Law invariant convex risk measures for portfolio vectors

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Abstract

The class of all law invariant, convex risk measures for portfolio vectors is characterized. The building blocks of this class are shown to be formed by the maximal correlation risk measures. We introduce some classes of multivariate distortion risk measures and relate them to multivariate quantile functionals and to an extension of the average value at risk measure.

1 Introduction

This paper is concerned with an extension of representation results for one dimensional law invariant, convex risk measures to the multivariate case. As reference for one dimensional risk measures we refer to the unifying presentation in Föllmer and Schied (2004) but several of the results go back to earlier and independent sources. We mention in particular Delbaen (2000, 2002), Kusuoka (2001), Wang, Young, and Panjer (1997), Wirch and Hardy (2000), Dhaene, Vanduffel, Tang, Goovaerts, Kaas, and Vynke (2004), Carlier and Dana (2003), Dana (2005) which include many further references.

Law invariant, convex risk measures on $L^\infty(P)$ have been characterized by a Kusuoka type representation of the form

$$\varrho(X) = \sup_{\mu \in M_1([0,1])} \left(\int_{(0,1]} AV@R_\lambda(X)\mu(d\lambda) - \beta(\mu) \right), \tag{1.1}$$

where $\varrho_\lambda(X) := AV@R_\lambda(X)$ is the average value at risk,

$$\beta(\mu) = \sup_{X \in \mathcal{A}_\varrho} \int_{[0,1]} AV@R_\lambda(X)\mu(d\lambda)$$

is the penalty function and $\mathcal{A}_\varrho = \{X \in L^\infty(P); \varrho(X) \leq 0\}$ is the acceptance set of ϱ (see Föllmer and Schied (2004, Theorem 4.57), Jouini, Schachermayer, and Touzi (2005), and Kusuoka (2001)). Thus in dimension $d = 1$ the average value at risk measures $(\varrho_\lambda)_{0<\lambda\le 1}$ are the basic building blocks of the class of all law invariant, convex risk measures.

The class of law invariant, convex risk measures can also be represented by the class of weighted quantiles $q_{-X}(t) = F^{-1}_{-X}(t)$ of $-X$,

$$\varrho(X) = \sup_{Q\in M(P)} \left(\int_0^1 q_{-X}(t) q_{\varphi_Q}(t)dt - \alpha(Q) \right) \tag{1.2}$$

where $M(P)$ is the class of probability measures, continuons *w.r.t.* P; $\varphi_Q = \frac{dQ}{dP}$ and q_{φ_Q}, q_{-X} are the quantiles of φ_Q, $-X$ (see Föllmer and Schied (2004, Theorem 4.54)). A further equivalent representation is known in terms of concave distortion risk measures or equivalently in terms of the Choquet expectation.

$$\varrho(X) = \sup_g \left(E_{c_g}(-X) - \gamma(g) \right), \tag{1.3}$$

where the sup ist over the class of all concave distortion functions g and

$$E_{c_g}X = \int_{-\infty}^0 (g \circ P(X>x) - 1)dx + \int_0^\infty g \circ P(X>x)dx \tag{1.4}$$

is the Choquet integral, defined in terms of the distortion functional $g \circ \overline{F}(x)$, $\overline{F}(x) = P(X > x)$ the survival functional (see Föllmer and Schied (2004, Corollary 4.72)). As consequence this implies that the law invariant, convex, comonotone additive risk measures are exactly those of the form

$$\varrho(X) = E_{c_g}(-X) = \int_0^1 AV@R_\lambda(X)d\mu(\lambda) \tag{1.5}$$

Risk measures are also naturally defined for portfolio vectors $X = (X_1,\dots,X_d) \in L_d^\infty(P)$. The aim of risk measures on the class of portfolio vectors is to measure not only the risk of the marginals separately but to quantify the risk of X caused by the variation of the components and at the same time by their possible dependence. The class of all convex risk measures on $L_d^\infty(P)$ has been characterized in Burgert and Rüschendorf (2005). In that paper also two concrete and easy to interprete classes of multivariate risk measures have been introduced and consistency *w.r.t.* various types of convex orderings has been studied.

In this paper we consider the question whether and in what form the basic classes of one dimensional risk measures can be extended to the multivariate case. What are analogs of the average value at risk measure, building the basic blocks of the law invariant risk measures. Are there senseful analogs of the distortion risk measures or of the weighted quantile representation? It will turn out however that only partially and less explicit forms of analogous classes of risk measures can be given in the multivariate case as there is no complete order, no obvious analog of quantiles and of distortions.

