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Keywords: recursive processes, depth first search process, random binary search tree, FIND process

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1 Introduction

In this paper we establish a limit theorem for recursively defined processes in L^p. The proof of this theorem is based on an extension of the contraction method which has turned out to be an effective method to establish limit theorems in the area of recursive structures and algorithms (see [RR01, NR04]).

One motivation for this paper comes from the fact that for several natural classes of recursive processes one can not expect convergence to hold in the space C of continuous functions or in the Skorohod space D supplied with uniform metric resp. with Skorohod topology. This has been observed for example in connection with a process F_n describing the number of comparison steps of the FIND algorithm by Grübel and Rösler (1996). [GR96] however managed to introduce a 'modification' F'_n of F_n and to establish convergence of F'_n to the 'FIND process' in $D[0,1]$.

For conditional Galton–Watson processes it has been shown in a series of papers that scaled versions of the depth first search process (DFS) as well as of the height process and of the width process converge to a Brownian excursion process, while the scaled profile process converges to the local time process of a Brownian excursion (see Aldous (1991, 1993), Gutjahr and Pflug (1992), Drmota and Gittenberger (1997), Kersting (1998), Le Gall and Le Jan (1998), Marckert and Mokkadem (2003) and further references therein); convergence holds in $C[0,1]$ resp. $D[0,1]$. For random binary search trees (rBST) however and more generally for random trees with logarithmic height corresponding convergence results in $D[0,1]$ can not be expected. As indication we give in Section 3 a formal proof that the DFS processes of random BSTs are not tight in $D[0,1]$.

We propose in this paper to imbed $D[0,1]$ into an L^p space (continuous imbedding) and then to consider convergence of the recursive processes in the weaker L^p-topology. As consequence one gets weak convergence for a smaller class of continuous real functions on L^p compared to a convergence result in $D[0,1]$. The main result of this paper gives a set of sufficient conditions for a sequence of recursive processes to converge in L^p. The conditions are of similar nature as corresponding conditions for related limit theorems for recursive random sequences in $\mathbb{R}$ or in $\mathbb{R}^p$ (see [NR04]).

It has been taken care in this paper in particular to postulate only pointwise convergence of the scaling operators describing the recursion, while to restrict the application of (uniform) operator norm only to the formulation of the contraction conditions for which pointwise conditions would not allow to control the development of the recursive processes.

Our convergence result serves as a general frame for convergence of recursive processes in the (weak) L^p-convergence sense. As an application we derive directly convergence of the FIND sequence F_n to the unmodified FIND process F. Note that Grübel and Rösler (1996) obtained the stronger $D[0,1]$-convergence result for the modified sequences F'_n. The DFS processes for conditional Galton–Watson trees and for random trees with logarithmic height do however not fit the conditions of our limit theorem. They lead to degenerate limit equations. The problem of degenerate limit equations is also present in the case of recursive sequences in $\mathbb{R}^p$. So far only for the case of normal limit theorems a solution for this problem has been found (see [NR02]). This is a problem of considerable further interest in $\mathbb{R}^p$ as well as in L^p.

As further application we show that the existence and uniqueness results on fractal processes in Hutchinson and Rüschendorf (2000) are special cases of the recursive frame and the limit theorem in this paper. In fact this paper can be seen as an extension of [HR00]. We are convinced that as in the case of the contraction method in $\mathbb{R}^p$ the result in this paper for convergence in L^p is a first step of further developments and will be of interest for a series of further applications.

Independently of this paper which has been circulated in April 2006 and which is based on the diploma thesis of Eickmeyer (2005) a related convergence result for recursive random sequences in Hilbert spaces has been established recently in Drmota, Janson, and Neininger (2006, Theorem 6.1). In their paper they apply the limit theorem to obtain a convergence result for the profile process of some class of random search trees including binary search trees. The proof is based on convergence of the profile polynomials to some analytic processes using convergence in the Bergman–Hilbert space which implies strong pointwise and uniform convergence. There are some differences between our limit theorem to that in [DJN06]. First we consider more general recursive sequences (allowing homomorphisms in the Banach spaces L^p compared to [DJN06] who considered random affine transformations in the frame of Hilbert spaces). Our results are based on the minimal L_1-metrics on L^p. We could extend this to ℓ_r-metrics but the conditions seem to be easier to apply w.r.t. ℓ_1. [DJN06] instead use in the Hilbert space case the more flexible Zolotarev metric. In comparison to [DJN06] we avoid convergence conditions on the coefficients in the (uniform) operator norm. But their application to the profile process speaks for itself.

After introducing our frame of L^p-convergence, the minimal ℓ_r-metrics on L^p together with scaling operators and some connection to D-convergence in Section 2 we formulate the recursive sequences considered in this paper in Section 3 and establish a general convergence theorem for them. In Section 4.2 we give an application to the FIND convergence result of Grübel and Rösler (1996) and formally prove that the DFS processes of random trees with logarithmic height are not tight in the Skorohod topology on $D[0, 1]$. It turns out however that DFS processes lead for the interesting cases of trees with logarithmic height as well as for conditional Galton–Watson trees to degenerate limit equations, so that our theorem is not applicable to them. Still we are convinced that the convergence result in this paper is of interst in itself and will be useful for further development and examples. In fact in the case of random fractal measures a similar frame has been developed in Hutchinson and Rüschendorf (2000) for random measures and the corresponding convergence result for random measures gives as consequence existence and uniqueness of random fractal measures (see [HR00, Theorems 3 and 4]). Also in Theorem 6 of that paper a similar existence and uniqueness result for fractal processes is proved based on ℓ_p-contraction arguments as in this paper. In particular Brownian bridges are characterized by fractal scaling properties in that paper. The results in this paper can be seen as an extension of that paper to general recursive sequences of processes including the case of fractal processes as particular case.

