqq i \leqq n, \quad \sum_{i=1}^n f_i^{(k)} \circ \pi_i \leqq \varphi$$

and

$$\lim_{k\to\infty} \sum_{i=1}^n \int f_i^{(k)} \mathrm{d}P_i = m .$$

Then since $\bigotimes_{i=1}^n B_K[0,1]$ is a sequentially compact subset of $\bigotimes_{i=1}^n L^1(P_i)$ supplied with the weak topology $\sigma\left(\bigotimes_{i=1}^n L^1(P_i), \bigotimes_{i=1}^n L^\infty(P_i)\right)$ there exist $(\bar f_1, \ldots, \bar f_n) \in \bigotimes_{i=1}^n B_K$ $[0,1]$ and a subsequence of $(f_1^{(k)}, \ldots, f_n^{(k)})_{k\in N}$ converging to $(\bar f_1, \ldots, \bar f_n)$ w.r.t. the weak topology. Now one can proceed as in LANDERS, ROGGE [12]. ∎

Corollary 3: *Let $P^* \in \mathfrak{H}(P_1, \ldots, P_n)$. Then P^* is a solution of* (1) *if and only if there exist $f_1^*, \ldots, f_n^* \in B[0,1]$ with $\sum_{i=1}^n f_i^* \circ \pi_i \leqq \varphi$ and $\sum_{i=1}^n f_i^* \circ \pi_i = \varphi[P^*]$.*

Proof: Let $f_1^*, \ldots, f_n^* \in B[0,1]$ with $\sum_{i=1}^n f_i^* \circ \pi_i \leqq \varphi$ and $\sum_{i=1}^n \int f_i^* \mathrm{d}P_i = m$. Then $P^* \in \mathfrak{H}(P_1, \ldots, P_n)$ is a solution of (1) if and only if $\int\left(\varphi - \sum_{i=1}^n f_i^* \circ \pi_i\right) \mathrm{d}P^* = 0$ which is equivalent to $\sum_{i=1}^n f_i^* \circ \pi_i = \varphi[P^*]$. ∎

Remark 1: Clearly Theorem 1, Proposition 2, and Corollary 3 remain true if the interval $[0,1]$ is replaced by an arbitrary compact metric space E. Then φ is a continuous real function on E^n, $P_1, \ldots, P_n$ are probability measures on $(E, \mathfrak{B}(E))$, $\mathfrak{B}(E) =$ BOREL σ-Algebra on E, and $\mathfrak{H}(P_1, \ldots, P_n)$ is the set of all probability measures on $(E^n, \mathfrak{B}(E^n))$ with marginals $P_1, \ldots, P_n$. As a special case let $P_1, \ldots, P_n$ be probability measures on $(R^1, \mathfrak{B}^1)$. By $P_i\{+\infty\} = P_i\{-\infty\} = 0$, $1 \leqq i \leqq n$, the P_n are extended on $(\bar R^1, \mathfrak{B}^1)$. Taking $E = \bar R^1$ and observing that $C(\bar R^n)$ can be identified with the set of functions

$$C'(R^n) := \{f \in C(R^n) \mid \lim_{\substack{x\in R^n\\ x\to x_0}} f(x) \text{ exists and is finite for each } x_0 \in \bar R^n \setminus R^n\}$$

we have by Theorem 1 and Proposition 2 for $\varphi \in C'(R^n)$

$$\begin{aligned}
&\inf\{\textstyle\int \varphi \mathrm{d}P \mid P \in \mathfrak{H}(P_1, \ldots, P_n)\}\\
&= \sup\left\{\sum_{i=1}^n \int f_i \mathrm{d}P_i \mid f_i \in C'(R^1),\ 1 \leqq i \leqq n,\ \sum_{i=1}^n f_i \circ \pi_i \leqq \varphi\right\}\\
&= \sup\left\{\sum_{i=1}^n \int f_i \mathrm{d}P_i \mid f_i \in B(R^1),\ 1 \leqq i \leqq n,\ \sum_{i=1}^n f_i \circ \pi_i \leqq \varphi\right\},
\end{aligned}$$

where $\mathfrak{H}(P_1, \ldots, P_n) =$ set of all probability measures P on $(R^n, \mathfrak{B}^n)$ with marginals $P_1, \ldots, P_n$ and $B(R^1)$ is the set of real, bounded measurable functions on R^1.