In section 2 we establish that the **maximal correlation** risk measures play in the multvariate case the role of basic building blocks of convex, law invariant risk measures. We then in section 3 introduce some natural extensions of multivariate quantile type and distortion type risk measures.

We denote by $(\Omega, \mathcal{A}, P)$ the underlying probability space which we generally assume to be non atomic (even if not always needed). $M(P)$ denotes the class of P-continuous probability measures on $(\Omega, \mathcal{A})$, $ba(P)$ denotes the corresponding class of finitely additive normed, P-continuous measures. $L_d^\infty(P)$ denotes the set of essentially bounded portfolio vectors $X = (X_1, \dots, X_d)$, i.e. $L_d^\infty(P) = \prod_{i=1}^d L^\infty(P)$ and $M_d(P)$ resp. $ba_d(P)$ denote the class of σ-additive resp. addive, P-continuous, normed measures on $L_d^\infty(P)$. $M_d(P)$ can equivalently be described by the corresponding class of P-densities

$$D := \{Y = (Y_1, \dots, Y_d);\ Y_i \geq 0\,[P],\ E_P Y_i = 1,\ 1 \leq i \leq d\}. \tag{1.6}$$

2 Law invariant, convex risk measures

A risk functional $\varrho : L_d^\infty(P) \to \mathbb{R}$ on the class of portfolio vectors $X = (X_1, \dots, X_d) \in L_d^\infty(P)$ is called **convex risk measure** if

C1) $X \geq Y \Rightarrow \varrho(X) \leq \varrho(Y)$ (2.1)

C2) $\varrho(X + me_i) = -m + \varrho(X), \quad \forall m \in \mathbb{R}$ and $1 \leq i \leq d$

C3) $\varrho(\alpha X + (1-\alpha)Y) \leq \alpha\varrho(X) + (1-\alpha)\varrho(Y)$ for all $\alpha \in (0,1)$.

Here $x \geq y$ denotes the usual componentwise ordering on $\mathbb{R}^d$ and e_i denotes the i-th unit vector. We denote throughout this paper by

$$\Psi(X) = \varrho(-X) \tag{2.2}$$

the corresponding insurance risk functional, which is monotone in the usual ordering. For financial risk measures $-X$ denotes the liability and therefore plays the essential role. This class of convex risk measures was characterized in Burgert and Rüschendorf (2005, Theorem 3.4), by a representation of the form

$$\varrho(X) = \sup_{Q \in ba_d(P)} (E_Q(-X) - \alpha(Q)), \tag{2.3}$$

where the penalty function α can be chosen as Legendre-Fenchel inverse of ϱ,

$$\alpha(Q) = \sup_{X \in L_d^\infty(P)} (E_Q(-X) - \varrho(X)) = \sup_{X \in A_\varrho} E_Q(-X).$$

For Fatou-continuous ϱ the class $ba_d(P)$ can be replaced by the class $M_d(P)$ of probability measures on $L^\infty(P)$ or, equivalently, by the class D of corresponding $P-$densities;

$$\varrho(X) = \sup_{Y \in D} (E(-X) \cdot Y - \alpha(Y)) \tag{2.4}$$

where

$$\begin{aligned}\alpha(Y) &= \sup_{X\in L_d^\infty(P)} (E(-X)\cdot Y - \varrho(X))\\ &= \sup_{X\in A_\varrho} E(-X)\cdot Y.\end{aligned}$$

A similar representation result holds true also on $L_d^p(P)$ instead of $L_d^\infty(P)$. Let for $X\in L_d^\infty(P)$

$$A(X) := \{\widetilde{X}\in L_d^\infty(P) : \widetilde{X}\sim X\} \tag{2.5}$$

be the class of all $\widetilde{X}$ with the same distribution as X. A risk measure ϱ is **law invariant** if

$$\varrho(X) = \varrho(\widetilde{X}), \quad \forall \widetilde{X}\in A(X). \tag{2.6}$$

Proposition 2.1 *If ϱ is a convex risk measure on $L_d^\infty(P)$, then*

$$\widehat{\varrho}(X) := \sup\{\varrho(\widetilde{X}); \widetilde{X}\in A(X)\} \tag{2.7}$$

is a convex, law invariant risk measure and

$$\varrho \text{ is law invariant } \Leftrightarrow \varrho = \widehat{\varrho} \tag{2.8}$$