2 Weak convergence, scalings and L^p-spaces

In this section we introduce the L^p-spaces together with the minimal ℓ_r-metrics on them and the induced convergence notion. We also introduce the basic homomorphisms on L^p which are used for formulating the recursive equations in this paper. A case of particular interst are scaling operators which allow to scale in space and in time. In the first part of this section we consider some properties of weak convergence in the Skorohod topology on $D = D[0,1]$ and discuss a continuous imbedding of D into the $L^p = L^p[0,1]$ space supplied with the usual L^p-topology. This is of interest when convergence of processes on D cannot be expected as in the case of DFS processes for trees with logarithmic height (see Section 3).

The space D of càdlàg functions supplied with the Skorohod topology is studied in detail in Billingsley (1968). A subset $K \subset D$ is relatively compact iff K is uniformly bounded, i.e.

$$\sup_{f\in K} \sup_{t\in[0,1]} |f(t)| < \infty \tag{2.1}$$

and

$$\lim_{\delta\to 0} \sup_{f\in K} \omega'_f(\delta) = 0 \tag{2.2}$$

where $\omega'_f(\delta) = \inf_{t_i - t_{i-1} > \delta} \max_{1\le i\le n} \omega_f([t_{i-1}, t_i))$, the inf over all decompositions (t_i) of $[0,1]$, and where $\omega_f(A)$ is the continuity modulus of f on A.

A consequence of this criterion is the following necessary condition for tightness.

Lemma 2.1 *If $K \subset D$ is relatively compact, then for any $\varepsilon > 0$, there exists some $n_\varepsilon = n_{\varepsilon,K} \in \mathbb{N}$ such that for all $f \in K$ and any $0 \le t_1 < \cdots < t_{n_\varepsilon+1} \le 1$ holds $\min_{1\le i\le n_\varepsilon} |f(t_i) - f(t_{i+1})| \le \varepsilon$, i.e. any $f \in K$ crosses intervals of length ε at most n_ε times.*

For functions f defined on an interval I we introduce the *scaling operator* $\mathfrak{T}_{a,b}$, $a > 0$, $b \in \mathbb{R}^1$ by

$$\mathfrak{T}_{a,b} f(t) := f(a(t-b)), \tag{2.3}$$

where $f(a(t-b))$ is defined to be zero if f is not defined there. This definition extends similarly to functions on more general domains. In particular $\mathfrak{T}_{a,0}$ describes a scaling of time from domain $I = [0, \frac{1}{a}]$ to $[0, 1]$.

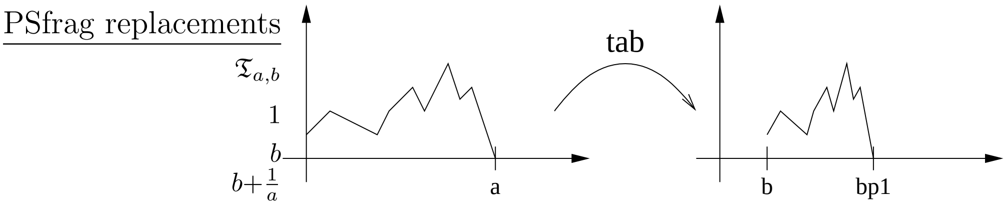

Figure 2.1 The operator $\mathfrak{T}_{a,b}$.

Scaling operators are a basic class for formulating recursive sequences. In particular they are used to introduce fractal random measures and fractal processes which are particular instances of random recursive processes (see [HR00]).

In order to obtain weak convergence results for processes in C or in D we have sometimes to consider a weakening of the Skorohod topology on C, D by embedding these spaces in the Banach space $L^p = L^p[0,1]$ supplied with the usual L^p-norm $\|f\|_{\mathrm{p}} = \left(\int_0^1 |f|^p dx\right)^{1/p}$, $1 \le p < \infty$. $\mathfrak{T}_{a,b}$ is well defined on L^p and

$$\|\mathfrak{T}_{a,b} f\|_{\mathrm{p}} \le \left(\frac{1}{a}\right)^{1/p} \|f\|_{\mathrm{p}}, \tag{2.4}$$

with equality holding for $a \in (0,1)$ and $b = 0$.

Let for a continuous endomorphism $\mathfrak{A}$ on L^p

$$\|\mathfrak{A}\|_{\mathrm{op}} := \sup_{\|f\|_{\mathrm{p}}=1} \|\mathfrak{A} f\|_{\mathrm{p}} \tag{2.5}$$

denote the operator norm. Then using (2.4) and approximation by continuous functions one sees

Lemma 2.2 *The mapping* $\mathfrak{T} : \mathbb{R} \times \mathbb{R} \times L^p \to L^p$, $\quad (a, b, f) \to \mathfrak{T}_{a,b} f$ *is continuous.*

Since elements $f \in D$ are bounded, one can embed $D \subset L^p$. With respect to the L^p-topology D is not complete. In the converse direction one obtains (for details see [Eic05]).