Example 1: a) Let $P_1=P_2=Q$ be the uniform distribution on $[0, 1]$ $\mathfrak{B}^1$ and $\varphi(x_1, x_2)=\varphi_1(x_1)\cdot\varphi_2(x_2)$, where φ_1 and φ_2 are continuous increasing functions on $[0, 1]$. We consider the primal problem of minimizing

$$\int \varphi_1(x_1)\,\varphi_2(x_2)\,\mathrm{d}P(x_1, x_2), \quad P\in\mathfrak{H}(Q, Q)\,.$$

This includes the well-known problem of finding random variables X_1 and X_2 with prescribed distribution functions F_1 and F_2, such that EX_1X_2 gets a minimum, where the F_i are continuous, strictly increasing on a compact interval $[a_i, b_i]$, and $F_i(t)=0$ for $t\leqq a_i$, $F_i(t)=1$ for $t\geqq b_i$, $i=1, 2$. Then $EX_1X_2= =\int F_1^{-1}(x_1)\,F_2^{-1}(x_2)\,\mathrm{d}P$, where P is the distribution of $(F_1(X_1), F_2(X_2))$, and hence $P\in\mathfrak{H}(Q, Q)$.

Since the φ_i are increasing functions we get by an heuristic argument that $\varphi_1(x_1)$ and $\varphi_2(x_2)$ must be ordered in opposite senses on the support of an optimal measure $P^*\in\mathfrak{H}(Q, Q)$. Hence P^* must be the distribution of $(U, 1-U)$, where U is a $R(0, 1)$-distributed random variable. To prove this by Corollary 3 we have to look for functions $f_1, f_2\in B[0, 1]$ such that

$$f_1(x_1)+f_2(x_2)\leqq\varphi_1(x_1)\,\varphi_2(x_2), \quad x_1, x_2\in[0, 1]\,, \tag{5}$$

with equality for $x_2=1-x_1$. Assuming the existence of the derivatives f_i', φ_i' for the moment and putting $H(x_1, x_2):=f_1(x_1)+f_2(x_2)-\varphi_1(x_1)\,\varphi_2(x_2)$ we get from (5):

$$\frac{\partial}{\partial x_1}H(x_1, x_2)\Big|_{x_2=1-x_1}=0, \quad \frac{\partial}{\partial x_2}H(x_1, x_2)\Big|_{x_2=1-x_1}=0\,,$$

and hence

$$f_1'(x_1)=\varphi_1'(x_1)\,\varphi_2\,(1-x_1), \quad f_2'\,(1-x_1)=\varphi_1(x_1)\,\varphi_2'\,(1-x_1)$$

or

$$f_1(x_1)=c_1+\int_0^{x_1}\varphi_2\,(1-t)\,\mathrm{d}\varphi_1(t)\,, \tag{6}$$

$$f_2(x_2)=c_2+\int_0^{x_2}\varphi_1\,(1-t)\,\mathrm{d}\varphi_2(t)\,.$$

Now let the functions $f_1, f_2\in C[0, 1]$ be given by (6) in terms of STIELTJES integrals (which do not require the differentiability conditions on the φ_i), where the constants c_i will be chosen according to (5). By partial integration we get

$$\int_0^{x_1}\varphi_2\,(1-t)\,\mathrm{d}\varphi_1(t)=\varphi_1(x_1)\,\varphi_2\,(1-x_1)-\varphi_1(0)\,\varphi_2(1)+\int_{1-x_1}^{1}\varphi_1\,(1-t)\,\mathrm{d}\varphi_2(t)$$

and hence

$$f_1(x_1)+f_2(x_2)=c_1+c_2+\varphi_1(x_1)\,\varphi_2\,(1-x_1)-\varphi_1(0)\,\varphi_2(1) + \int_{1-x_1}^{1}\varphi_1\,(1-t)\,\mathrm{d}\varphi_2(t)+\int_0^{x_2}\varphi_1\,(1-t)\,\mathrm{d}\varphi_2(t)\,.$$