Proof: Obviously $\widehat{\varrho}$ is law invariant. For $X, Y\in L_d^\infty(P), \alpha\in(0,1)$ and with $Z := \alpha X + (1-\alpha)Y$ holds

$$\widehat{\varrho}(\alpha X + (1-\alpha)Y) = \sup_{\widetilde{Z}\sim Z} \varrho(\widetilde{Z}).$$

Since $\widetilde{Z}\sim h(X,Y)$, with $h(x,y) := \alpha x + (1-\alpha)y$, by a result on solutions of stochastic equations (see Rüschendorf (1985)) there exists $rv's$ $(\widetilde{X},\widetilde{Y})\sim(X,Y)$ such that $\widetilde{Z} = h(\widetilde{X},\widetilde{Y})\,[P]$.

Therefore,

$$\begin{aligned}&\widehat{\varrho}(\alpha X + (1-\alpha)Y) \qquad (2.9)\\ &= \sup\{\varrho(\alpha\widetilde{X} + (1-\alpha)\widetilde{Y}); (\widetilde{X},\widetilde{Y})\sim(X,Y) \text{ and } \widetilde{Z} := \alpha\widetilde{X} + (1-\alpha)\widetilde{Y}\sim Z\}\\ &\le \sup\{\alpha\varrho(\widetilde{X}) + (1-\alpha)\varrho(\widetilde{Y}); \widetilde{X}\sim X, \widetilde{Y}\sim Y \text{ and } \alpha\widetilde{X} + (1-\alpha)\widetilde{Y}\sim Z\}\\ &\le \alpha\widehat{\varrho}(X) + (1-\alpha)\widehat{\varrho}(Y),\end{aligned}$$

i.e. $\widehat{\varrho}$ is a convex, law invariant risk measure. (2.8) is obvious. □

Thus for any risk measure ϱ resp. Ψ we obtain by the process in (2.7) a law invariant risk measure and the mapping $\varrho\to\widehat{\varrho}$ from the class of convex risk measures to the class of convex, law invariant risk measures is surjective.

Example 2.2 ***maximal correlation risk meassure***
For $Y \in D$ define the risk measure

$$\Psi_Y : L_d^\infty(P) \to \mathbb{R}, \Psi_Y(X) = EX \cdot Y \tag{2.10}$$

$\Psi_Y(X)$ is (up to the normalization) the correlation coefficient of X, Y. Ψ_Y is a convex even coherent risk measure. The corresponding law invariant risk measure

$$\widehat{\Psi}_Y(X) = \sup_{\widetilde{X} \sim X} E\widetilde{X} \cdot Y \tag{2.11}$$

is called the maximal correlation risk measure in direction Y. Correspondingly, $\widehat{\varrho}_Y(X) := \widehat{\Psi}_Y(-X)$ is the financial version of the maximal correlation risk measure. Some properties of the maximal correlation risk measure are:

a)
$$\widehat{\Psi}_Y(X) = \sup_{\substack{\widetilde{X} \sim X \\ \widetilde{Y} \sim Y}} E\widetilde{X} \cdot \widetilde{Y} = \widehat{\Psi}_{\widetilde{Y}}(X) \quad \text{for all } \widetilde{Y} \sim Y. \tag{2.12}$$

b) $\widehat{\Psi}_Y$ is consistent w.r.t. increasing convex ordering $\leq_{icx}$, i.e.

$$X_1 \leq_{icx} X_2 \text{ implies } \widehat{\Psi}_Y(X_1) \leq \widehat{\Psi}_Y(X_2) \tag{2.13}$$

This follows from Theorem 3.10 in Burgert and Rüschendorf (2005).

c) In dimension $d = 1$ an explicit representation of $\widehat{\Psi}_Y$ is known;

$$\widehat{\Psi}(X, Y) = \widehat{\Psi}_Y(X) = \int_0^1 F_X^{-1}(U) F_Y^{-1}(U) dP, \tag{2.14}$$

where $U \sim U(0,1)$ is uniformly distributed on $(0,1)$. In $d \geq 1$ the basic result for solutions $X^* \sim X, Y^* \sim Y$ of (2.12) which are called optimal couplings is the following (see Rüschendorf and Rachev (1990)):

$$\widehat{\Psi}(X, Y) = EX^* \cdot Y^* \text{ if and only if } Y^* \in \partial f(X^*)[P] \tag{2.15}$$

for some convex, lower semicontinuous function f, where ∂f is the subgradient of f. The typical case is, where $Y^* = T(X^*)$ respectively $X^* = T^*(Y^*)$ for some cyclycally monotone transformation T resp. T^*.