Lemma 2.3 *The imbedding $D \to L^p$ is continuous, where D is supplied with the Skorohod topology, i.e., convergence w.r.t. the Skorohod topology implies convergence in L^p.*

To describe weak convergence of stochastic processes in the Banach space $(L^p, \| \|_{\mathrm{p}})$ we define for $1 \le r < \infty$ the minimal L_r-metric ℓ_r on the set $M_{p,r}$ of all probability measures μ on L^p with finite r-th moments $\int \|f\|_{\mathrm{p}}^r d\mu(f) < \infty$ by

$$\ell_r(\mu, \nu) = \inf\{(E\|X - Y\|_{\mathrm{p}}^r\}^{1/r}; \quad X \sim \mu, Y \sim \nu\}. \tag{2.6}$$

Here $X \sim \mu$ means that the stochastic process X has distribution μ. It is well known that $(M_{p,r}, \ell_r)$ is a complete metric space and for μ_n, $\mu \in M_{p,r}$, convergence $\ell_r(\mu_n, \mu) \to 0$ is equivalent to

$$\mu_n \xrightarrow{\mathcal{D}} \mu \quad \text{and} \quad \int \|f\|_{\mathrm{p}}^r d\mu_n \to \int \|f\|_{\mathrm{p}}^r d\mu \tag{2.7}$$

or to

$$\int \phi(f) d\mu_n \to \int \phi(f) d\mu \tag{2.8}$$

for all continuous $\phi : L^p \to \mathbb{R}$ with $|\phi(f)| = O(\|f\|_{\mathrm{p}}^r)$.

For $X \sim \mu, Y \sim \nu$ we use the notation $\ell_r(X, Y) = \ell_r(\mu, \nu)$. For continuous endomorphisms $\mathfrak{A} : L^p \to L^p$ holds the estimate

$$\ell_r(\mathfrak{A}X, \mathfrak{A}Y) \le \|\mathfrak{A}\|_{\mathrm{op}} \ell_r(X, Y) \tag{2.9}$$

while for the particular scalings $\mathfrak{A}_\alpha f = \alpha f$ equality holds. We shall use continuous endomorphisms to formulate the recursive sequences in Section 3. This gives additional degrees of freedom compared to using random affine transformations only as is quite typical in recursive algorithms. One may e.g. include kernel transforms $\int K(t, s) X_s ds$ of a process X in this framework. In the following we concentrate on the case $r = 1$ and define $M_1 := M_{p,1} = M_1(L^p)$ the class of all distributions of L^p processes Z with finite norm $E\|Z\|_p < \infty$.

Lemma 2.4 (Random scaling) *Let A, B, C be real random variables, $A > 0$ and for $X \sim \mu \in M_p$, X independent of (A, B, C) consider the random scaled process $Y := C\mathfrak{T}_{A,B}X$. Then*

$$E\|Y\|_{\mathrm{p}} \le cE\|X\|_{\mathrm{p}} \quad \textit{with } c := E\frac{|C|}{A^{1/p}}. \tag{2.10}$$

Proof: Using the estimate (2.4) and the independence assumption we obtain $E\|Y\|_{\mathrm{p}} = E\|C\mathfrak{T}_{A,B}X\|_{\mathrm{p}} \le E(|C| \, \|\mathfrak{T}_{A,B}\|_{\mathrm{op}} \|X\|_{\mathrm{p}}) \le E\Big|\frac{C}{A^{1/p}}\Big| E\|X\|_{\mathrm{p}}$. □

3 A limit theorem for recursively defined processes

In this section we introduce a class (Z_n) of recursive processes in $L^p = L^p[0,1]$ and establish sufficient conditions for a limit theorem for (Z_n). In more detail we consider a sequence (Z_n) of $L^p[0,1]$-valued processes satisfying the following recursive structure

$$Z_n \stackrel{d}{=} b^{(n)} + \sum_{r=1}^{K} \mathfrak{A}_r^{(n)} Z^{(r)}_{I_r^{(n)}}, \quad n \geq n_0 \tag{3.1}$$

with L^p-valued processes $b^{(n)}$ and random continuous endomorphisms $\mathfrak{A}_r^{(n)}$ of L^p. $(Z_n^{(1)}), \dots, (Z_n^{(K)})$ are independent copies of (Z_n) which are also independent of $((I_r^{(n)}), (\mathfrak{A}_1^{(n)}, \dots, \mathfrak{A}_K^{(n)}), b^{(n)})$. Further the 'subgroup sizes' $(I_r^{(n)})$ are assumed to be distributed on $\{0, \dots, n\}$.

Thus the process at 'time' n is formed by applying some continuous homomorphisms as e.g. scaling transforms or kernel integral transforms $\mathfrak{A}_r^{(n)}$ to some independent random copies of the processes at random previous 'time points' $I_r^{(n)}$ adding them up and adding an L^p-process $b^{(n)}$. In random tree examples the $I_r^{(n)}$ typically are subgroup sizes of the subtrees below the root. Conditioned on the numbers $I_r^{(n)}$ these subtrees are independent and are trees of the same type. This is a quite common structure in algorithms (see [NR04] for examples).

Our main result is the following limit theorem. We denote by $Z \in M_1 = M_p(L^p)$ that $E\|Z\|_{\mathrm{p}} < \infty$.