Choosing $c_1+c_2=\varphi_1(0)\,\varphi_2(1)-\int_0^1 \varphi_1(1-t)\,\mathrm{d}\varphi_2(t)$ we have $f_1(x_1)+f_2(1-x_1)=\varphi_1(x_1)\varphi_2(1-x_1)$, and for $x_2>1-x_1$ (the case $x_2<1-x_1$ is analogue):

$$\begin{aligned} f_1(x_1)+f_2(x_2) &= \varphi_1(x_1)\,\varphi_2(1-x_1)+\int_{1-x_1}^{1} \varphi_1(1-t)\,\mathrm{d}\varphi_2(t)-\int_{x_2}^{1} \varphi_1(1-t)\,\mathrm{d}\varphi_2(t) \\ &= \varphi_1(x_1)\,\varphi_2(1-x_1)+\int_{1-x_1}^{x_2} \varphi_1(1-t)\,\mathrm{d}\varphi_2(t) \\ &\leqq \varphi_1(x_1)\,\varphi_2(1-x_1)+\varphi_1(x_1)\,(\varphi_2(x_2)-\varphi_2(1-x_1)) \\ &= \varphi_1(x_1)\,\varphi_2(x_2). \end{aligned}$$

b) Let $P_1=\ldots=P_n=Q$ be the uniform distribution on $[0,1]\,\mathfrak{B}^1$ and $\varphi(x_1,\ldots,x_n)=-\prod_{i=1}^{n}\varphi_i(x_i)$ with $\varphi_i\in C[0,1]$, φ_i increasing, $\varphi_i\geqq 0$, $1\leqq i\leqq n$, and $\varphi_i(0)=0$ for at least one $i\in\{1,\ldots,n\}$. Again an heuristic approach to the primal problem

$$\inf\left\{-\int\left(\prod_{i=1}^{n}\varphi_i(x_i)\right)\mathrm{d}P(x_1,\ldots,x_n)\mid P\in\mathfrak{H}(Q,\ldots,Q)\right\}$$

leads to a solution P^* which orders $\varphi_1(x_1),\ldots,\varphi_n(x_n)$ in the same sense on its support. Hence P^* must be the distribution of $(U,\ldots,U)$, where U is $R(0,1)$ distributed. By Corollary 3 we have to determine functions $f_1,\ldots,f_n\in B[0,1]$ with

$$\sum_{i=1}^{n} f_i(x_i)\leqq-\prod_{i=1}^{n}\varphi_i(x_i) \quad \text{for all} \quad x_1,\ldots,x_n\in[0,1], \tag{7}$$

with equality for $x_1=\ldots=x_n$. Assuming the existence of the derivatives f_i', φ_i' for the moment we get from (7)

$$\frac{\partial}{\partial x_i}H(x_1,\ldots,x_n)\mid_{x_1=\ldots=x_n=u}=0, \quad 1\leqq i\leqq n,$$

where

$$H(x_1,\ldots,x_n):=\sum_{i=1}^{n} f_i(x_i)+\prod_{i=1}^{n}\varphi_i(x_i),$$

and hence

$$f_i'(u)=-\varphi_i'(u)\prod_{\substack{j=1\\ j\neq i}}^{n}\varphi_j(u),$$

or

$$f_i(u)=c_i-\int_0^u\left(\prod_{\substack{j=1\\ j\neq i}}^{n}\varphi_j(t)\right)\mathrm{d}\varphi_i(t), \quad 1\leqq i\leqq n. \tag{8}$$

Now we define the functions $f_i\in C[0,1]$ by (8) (without differentiability conditions on the φ_i), where the constants c_i will be determined according to (7).

We get from (8):

$$\sum_{i=1}^{n} f_i(u) = \sum_{i=1}^{n} c_i - \sum_{i=1}^{n} \int_0^u \Big(\prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(t)\Big)\, d\varphi_i(t) = \sum_{i=1}^{n} c_i - \prod_{i=1}^{n} \varphi_i(u)\,,$$

and consequently

$$\sum_{i=1}^{n} f_i(u) = -\prod_{i=1}^{n} \varphi_i(u) \quad \text{by choosing} \quad \sum_{i=1}^{n} c_i = 0\,.$$

Then the inequality (7) for $x_1, \ldots, x_n \in [0, 1]$ becomes

$$\sum_{i=1}^{n} \int_0^{x_i} \Big(\prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(t)\Big)\, d\varphi_i(t) \geqq \prod_{i=1}^{n} \varphi_i(x_i)\,,$$

which is an extension of YOUNG's inequality due to OPPENHEIM (cf. [13], pg. 50).