d) The maximal correlation risk measure has the following interpretation. It describes the maximal possible risk over all possible distributional versions $\widetilde{X} \sim X$ averaged over all directions y according to the scenario measure $Q = P^Y$. This interpretation results from the presentation

$$\widehat{\Psi}_Y(X) = \sup_{\widetilde{X} \sim X} \int E(\widetilde{X}|y) \cdot y \, dP^Y(y). \tag{2.16}$$

Here $E(\widetilde{X}|y) \cdot y$ is the conditional risk of $\widetilde{X}$ in direction $y \in \mathbb{R}_+^d$. Thus $\widehat{\Psi}_Y(X)$ describes the risk of X in random direction Y.

By a recent result of Jouini, Schachermayer, and Touzi (2005) law invariant, convex risk measures in $d = 1$ are Fatou continuous. We make use in the following of an extension of this result for $d \geq 1$. We denote for a convex risk measure with penalty function α by

$$D_0 := \{Y \in D;\ \alpha(Y) < \infty\} \tag{2.17}$$

the set of all densities of scenario measures $Q \in M_d(P)$. Then we obtain that the maximal correlation risk measures in Example 2.2 are the basic building blocks of convex law invariant risk measures.

Theorem 2.3 *Let Ψ be a convex risk measure on $L_d^\infty(P)$ with penalty function α, then it holds:*

Ψ *is law invariant*

$\Leftrightarrow$ *The penalty function $\alpha : D_0 \to \mathbb{R}$ of Ψ with D_0 the scenario set is law invariant i.e. $\alpha(Y) = \alpha(\widetilde{Y})$ if $\widetilde{Y} \sim Y$.*

$\Leftrightarrow$ *Ψ has a representation of the form*

$$\Psi(X) = \sup_{Y \in D_0} \Big(\widehat{\Psi}_Y(X) - \alpha(Y)\Big) \tag{2.18}$$

with law invariant penalty function α which can be chosen as

$$\alpha(Y) = \sup_{X \in \mathcal{A}_\Psi} \widehat{\Psi}_Y(X) = \sup_{\substack{\widetilde{Y} \sim Y \\ X \in \mathcal{A}_\Psi}} \widehat{\Psi}_{\widetilde{Y}}(X).$$

Proof: If Ψ is law invariant, then

$$\Psi(X) = \widehat{\Psi}(X) = \sup_{\widetilde{X} \sim X} \sup_{Y \in D} \Big(E\widetilde{X} \cdot Y - \alpha(Y)\Big) = \sup_{Y \in D} \Big(\widehat{\Psi}_Y(X) - \alpha(Y)\Big)$$

i.e. representation (2.18) holds. Furthermore,

$$\begin{aligned}
\alpha(Y) &= \sup_{X \in L_d^\infty} (EX \cdot Y - \Psi(X)) \\
&= \sup_{X \in L_d^\infty} \sup_{\widetilde{X} \sim X} \Big(E\widetilde{X} \cdot Y - \Psi(\widetilde{X})\Big) \\
&= \sup_{X \in L_d^\infty} \Big(\widehat{\Psi}_Y(X) - \Psi(X)\Big) \\
&= \alpha(\widetilde{Y}) \quad \text{for all } \widetilde{Y} \sim Y
\end{aligned}$$

since by (2.12) $\widehat{\Psi}_Y(X) = \widehat{\Psi}_{\widetilde{Y}}(X)$. Thus α is law invariant.