Theorem 3.1 *Let (Z_n) be a recursively defined sequence of stochastic processes in $L^p = L^p[0,1]$ as in (3.1). Further we assume the following conditions for any $f \in L^p$, $\ell \geq 1$ and $1 \leq r \leq K$ holding for some $b^* \in L^p$ and random endomorphisms $\mathfrak{A}_r^*$ on L^p:*

Let $Z_0, \dots, Z_{n_0-1} \in M_1$, $b^{(n)}, b^* \in M_1$ for $n \in \mathbb{N}$ such that

(1) $E\|b^{(n)} - b^*\|_{\mathrm{p}} \to 0$ (2) $E\|\mathfrak{A}_r^{(n)} f - \mathfrak{A}_r^* f\|_{\mathrm{p}} \to 0$

(3) $E\sum_{r=1}^K \|\mathfrak{A}_r^*\|_{\mathrm{op}} < 1$ (4) $E\Big(1_{\{I_r^{(n)} \leq \ell\} \cup \{I_r^{(n)} = n\}} \|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}}\Big) \to 0$

(5) $\overline{\lim} E\sum_{r=1}^K \|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}} < 1$

(6) $\|\mathfrak{A}_r^*\|_{\mathrm{op}} < M$ *and* $\|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}} < M$ *for* $1 \leq r \leq K$, $n \in \mathbb{N}$ *and some* $M > 0$.

Then $\ell_1(Z_n, Z_*) \to 0$, where Z_* is the unique solution (in distribution) of the *recursive equation*

$$Z_* \stackrel{d}{=} \sum_{r=1}^K \mathfrak{A}_r^* Z^{(r)} + b^* \tag{3.2}$$

in L^p with $E\|Z\|_p < \infty$.

Proof: For the proof we define the linear operator $T : M_1 \to M_1$ by $TZ \stackrel{d}{=} \sum_{r=1}^K \mathfrak{A}_r^* Z^{(r)} + b^*$. Let $X^{(1)}, \ldots, X^{(K)} \stackrel{d}{=} X$ and $Y^{(1)}, \ldots, Y^{(K)} \stackrel{d}{=} Y$ be pairwise optimal couplings, i.e. $\ell_1(X,Y) = E\|X^{(r)} - Y^{(r)}\|_{\mathrm{p}}$, $1 \le r \le K$ and let $(X^{(1)}, Y^{(1)}), \ldots, (X^{(K)}, Y^{(K)}), (\mathfrak{A}_1^*, \ldots, \mathfrak{A}_K^*, b^*)$ be independent. Then we obtain by Minkowski's inequality and using (2.9)

$$\begin{aligned} \ell_1(TX, TY) &\le E\Big\|\Big(\sum_{r=1}^K \mathfrak{A}_r^* X^{(r)} + b^*) - \Big(\sum_{r=1}^K \mathfrak{A}_r^* Y^{(r)} + b^*\Big)\Big\|_p \\ &= E\Big\|\sum_{r=1}^K \mathfrak{A}_r^*(X^{(r)} - Y^{(r)})\Big\|_p \\ &\le E\Big(\sum_{r=1}^K \|\mathfrak{A}_r^*\|_{\mathrm{op}}\Big)\ell_1(X,Y). \end{aligned} \tag{3.3}$$

By assumption (3) $c := E\sum_{r=1}^K \|\mathfrak{A}_r^*\|_{\mathrm{op}} < 1$ and thus T is a contractive operator on the complete metric space (M_1, ℓ_1). By Banach's fixpoint theorem there exists a unique fixpoint Z_* of T in M_1, i.e. a unique solution of the recursive stochastic equation (3.2).

To prove convergence of Z_n to Z_* we introduce

$$Q_n := \sum_{r=1}^K \mathfrak{A}_r^{(n)} Z_*^{(r)} + b^{(n)}, \quad n \ge n_0, \tag{3.4}$$

where $Z_*^{(r)}$ are independent copies of Z_*. By the triangle inequality we obtain

$$\ell_1(Z_n, Z_*) \le \ell_1(Z_n, Q_n) + \ell_1(Q_n, Z_*) =: a_n + b_n. \tag{3.5}$$

From the fixpoint equation (3.2) we obtain

$$\begin{aligned} b_n = \ell_1(Q_n, Z_*) &\le E\Big\|\Big(\sum_{r=1}^K \mathfrak{A}_r^{(n)} Z_*^{(r)} + b^{(n)}\Big) - \Big(\sum_{r=1}^K \mathfrak{A}_r^* Z_*^{(r)} + b^*\Big)\Big\|_{\mathrm{p}} \\ &\le \sum_{r=1}^K E\|(\mathfrak{A}_r^{(n)} - \mathfrak{A}_r^*)Z_*\|_{\mathrm{p}} + E\|b^{(n)} - b^*\|_{\mathrm{p}} \end{aligned} \tag{3.6}$$

The second term converges by assumption (1) to zero. Using the independence assumptions and a conditioning argument we obtain

$$\begin{aligned} E\|(\mathfrak{A}_r^{(n)} - \mathfrak{A}_r^*)Z_*\|_p &= \int E\big[\|(\mathfrak{A}_r^{(n)} - \mathfrak{A}_r^*\|_p \mid Z_* = f\big]dP^{Z_*}(f) \\ &= \int E\|(\mathfrak{A}_r^{(n)} - \mathfrak{A}_r^*)f\|_{\mathrm{p}} dP^{Z_*}(f). \end{aligned}$$