For $n=2$ the assumptions "$\varphi_i \geqq 0$, $1 \leqq i \leqq 2$, and $\varphi_i(0)=0$ for at least one $i \in \{1, 2\}$" can be omitted. Choosing $c_1 + c_2 = -\varphi_1(0)\,\varphi_2(0)$ the functions f_1, f_2 given by (8) satisfy

$$f_1(x_1) + f_2(x_2) \leqq -\varphi_1(x_1)\,\varphi_2(x_2)$$

with equality for $x_1 = x_2$ which can be proved as in Example 1. a).

c) For $\varphi(x_1, \ldots, x_n) = -\max\{x_i \mid 1 \leqq i \leqq n\}$, $x = (x_1, \ldots, x_n) \in [0, 1]^n$, and $P_1 = P_2 = \ldots = P_n$ define

$$f_i(t) := -\frac{a_n}{n} - (t - a_n)_+, \quad t \in R^1, \quad 1 \leqq i \leqq n\,,$$

where $a_n \in R^1$. Then,

$$-\sum_{i=1}^{n} f_i(x_i) = a_n + \sum_{i=1}^{n} (x_i - a_n)_+ \geqq \max_{1 \leqq i \leqq n} x_i\,.$$

Therefore, if there exist $a_n \in R^1$ and random variables $X_1', \ldots, X_n'$, with distributions P_1 such that $a_n + \sum_{i=1}^{n} (X_i' - a_n)_+ = \max_{1 \leqq i \leqq n} X_i'$, then $(X_i', \ldots, X_n')$ yields an optimal solution for the problem: $E \max_{1 \leqq i \leqq n} X_i = \max!$ under the condition that X_i have distributions P_i, $1 \leqq i \leqq n$. This condition is equal to condition 2.4 of LAI and ROBBINS [11], who construct random variables satisfying this condition. ▮

Remark 2: If $P_1 = \ldots = P_n = Q$ and φ is a symmetric function, i.e. $\varphi(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \varphi(x_1, \ldots, x_n)$ for all $x = (x_1, \ldots, x_n) \in [0, 1]^n$ and for all permutations σ, then the dual problem (2) is equivalent to

$$\sup\Big\{n \int f dQ \mid f \in C[0, 1],\ \sum_{i=1}^{n} f \circ \pi_i \leqq \varphi\Big\}\,.$$

This can be seen as follows: For $f_1, \ldots, f_n \in C[0, 1]$ with $\sum_{i=1}^{n} f_i \circ \pi_i \leqq \varphi$ we put

$f(t) := \frac{1}{n} \sum_{i=1}^{n} f_i(t)$. Then $f \in C[0, 1]$,

$$\sum_{i=1}^{n} f(x_i) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} f_j(x_i) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} f_j(x_{\sigma_i(j)})$$
$$\leqq \frac{1}{n} \sum_{i=1}^{n} \varphi(x_{\sigma_i(1)}, \ldots, x_{\sigma_i(n)}) = \varphi(x_1, \ldots, x_n) ,$$

where the σ_i are the cyclic permutations $\sigma_i(j) = j + i \pmod n$. So we have $\sum_{i=1}^{n} f \circ \pi_i \leqq \varphi$, and $n \int f \mathrm{d}Q = \int \left(\sum_{i=1}^{n} f_i \right) \mathrm{d}Q = \sum_{i=1}^{n} \int f_i \mathrm{d}P_i$.

3. Extremal Variance of Sum Variables

A problem which stems from Monte Carlo theory is the problem of variance reduction. Let $P_1, \ldots, P_n$ be n probability measures on $(R^1, \mathfrak{B}^1)$. The question is to construct a random variable $X = (X_1, \ldots, X_n)$ with distribution in $\mathfrak{H}(P_1, \ldots, P_n)$ such that the variance of $\sum_{i=1}^{n} X_i$ is as large or as small as possible.

The background of this question is that one looks for estimators of a parameter which depend on the sum of random variables and which should be unbiased and have smallest variance. The first method which leads to the question of largest variance of $\sum_{i=1}^{n} X_i$ is called the method of *antithetic variates*. The second method concerning the smallest variance is called the method of control variates (cf. Hammersley, Handscomb [6], pg. 59, Fishman [4], Handscomb [7], Whitt [15]).