If conversely α is law invariant, then for $\widetilde{X} \sim X$ holds

$$\begin{aligned}\Psi(X) &= \sup_{Y\in D}(EX\cdot Y-\alpha(Y))\\ &= \sup_{Y\in D}\left(\sup_{\widetilde{Y}\sim Y} EX\cdot\widetilde{Y}-\alpha(Y)\right)\\ &= \sup_{Y\in D}\left(\widehat{\Psi}_Y(X)-\alpha(Y)\right)\\ &= \sup_{Y\in D}\left(\widehat{\Psi}_Y(\widetilde{X})-\alpha(Y)\right) \qquad \text{by Proposition 2.1}\\ &= \Psi(\widetilde{X}).\end{aligned}$$

Thus Ψ is law invariant and the presentation of Ψ in (2.18) holds. □

Corollary 2.4 *The class of law invariant, coherent risk measures on L_d^∞ is given by $\{\Psi_A; A\subset D\}$ where*

$$\Psi_A(X) = \sup_{Y\in A}\widehat{\Psi}_Y(X) \tag{2.19}$$

is a supremum of maximal correlation risk measures.

By Theorem 3.10 in Burgert and Rüschendorf (2005) law invariant, convex risk measure Ψ are consistent w.r.t. the increasing convex order $\leq_{icx}$ and w.r.t. the convex order $\leq_{cx}$ i.e.

$$X \leq_{icx} Y \text{ implies } \Psi(X)\leq\Psi(Y). \tag{2.20}$$

Therefore, as consequence we obtain

Corollary 2.5 *Let Ψ be a convex risk measure on $L_d^\infty(P)$. Then the following are equivalent:*

Ψ is law invariant

$\Leftrightarrow$ Ψ is $\leq_{icx}$ – consistent

$\Leftrightarrow$ Ψ is $\leq_{cx}$ – consistent

$\Leftrightarrow$ Ψ has a representation as in 2.18

Remark 2.6 *For $d=1$ the maximal correlation risk measures $\widehat{\Psi}_Y$ have an explicit representation as mixtures of quantiles (see 2.14), which leads to a Kusuoka type representation as in (1.1) and also to a distortion type repesemtation as in (1.4) (see Dana (2005), Föllmer and Schied (2004), Carlier and Dana (2003)).*

For $d\geq 1$ only for some special classes of distributions optimal coupling results are known in explicit form. In contrast the class of cyclically monotone functions is quite well studied. It includes e.g. radial transformations $r(X)\frac{X}{||X||}$, $r(X)\uparrow$ real *and transformations $T(x)$ with symmetric, positive semidefinite functional matrix $\left(\frac{\partial T_i}{\partial x_i}\right) = DT$. If e.g. $Y \sim N(0, \Sigma_0)$ is multivariate normal and $TX = AX$, A positive semidefinite, then for $X \sim N(0, \Sigma_1)$, $\Sigma_1 = A^T \Sigma_0 A$, the pair $X^* := TY$, Y is an optimal coupling and the maximal correlation is given by*

$$\widehat{\Psi}_Y(X) = EY \cdot AY = tr\ \Sigma_0^{1/2} A \Sigma_0^{1/2}.$$

3 Multivariate distortion type risk measures and quantile functionals

In dimension 1 the representation of law invariant, convex risk measures (see (2.18)) leads to the representation of ϱ by weighted quantiles in (1.2) and also to the representation as distortion risk measure in (1.4). Both types of representation are senseful and have a natural interpretation. Since the general representation result in section 2 in terms of maximal correlation risk measures is only qualitatively good to interprete (see Example 2.2 b) but in general difficult to determine explicitly we discuss in this section some extensions of quantile based and distortion based risk measures to the multivariate case.

Here the main aim is not to obtain complete mathematical representation results as in $d = 1$ but to define risk measures which have a clear motivation and which can be calculated (in principle at least). This supplements the proposal of concrete risk measures in Burgert and Rüschendorf (2005) where the main idea is to measure the risk of some real aspects of X.

We concentrate in this section to nonnegative risk vectors $X \geq 0$. In analogy to the one dimensional case $d = 1$ we define for $d \geq 1$ **distortion type risk measures** of the form

$$\Psi_\mu(X) = \int_{\mathbb{R}_+^d} g(1 - F(x)) d\mu(x), \tag{3.1}$$

where $F = F_X$ is the multivariate distribution function of X, g is a distortion function and μ is some weighting measure. More generally one could consider $\Psi_A(X) := \sup_{\mu \in A} \Psi_\mu(X)$, the sup over some class of weighting measures resp. the convex variant of these with penalty functions.