By assumption (6) the integrand is bounded above by $2M\|f\|_{\mathrm{p}}$. Since $Z_* \in M_1$ we conclude by the dominated convergence theorem that

$$b_n \to 0. \tag{3.7}$$

To estimate the first term $a_n = \ell_1(Z_n, Q_n)$ we assume that $(Z_*^{(r)}, Z_n^{(r)})$ are independent optimal couplings of (Z_*, Z_n). By assumption (4) $p_n := E\sum_{r=1}^K (1_{\{I_r^{(n)}=n\}}\|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}}) \to 0$. This implies that

$$\begin{aligned} a_n = \ell_1(Z_n, Q_n) &\leq E\Big\|\Big(b^{(n)} + \sum_{r=1}^K \mathfrak{A}_r^{(n)} Z_{I_r^{(n)}}^{(r)}\Big) - \Big(b^{(n)} + \sum_{r=1}^K \mathfrak{A}_r^{(n)} Z_*^{(r)}\Big)\Big\|_{\mathrm{p}} \\ &= E\Big\|\sum_{r=1}^K \mathfrak{A}_r^{(n)}(Z_*^{(r)} - Z_{I_n^{(r)}}^{(r)})\Big\|_{\mathrm{p}} \\ &\leq E\Big\|\sum_{r=1}^K 1_{\{I_r^{(n)} \neq n\}} \mathfrak{A}_r^{(n)}(Z_*^{(r)} - Z_{I_n^{(r)}}\Big\|_{\mathrm{p}} \\ &\qquad + E\Big[1_{\{I_r^{(n)}=n\}}\Big(\sum_{r=1}^K \|\mathfrak{A}_r^{(n)}\|\Big)\ell_1(Z_n, Z_*)\Big] \\ &\leq p_n \ell_1(Z_n, Z_*) + E\Big\|\sum_{r=1}^K 1_{\{I_r^{(n)} \neq n\}} \mathfrak{A}_r^{(n)}(Z_*^{(r)} - Z_{I_n^{(r)}}^{(r)})\Big\|_{\mathrm{p}}. \end{aligned} \tag{3.8}$$

From the triangle inequality in (3.5) we thus obtain

$$\begin{aligned} \ell_1(Z_n, Z_*) &\leq \frac{1}{1-p_n}\Big[\Big(E\sum_{r=1}^K \|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}}\Big) \sup_{0 \leq j \leq n-1} \ell_1(Z_j, Z_*) + \ell_1(Q_n, Z_n)\Big] \\ &\leq \frac{1}{1-p_n}\Big[\Big(E\sum_{r=1}^K \|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}}\Big) \sup_{0 \leq j \leq n-1} \ell_1(Z_j, Z_*) + o(1)\Big]. \end{aligned} \tag{3.9}$$

Using assumption (5) we obtain that the sequence $\ell_1(Z_n, Z_*)$ is bounded, i.e.

$$\overline{\eta} := \sup_{j \geq 0} \ell_1(Z_j, Z_*) < \infty. \tag{3.10}$$

Thus $\eta := \overline{\lim}\ell_1(Z_j, Z_*) \le \overline{\eta} < \infty$ and for all $\eta > 0$ holds $\ell_1(Z_n, Z_*) < \eta + \varepsilon$ for $n \ge n_0$. This implies for $n > n_1$ using (3.6)

$$\begin{aligned}\ell_1(Z_n, Z_*) &\le \frac{1}{1-p_n} E\Big\|\Big(\sum_{r=1}^K \mathbb{1}_{\{I_r^{(n)} \ne n\}} \mathfrak{A}_r^{(n)}\big(Z_*^{(r)} - Z_{I_n^{(r)}}^{(r)}\big)\Big)\Big\|_{\mathrm{p}} + o(1)\\ &\le \frac{1}{1-p_n} E\Big[\Big(\sum_{r=1}^K \|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}}\Big)\ell_1(Z_{I_r^{(n)}}, Z_*)\Big] + o(1)\\ &\le \frac{\overline{\eta}}{1-p_n} E\Big(\sum_{r=1}^K \mathbb{1}_{\{0 \le I_r^{(n)} \le n_0\}} \|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}}\Big)\\ &\qquad + \frac{\eta+\varepsilon}{1-p_n} E\Big(\sum_{r=1}^K \|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}}\Big) + o(1)\\ &\le \Big(E\sum_{r=1}^K \|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}}\Big)\frac{\eta+\varepsilon}{1-p_n} + o(1).\end{aligned}$$

As consequence this implies

$$\eta \le \overline{\lim} E\Big(\sum_{r=1}^K \|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}}\Big)(\eta+\varepsilon) \tag{3.11}$$

holding for all $\varepsilon > 0$. Thus assumption (5) implies that $\eta = 0$. □

Remark 3.2 *a) We remark that pointwise convergence of the operators* $\mathfrak{A}_r^{(n)} f \to \mathfrak{A}_r^* f$ *in* L^p *does in general not imply that* $\|\mathfrak{A}_r^{(n)} - \mathfrak{A}_r^*\|_{\mathrm{op}} \to 0$. *In applications the postulate of convergence of* $\mathfrak{A}_r^{(n)} \to \mathfrak{A}_r^*$ *in operator norm would be too strong. As consequence we have to add to condition (3) which is quite common in the application of the contraction method (see e.g. [RR01, Rös04, NR04]) the asymptotic condition (5) on the* $\mathfrak{A}_r^{(n)}$.