From Example 1a) it follows that $(F_1^{-1}(U), F_2^{-1}(1-U))$ is a solution of the problem of minimum variance in the case $n=2$. This result is already known in the literature. From example 1b) (applied on each pair of components separately) we obtain that $(F_1^{-1}(U), F_2^{-1}(U), \ldots, F_n^{-1}(U))$, where F_i is the distribution function of P_i, is a solution of the problem of maximum variance. For general n and F_i we are not able to solve problem (1) of Sec. 1 in the case of minimum variance i.e. $\varphi(x_1, \ldots, x_n) = \sum_{i<j} x_i x_j$.

We want to isolate the following more special question: For which $P_1, \ldots, P_n$ does there exist a random variable $X = (X_1, \ldots, X_n)$ with distribution in $\mathfrak{H}(P_1, \ldots, P_n)$ such that

$$\sum_{i=1}^{n} X_i = c, \quad c := \sum_{i=1}^{n} EX_i. \tag{9}$$

For $n=2$ it is necessary and sufficient that

$$F_1(t) = 1 - F_2((c-t)-), \quad \forall t \in R^1.$$

For continuous distribution functions F_i, $1 \leq i \leq n$, question (9) can be reduced to the question of the existence of an uniform distribution with support in $\left\{\sum_{i=1}^{n} F_i^{-1}(U_i)=c\right\}$. The proof of the following lemma is immediate.

Lemma 4: *Let $P_1, \ldots, P_n$ be probability measures on $(R^1, \mathfrak{B}^1)$ with continuous distribution functions $F_1, \ldots, F_n$. Then it holds: There exists a random variable X with distribution in $\mathfrak{H}(P_1, \ldots, P_n)$ such that $\sum_{i=1}^{n} X_i=c$ if and only if there exists a random variable $U=(U_1, \ldots, U_n)$ with distribution in $\mathfrak{H}(P_0, \ldots, P_0)$ such that $\sum_{i=1}^{n} F_i^{-1}(U_i)=c$, where P_0 is the uniform distribution on $[0, 1]\ \mathfrak{B}^1$.*

Lemma 4 can be used to give a negative answer to (9) in some cases.

Example 2: Let $P_1=\ldots=P_n$ be the exponential distribution with distribution function

$$F(x)=(1-e^{-x})\, 1_{[0,\infty)}(x), \quad x\in R^1.$$

Assume that there exists a random variable X with distribution in $\mathfrak{H}(P_1,\ldots,P_n)$ such that $\sum_{i=1}^{n} X_i=n=n\int x\mathrm{d}P_1$. Since

$$F_i^{-1}(t)=-\ln(1-t), \quad t\in[0, 1], \quad 1\leq i\leq n\ ,$$

by Lemma 4 there would exist $R(0, 1)$-distributed U_i, $1\leq i\leq n$ with $-\sum_{i=1}^{n}\ln(1-U_i)=n$. Therefore, $\prod_{i=1}^{n}(1-U_i)=\mathrm{e}^{-n}$ which implies

$$1-U_i\geq \mathrm{e}^{-n} \quad \text{or} \quad U_i\leq 1-\mathrm{e}^{-n}.$$

This is a contradiction to the assumption that U_i are $R(0, 1)$-distributed. So there does not exist a random variable X with distribution in $\mathfrak{H}(P_1, \ldots, P_n)$ and $\sum_{i=1}^{n} X_i=n$.

A characterization of (9) gives the following theorem.

Theorem 5: *There exists a random variable $X=(X_1, \ldots, X_n)$ with distribution in $\mathfrak{H}(P_1, \ldots, P_n)$ and $\sum_{i=1}^{n} X_i=c$ if and only if*

$$\sum_{i=1}^{n}\int f_i\mathrm{d}P_i\leq \sup\left\{\sum_{i=1}^{n} f_i(x_i) \mid x_1, \ldots, x_n\in R^1,\ \sum_{i=1}^{n} x_i=c\right\} \tag{10}$$

for all continuous bounded functions $f_1, \ldots, f_n$ on R^1.