We denote by $\bar{F}_\mu^{-1}$ the **multivariate quantile functional**

$$\bar{F}_\mu^{-1}(t) := \mu(\{x \in \mathbb{R}_+^d; \bar{F}(x) < t\}, \tag{3.2}$$

$\bar{F}(x) := 1 - F(x)$. Then

$$\Psi_\mu(X) = \int_0^\infty g(t) d\bar{F}_\mu^{-1}(t) \tag{3.3}$$

In dimension $d = 1$ and for $\mu = \lambda^1$ holds

$$\bar{F}_\mu^{-1}(t) = \bar{F}^{-1}(t), 0 \leq t \leq 1 \tag{3.4}$$

is the generalized inverse of $\bar{F}$, and is indentical to the quantile functional $\bar{F}^{-1}(t) = F^{-1}(1-t)$. Under a corresponding integrabilty condition we obtain from partial integration in the case $d = 1, \mu = \lambda^1$ a representation as weighted quantile risk measure

$$\Psi(X) = -\int_0^1 \bar{F}^{-1}(t)dg(t) \tag{3.5}$$

resp. for $d \geq 1$ and for any probalility weighting measure μ the corresponding representation

$$\Psi_\mu(X) = -\int_0^1 \bar{F}_\mu^{-1}(t)dg(t). \tag{3.6}$$

Thus the distortion risk measure Ψ_μ defined in (3.1) also for $d \geq 1$ has a representation as a weighted, linear functional of the generalized quantile functional $\bar{F}_\mu^{-1}$ if g is a concave distortion function. If g is absolutely continuous and concave, then we can write (3.6) also in the form

$$\Psi_\mu(X) = \int_{(0,1)} AV@R_s^\mu(X)d\nu(s) \tag{3.7}$$

where

$$AV@R_s^\mu(X) := \frac{1}{s}\int_0^s \bar{F}_\mu^{-1}(1-u)du. \tag{3.8}$$

Hier ν is a probalility measure defined by

$$d\nu(s) = s\, d\widetilde{\nu}(s), \tag{3.9}$$

where $\widetilde{\nu}(s,1] := g'(s)$. Note that

$$\int_0^1 t\, d\widetilde{\nu}(t) = \int_0^1 \widetilde{\nu}(s,1]\, ds = \int_0^1 g'(s)\, ds = g(1) - g(0) = 1.$$

(Compare also a similar argument in Föllmer and Schied (2004, p. 186) in $d = 1$.) Thus $AV@R_s^\mu(X)$ plays the role of an average value at risk measure in $d \geq 1$.

Example 3.1 *In this example we consider the special case that μ is the Lebesgue measure on the positive diagonal, $\mu = \lambda^\pi$, where $\pi : [0,\infty) \to \mathbb{R}_+^d$, $t \to t \cdot 1$ is a parametrization of the diagnoal.*

Then

$$\psi_\mu(X) = \int_0^\infty g(1 - F(t\cdot 1))dt \tag{3.10}$$

and

$$\bar{F}_\mu(t\cdot 1) = 1 - F(t\cdot 1) = P(\max_{i\leq d} X_i > t) = \bar{F}_{\max X_i}(t). \tag{3.11}$$

Further

$$\bar{F}_\mu^{-1}(u) = \lambda^1(\{0 \leq t : \bar{F}_\mu(t\cdot 1) \leq u\} = \bar{F}_{\max X_i}^{-1}(u)$$

and thus we obtain

$$\Psi_\mu(X) = -\int_0^1 \bar{F}^{-1}_{\max X_i}(u)\, dg(u) \tag{3.12}$$

Thus in this special case $\Psi_\mu(X)$ is dentical to the one dimensional distortion risk measure applied to $\max_{i\leq d} X_i$.

The class of distortion type risk measures defined in (3.1) is weighting the risk sets

$$A_x := \{X \leq x\}^c, \quad x \in \mathbb{R}^d_+ \tag{3.13}$$

by the distortion

$$c_g := g \circ P \tag{3.14}$$

of the probability measure P. In the multivariate case the distribution function is however no longer simple to calculate and thus the calculation of Ψ_μ in (3.1) poses a considerable problem. It is also not the case that the risk sets of the form A_x represent the only relevant class of risk sets.