b) As consequence of Theorem 3.1 we obtain that

$$\phi(Z_n) \xrightarrow{\mathcal{D}} \phi(Z_*) \tag{3.12}$$

for any continuous real function $\phi : L^p \to \mathbb{R}$. *Thus in particular for continuous linear functionals* Λ *on* L^p *of the form*

$$\Lambda f = \int f(x)g(x)dx \tag{3.13}$$

for some $g \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$ *convergence* $\ell_1(Z_n, Z_*) \to 0$ *implies*

$$\Lambda Z_n \xrightarrow{\mathcal{D}} \Lambda Z_* \tag{3.14}$$

as well as

$$E\|Z_n\|_{\mathrm{p}} \to E\|Z_*\|_{\mathrm{p}} \tag{3.15}$$

Further for linear functionals Λ *as in* (3.13) *and* A, B *real rv's with* $A \geq a > 0$, $E\Lambda \circ \mathfrak{T}_{A,B}$ *defines a linear functional on* L^p *and*

$$\begin{aligned}(E\Lambda \circ \mathfrak{T}_{A,B})f &= E \int f(A(x-B))g(x)dx \\ &= E \int f(y)g\Big(\frac{y}{A} + B\Big)\frac{dy}{A} \\ &= \int f(y)h_{A,B}(y)dy \end{aligned} \tag{3.16}$$

with $h_{A,B}(h) := E\frac{1}{A}g(\frac{y}{A} + B) \in L^q$.

4 Some applications and remarks

4.1 FIND process

The FIND process has been introduced in Grüebel and Rösler (1996) when studying the FIND-algorithm. The FIND-algorithm determines in a list L with n linearly ordered elements the k-th order statistic in the following recursive way. An element $p \in L$ is randomly chosen (p = pivot) and the list L is subdivided into 3 sublists $L_<$ the list of elements smaller then p, $L_>$ the list of elements larger than p, and $L_= = \{p\}$. If $|L_<| = k-1$, then p is the searched k-th order statistic, if not, the algorithm continues to search in $L_<$ or in $L_>$.

Let $X_k^{(n)}$ denote the number of comparisons of FIND needed to find the k-th order statistic and define the step function F_n in $D\,[0,\infty)$

$$F_n(t) := \begin{cases} X_{\lfloor t \rfloor + 1}^{(n)} & \text{if } t < n, \\ 0 & \text{else,} \end{cases} \tag{4.1}$$

then the stochastic process F_n satisfies the recursive equation

$$\begin{aligned} F_0 &\stackrel{d}{=} F_1 \stackrel{d}{=} 0 \\ F_n &\stackrel{d}{=} (n-1)1_{[0,n)} + \mathfrak{T}_{1,0}F_{L_n} + \mathfrak{T}_{1,L_n+1}\overline{F}_{R_n} \end{aligned} \tag{4.2}$$

where $L_n = |L_<|$ is uniformly distributed on $\{0, \dots, n-1\}, R_n = |L_>| = n - 1 - L_n$, $F_n \stackrel{d}{=} \overline{F}_n$, and (F_n), $(\overline{F}_n)$ are independent of each other and of (L_n) resp. (R_n).

Grübel and Rösler (1996) have shown that the FIND algorithm process (F_n) does not converge in Skorohod topology. [GR96] introduced a modification F'_n of the FIND-algorithm process, where the list L is splitted into two lists $L_\leq = L_< \cup L_=$ and $L_>$, i.e. if the random pivot p is identical to the searched order statistic it is put into the list $L_\leq$ and the search continues. They proved that this modification converges after normalization in $D[0,1]$ to some limit F which is called FIND process.

$$\frac{1}{n}\mathfrak{T}_{n,0}F'_n \xrightarrow{\mathcal{D}} F. \tag{4.3}$$

Let $G_n := \frac{1}{n}\mathfrak{T}_{n,0}F_n$ denote the scaled version of the unmodified FIND algorithm process F_n, then for $n \geq 1$

$$G_n \stackrel{d}{=} \frac{n-1}{n}1_{[0,1)} + \frac{L_n}{n}\mathfrak{T}_{\frac{n}{L_n},0}G_{L_N} + \frac{R_n}{n}\mathfrak{T}_{\frac{n}{R_n},\frac{L_n+1}{n}}\overline{G}_{R_n}. \tag{4.4}$$

Thus (G_n) fits the recursive scheme in (3.1) with $K = 2$, $\left(I_1^{(n)}, I_2^{(n)}\right) = (L_n, R_n)$, $b^{(n)} = \frac{n-1}{n}1_{[0,1)}$, $\mathfrak{A}_1^{(n)} = \frac{L_n}{n}\mathfrak{T}_{\frac{n}{L_n},0}$ and $\mathfrak{A}_2^{(n)} = \frac{R_n}{n}\mathfrak{T}_{\frac{n}{R_n},\frac{L_n+1}{n}}$. We have convergence $\frac{1}{n}(L_n, R_n) \xrightarrow{\mathcal{D}} (U, 1-U)$ where U is uniformly distributed on $(0,1)$ and by Lemma 2.2.

$$\begin{aligned} &\mathfrak{A}_1^{(n)}f \xrightarrow{D} \mathfrak{A}_1^* f := U\mathfrak{T}_{\frac{1}{U},0}f \text{ and} \\ &\mathfrak{A}_2^{(n)}f \xrightarrow{D} \mathfrak{A}_2^* f := (1-U)\mathfrak{T}_{\frac{1}{1-U},U}f, \text{ for all } f \in L^p. \end{aligned} \tag{4.5}$$