Proof: Define

$$\Lambda:=\left\{P \mid P \text{ is a probability measure on } \left\{\sum_{i=1}^{n} x_i=c\right\}\right\}.$$

Then we get

$$\sup\left\{\int \sum_{i=1}^{n} f_i \circ \pi_i \mathrm{d}P \mid P \in \Lambda\right\} = \sup\left\{\sum_{i=1}^{n} f_i(x_i) \mid \sum_{i=1}^{n} x_i = c\right\}, \tag{11}$$

where π_i is the projection on the i-th coordinate. Clearly the left hand side of (11) is smaller than or equal to the right hand side of (11). On the other hand there are $(x_1', \ldots, x_n')$ with $\sum_{i=1}^{n} x_i' = c$ such that

$$\sum_{i=1}^{n} f_i(x_i') \geqq \sup\left\{\sum_{i=1}^{n} f_i(x_i) \mid \sum_{i=1}^{n} x_i = c\right\} - \varepsilon.$$

Taking as P the one point measure concentrated in $(x_1', \ldots, x_n')$ we also get the other inequality in (11). Now (10) follows from Theorem 7 of STRASSEN [14] (resp. his generalization of Theorem 7 on pg. 437). ∎

Example 3: We give a solution to question (9) in the case that P_i, $1 \leqq i \leqq n$, are uniform distributions on $[0, 1]$. For $n=2$ the distribution of $(U, 1-U)$, where U is $R(0, 1)$-distributed, solves this question.

For $n=3$ a solution is given by the distribution of (V_1, V_2, V_3), where

$$V_1 := U,$$

$$V_2 := U + \frac{1}{2}\, 1_{\left[0, \frac{1}{2}\right]}(U) - \frac{1}{2} \cdot 1_{\left[\frac{1}{2}, 1\right]}(U)$$

and

$$V_3 := -2U + 1_{\left[0, \frac{1}{2}\right]}(U) + 2 \cdot 1_{\left[\frac{1}{2}, 1\right]}(U).$$

For general n we obtain a solution by combination of the cases $n=2, 3$.

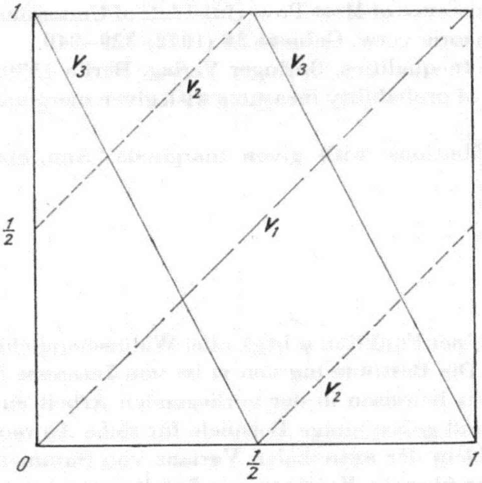

Fig. 1

From Example 3 and Theorem 5 we get the following corollary.

Corollary 6: *Let $f: [0, 1] \to R^1$ be continuous. Then*

$$\int_0^1 f(x)\,dx \leqq \sup \left\{ \frac{1}{n} \sum_{i=1}^n f(x_i) \mid x_1, \ldots, x_n \in [0, 1], \quad \sum_{i=1}^n x_i = \frac{n}{2} \right\}$$

for all $n \geqq 2$.

References

[1] BILLINGSLEY, P.: Convergence of probability measures. Wiley, New York 1968.

[2] DIETER, U.: Optimierungsaufgaben in topologischen Vektorräumen I: Dualitätstheorie. Z. Wahrscheinlichkeitstheorie verw. Gebiete **5** (1966) 89–117.

[3] DUNFORD, N., SCHWARTZ, I. T.: Linear Operators, Part I: General Theory. Interscience Publishers, inc. New York 1957.

[4] FISHMAN, G. S.: Variance reduction in simulation studies. J. Statist. Comp. and Simulation **1** (1972) 173–182.

[5] GOLSTEIN, E. G.: Dualitätstheorie in der nichtlinearen Optimierung und ihre Anwendung. Akademie-Verlag, Berlin 1975.

[6] HAMMERSLEY, I. M., HANDSCOMB, D. C.: Monte Carlo Methods. Methuen, London 1964.

[7] HANDSCOMB, D. C.: Proof of the antithetic-variates theorem for $n \geqq 2$. Proc. Comb. Philos, Soc. **54** (1958) 300–301.