We next consider an extension of the class of distortion risk measures defined in (3.1) by allowing more general classes of relevant risk sets. We restrict to one parametric classes of risk sets $(A_t)_{t\geq 0} \subset \mathbb{R}^d_+$ in order to induce the order structive from $\mathbb{R}_+$ and to get not to complicated expressions. We assume the following conditions on the class of risk sets $(A_t)_{t\geq 0} \subset \mathbb{R}^d_+$

Risk sets: $(A_t)_{t\geq 0} \subset \mathbb{R}^d_+$ is called family of risk sets, if

R1) A_t are monotone sets, $t \geq 0$, i.e. $x \in A_t$ and $y \geq x \Rightarrow y \in A_t$

R2) (A_t) is decreasing in t

R3) $A_0 = \mathbb{R}^d_+$, $\lim_{t\to\infty} A_t = \emptyset$

R4) (A_t) is right continous, i.e. $A_{t+\varepsilon} \uparrow A_t$ as $\varepsilon \to 0$.

As consequence of R1) – R4) we may introduce a generating **risk function** $U : \mathbb{R}^d_+ \to \mathbb{R}_+$ by

$$U(x) := \inf\{s : x \in A_s\}. \tag{3.15}$$

We obtain a representation of the risk sets A_t as level sets of the risk function U:

$$A_t = \{U \geq t\}. \tag{3.16}$$

Our generalized class of multivariate distortion risk measures is induced by a class of risk sets (A_t) satisfying R1) – R4) and by a concave distortion function g. It is given by

$$\Psi^g(X) := \int_0^\infty g \circ P(X \in A_t) d\lambda\!\!\lambda(t). \tag{3.17}$$

Since

$$\bar{F}(t) := P(X \in A_t) = P(U(X) \geq t) = \bar{F}_{U(X)}(t) \tag{3.18}$$

we obtain

$$\Psi^g(X) = \int_0^\infty g(\bar{F}_{U(X)}(t))dt = -\int_0^1 \bar{F}^{-1}_{U(X)}(t)dg(t) = \Psi_g(U(X)). \tag{3.19}$$

Thus $\Psi^g(X)$ is the one dimensional distortion risk measures applied to the (real) risk function $U(X)$ of X. As consequence Ψ^g can be subsumed in the classes Ψ_A, Ψ_M of risk measures introduced in Burgert and Rüschendorf (2005). If U is a convex function, then Ψ^g is a convex risk measure. Let e.g. $U(x) = \sum_{i=1}^n x_i^2$, then $A_t = \{x : \sum_{i=1}^d x_i^2 \geq t\}$ and Ψ^g is based on weighting the radial part of the risk X. Further interesting choices of U are $\sum a_i x_i^2$, or $\max a_i X_i, a_i \geq 0$, which lead to convex risk measures Ψ^g as well.

References

Burgert, C. and L. Rüschendorf (2005). Consistent risk measures for portfolio vectors. *To appear in: Insurance: Mathematics and Economics*.

Carlier, G. and R.-A. Dana (2003). Core of convex distortions of probability. *Journal of Econometric Theory 13*, 199–222.

Dana, R.-A. (2005). A presentation result for concave Schur concave functions. Preprint, Université Paris Dauphine.

Delbaen, F. (2000). Coherent risk measures. Cattedra Galileiana, Pisa: Scuola Normale Superiore.

Delbaen, F. (2002). Coherent risk measures on general probability spaces. In K. Sandmann et al. (Eds.), *Advances in Finance and Stochastics. Essays in honour of Dieter Sondermann*, pp. 1–37. Springer.

Dhaene, J., S. Vanduffel, Q. Tang, M. Goovaerts, R. Kaas, and D. Vynke (2004). Solvency capital, risk measures and monotonicity: A review. Research report or 0416, Department of Applied Economics, K.U. Leuven.

Föllmer, H. and A. Schied (2004). *Stochastic Finance* (2nd ed.). de Gruyter.

Jouini, E., M. Meddeb, and N. Touzi (2004). Vector-valued coherent risk measures. *Finance and Stochastics 4*, 531–552.

Jouini, E., W. Schachermayer, and N. Touzi (2005). Law invariant risk measures have the Fatou property. Preprint.

Kusuoka, S. (2001). On law-invariant coherent risk measures. *Advances in Mathematical Economics 3*, 83–95.

Rüschendorf, L. (1985). The Wasserstein distance and approximation theorems. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 70*, 117–129.

Rüschendorf, L. and S. T. Rachev (1990). A characterization of random variables with minimum L^2-distance. *Journal of Multivariate Analysis 1*, 48–54.

Wang, S., V. Young, and H. Panjer (1997). Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics 21*, 173–183.

Wirch, J. L. and M. R. Hardy (2000). Distortion risk measures: Coherence and stochastic dominance. Preprint.

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