Using couplings of (L_n) as e.g. $L_n = \lfloor nU \rfloor$ we obtain a.s. convergence of (L_n, R_n) and thus for any $f \in L^p$

$$\|\mathfrak{A}_r^{(n)}f - \mathfrak{A}_r^* f\|_{\mathrm{p}} \to 0 \text{ a.s.}, \quad r = 1, 2. \tag{4.6}$$

As

$$\|\mathfrak{A}_1^{(n)}f\|_{\mathrm{p}} \leq \|\mathfrak{A}_1^{(n)}\|_{\mathrm{op}}\|f\|_{\mathrm{p}} = \left(\frac{b_n}{n}\right)^{\frac{p+1}{p}} \|f\|_{\mathrm{p}} < \|f\|_{\mathrm{p}}$$

and

$$\|\mathfrak{A}_2^{(n)}f\|_{\mathrm{p}} \leq \left(\frac{R_n}{n}\right)^{\frac{p+1}{p}} \|f\|_{\mathrm{p}} < \|f\|_{\mathrm{p}}$$

and similarly for $\mathfrak{A}_1^*$, $\mathfrak{A}_2^*$, we conclude by dominated convergence

$$E\|\mathfrak{A}_r^{(n)}f - \mathfrak{A}_r^* f\|_{\mathrm{p}} \to 0, \quad r = 1, 2. \tag{4.7}$$

Further

$$E\|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}} = \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{i}{n}\right)^{\frac{p+1}{p}} \to \int_0^1 x^{\frac{p+1}{p}}\,dx = \frac{p}{2p+1} < \frac{1}{2}$$

and thus

$$E\Big(\|\mathfrak{A}_1^{(n)}\|_{\mathrm{op}} + \|\mathfrak{A}_2^{(n)}\|_{\mathrm{op}}\Big) \to E\Big(\|\mathfrak{A}_1^*\|_{\mathrm{op}} + \|\mathfrak{A}_2^*\|_{\mathrm{op}}\Big) = \frac{2p}{2p+1} < 1. \tag{4.8}$$

Finally for all $1 \le r \le K$ and $\ell \ge 1$ holds

$$E\,1_{\left\{I_r^{(n)} \le \ell\right\} \cup \left\{I_r^{(n)} = n\right\}} \|\mathfrak{A}_r^{(n)}\|_{\mathrm{op}} \le P\left(\left\{I_r^{(n)} \le \ell\right\} \cup \left\{I_r^{(n)} = n\right\}\right) \to 0 \text{ as } n \to \infty. \tag{4.9}$$

Thus all conditions of Theorem 3.1 are satisfied and we obtain as corollary.

Proposition 4.1 *The normalized FIND algorithm process (G_n) converges weakly in L^p to the FIND process F, and moreover $\ell_1(G_n, F) \to 0$. F is the unique solution in $M_1 = M_1(L^p)$ of the recursive equation*

$$Z \stackrel{d}{=} 1_{[0,1)} + U\mathfrak{T}_{\frac{1}{U},0}Z + (1-U)\mathfrak{T}_{\frac{1}{1-U},0}\overline{Z}. \tag{4.10}$$

Here $\overline{Z}$ is an independent copy of Z, $U \sim U[0,1]$ and Z, $\overline{Z}$, U are independent. Applying Remark 3.2 b) to the functional $\Lambda f = \int_0^1 f(x)dx$ we obtain

$$\Lambda G_n \xrightarrow{\mathcal{D}} \Lambda F, \tag{4.11}$$

where F is the FIND process as in (4.10). Using the case $p = 1$ we obtain further

$$E\int_0^1 G_n(t)dt \longrightarrow E\int_0^1 F(t)dt. \tag{4.12}$$

ΛG_n is the scaled version of the average search time of the FIND algorithm over all order statistics. From (4.10) we obtain

$$\Lambda F \stackrel{d}{=} \Lambda 1_{[0,1]} + U\Lambda(\mathfrak{T}_{\frac{1}{U},0}F) + (1-U)\Lambda\left(\mathfrak{T}_{\frac{1}{1-U},U}\overline{F}\right). \tag{4.13}$$

Thus $\Lambda\, 1_{[0,1]} = 1$, $U\Lambda(\mathfrak{T}_{\frac{1}{U},0}F) = U\int_0^1 F(\frac{x}{U})dx = U^2\int_0^{1/U} F(y)dy = U^2\Lambda F$. Similarly, $(1-U)\Lambda(\mathfrak{T}_{\frac{1}{1-U},U}\overline{F}) = (1-U)^2\Lambda\overline{F}$ and thus we obtain the recursive equation

$$\Lambda F \stackrel{d}{=} 1 + U^2\Lambda F + (1-U)^2\Lambda\overline{F}, \tag{4.14}$$

which characterizes uniquely the distribution of ΛF. In particular (4.14) implies that the expectation $E\Lambda F = 3$ and thus

$$E\Lambda G_n \to 3. \tag{4.15}$$

4.2 Depth first search processes

Depth first search processes have been a topic of intensive study in the last 15 years. For Galton–Watson trees conditioned on their total progeny it has been proved that the normalized depth first search process (DFS) and the height process converge in $C[0,1]$ to a Brownian excursion process while the profile process converges in $D[0,1]$ to the local time process of a Brownian excursion process. Convergence is w.r.t. the Skorohod topology on C resp. D. This implies in particular convergence results for simple or simply generated trees. For references to these kind of results we refer to Aldous (1991, 1993), Gutjahr and Pflug (1992), Drmota and Gittenberger (1997), Kersting (1998), Le Gall and Le Jan (1998), Marckert and Mokkadem (2003), and many references therein.