[8] ISII, K.: On sharpness of TSCHEBYCHEFF-type inequalities. Ann. Inst. Stat. Math. **14** (1963) 185–197.

[9] ISII, K.: Inequalities of the types of CHEBYCHEV and CRAMER-RAO and mathematical programming. Ann. Inst. Stat. Math. **16** (1964) 277–293.

[10] LAI, T. L., ROBBINS, H.: Maximally dependent random variables. Proc. Nat. Acad. Sci. USA **73** (1976) 286–289.

[11] LAI, T. L., ROBBINS, H.: A class of dependent random variables and their maxima. Z. Wahrscheinlichkeitstheorie verw. Gebiete **42** (1978) 89–111.

[12] LANDERS, D., ROGGE, L.: Existence of Most Powerful Tests of Undominated Hypotheses. Z. Wahrscheinlichkeitstheorie verw. Gebiete **24** (1972) 339–340.

[13] MITRINOVIC, D. S.: Analytic Inequalities. Springer Verlag, Berlin 1970.

[14] STRASSEN, V.: The existence of probability measures with given marginals. Ann. Math. Statist. **36** (1965) 423–439.

[15] WHITT, M.: Bivariate distributions with given marginals. Ann. Statist. **4** (1976) 1280–1289.

Zusammenfassung

Sei m das Infimum der Integrale einer Funktion φ bzgl. aller Wahrscheinlichkeitsmaße mit vorgegebenen Randverteilungen. Die Bestimmung von m ist von Interesse für eine Reihe von stochastischen Problemen. Wir beweisen in der vorliegenden Arbeit einen Dualitätssatz für die Bestimmung von m und geben einige Beispiele für seine Anwendung an. Wir betrachten insbesondere das Problem der extremalen Varianz von Summen von Zufallsvariablen und beweisen einen Satz über die Existenz von Zufallsvariablen mit vorgegebenen Randverteilungen und konstanter Summe.

Résumé

Soit m l'infimum des intégrales d'une fonction φ concernant tous les probabilités avec des marginals préscrits. La détermination de m est importante pour beaucoup des problèmes stochastiques d'optimization. Dans ce travail nous prouvons une théorème dual pour la détermination de m avec quelques examples de son application. Nous traitons spécialement le problème de la variance extremale des sommes de fonctions aléatoires et nous donnons une théorème d'existence de variables aléatoires avec de marginals préscrits et de somme constante.

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Book Review

J. Rosenmüller: **Extreme Games and Their Solutions.** Vol. 145. Springer-Verlag Berlin-Heidelberg-New York 1977, 126 S., DM 18,–.

The paper concerns the theory of cooperative games with convex (Chapter I) and superadditive (Chapter II) characteristic functions. The main problem discussed in the paper is the description of solution concepts, the core and stable sets for convex and superadditive games, respectively. First, some representations for convex and superadditive characteristic functions are obtained. A convex characteristic function is represented as an envelope of affine set functions (Sec. 1, Chpt. I). A superadditive characteristic function is represented as an envelope of step functions (Sec. 1, Chpt. II). For both respresentations some game-theoretical interpretations are given. If the normalization condition on a characteristic function is required then we have bases for cones of convex and superadditive games. The second problem studied in the paper is to characterize the extreme points of the bases (Sec. 2, Chpt. I and II). The representation theory developed in previous sections is applied. The extremality conditions for the convex and superadditive functions are essentially different. In the first case '*nondegeneracy*' of measures w.r.t. constants appearing in the representation is involved and in the second case '*homogeneity*' of measures and '*linearity*' of the step functions in the representation are needed. Now, the most interesting problem is the discussion of the nature of the core for extreme convex games and the main simple solution for extreme superadditive games. Thus, an extreme social situation described by the core is a division of the players in several classes (Sec. 3, Chpt. I). In case of a specific superadditive game a similar description is obtained. Nevertheless it seems that the study on extreme superadditive functions should be continued.

The cooperative game theory requires the insight in the structure of games. The paper is an interesting step in that direction. One could relate the results obtained by the author with some experimental data observed in social behaviour. The examples presented in Chapter III and concerning the so-called production games and selling goods in minimal quantities reveal the nature of extreme games.

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