For several further classes of random trees like random binary search trees (rBST), tries, digital search trees and others one cannot expect similar convergence results in $D[0,1]$ in the Skorohod topology. We demonstrate this claim in the following for the depth first search process (DFS) also called depth first walk (DFW) of an rBST.

Let T_n be a random binary search tree with n internal and $n+1$ external nodes. Let L_n, R_n denote the number of nodes in the left resp. right subtree of the root, then $R_n + L_n = n-1$. R_n is uniformly distributed on $\{0, \ldots, n-1\}$ and conditional on (L_n, R_n) the left subtree T_{L_n} and the right subtree T_{R_n} are independent and both are rBSTs. Let V_n denote the depth first search process of T_n (DFS) including the external nodes.The walk follows the contour of the tree from left to the right (see Figure 4.1).

There are $2(2n+1-1) = 4n$ steps in the definition of the DSF. Thus by linear interpolation V_n is defined in $C[0,\infty)$ with support in $[0,4n]$. The local maxima of V_n correspond to the external nodes. $V_n(2(2L_n+1)) = V_n(4L_n+2) = 0$ since the left subtree contains $2L_n+1$ nodes and the contour returns to the root after $2(2L_n+1)$ steps.

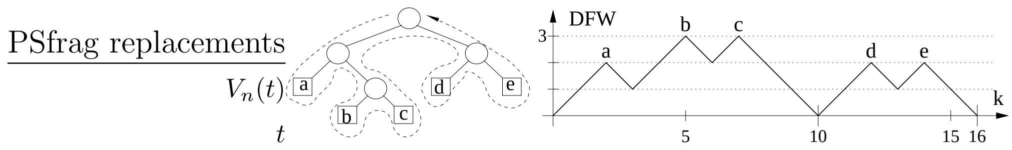

Figure 4.1 A binary search tree and its DFS-process ($n = 4$, $L_n = 2$, and $R_n = 1$).

The recursive structure of T_n is reflected by the following recursive structure of V_n:

$$\begin{cases} V_0 = 0 \\ V_n \stackrel{d}{=} b^{(n)}_{L_n} + \mathfrak{T}_{1,1} V_{L_n} + \mathfrak{T}_{1,4L_n+3} \overline{V}_{R_n} \end{cases} \tag{4.16}$$

where (L_n, R_n), (V_n), and $\overline{V}_n)$ are independent, $\overline{V}_n \stackrel{d}{=} V_n$, $L_n + R_n = n - 1$, and $L_n \stackrel{d}{=} \text{unif}\{0, \dots, n-1\}$. Further $b^{(n)}_{L_n} \in C[0, \infty)$ is one on $[1, 4L_n + 1] \cup [4L_n+3, 4n-1]$ and zero on $\{0, 4L_n+2\} \cup [4n, \infty)$ and is linearly interpolated else. We denote by X_n the rescaled DFS-process

$$X_n := \gamma(n) \mathfrak{T}_{4n,0} V_n, \tag{4.17}$$

where $\gamma(n)$ is a scaling function. The operator $\mathfrak{T}_{4n,0}$ scales V_n to the unit interval $[0, 1]$.

In this section we give a formal proof showing that the rescaled DFS-process of an rBST does not converge to some nontrivial limit when considering weak convergence $\stackrel{\mathcal{D}}{\to}$ on $C[0, 1]$ resp. on $D[0, 1]$ in the Skorohod topology. We show in the following that a nontrivial convergence result cannot be expected for any scaling sequence $\gamma(n)$.

Proposition 4.2 *a) If $\gamma(n) = o\Big(\frac{1}{\log n}\Big)$, then*

$$X_n = \gamma(n) \mathfrak{T}_{4n,0} V_n \stackrel{\mathcal{D}}{\to} 0. \tag{4.18}$$

b) If $\gamma(n) = \omega(\frac{1}{\log n})$, i.e. $\frac{1}{\log n} = o(\gamma(n))$, then (X_n) is not convergent in $D[0, 1]$ w.r.t. the Skorohod topology.

Proof: a) For binary search trees T_n the height $h(T_n)$ is of logarithmic order

$$\frac{h(T_n)}{\log n} \underset{P}{\to} c, \tag{4.19}$$

where $\underset{P}{\to}$ is convergence in probability and $c \approx 4.31107\ldots$ is the unique solution of the equation $\gamma \log(\frac{2e}{\gamma}) = 1$ bigger than 2 (see Devroye (1986)). Therefore,

$$\|X_n\|_\infty = \gamma(n)\|\mathfrak{T}_{4n,0}V_n\|_\infty = \gamma(n)h(T_n) = (\gamma(n)\log n)\frac{h(T_n)}{\log n} \underset{P}{\to} 0.$$

This implies (4.18).
b) Assume that $X_n \overset{\mathcal{D}}{\to} X$ for some process X in $D[0,1]$. Then $\|X_n\|_\infty = \gamma(n)h(T_n) \overset{\mathcal{D}}{\to} \|X\|_\infty$, where $\overset{\mathcal{D}}{\to}$ here denotes weak convergence in $\mathbb{R}^1$. But since $\gamma(n)\log n \to \infty$ we obtain by (4.19) a contradiction to tightness of $(\frac{h(T_n)}{\log n})$. □

Thus essentially it remains to check normalizing sequences of the order of magnitude $\gamma(n) \sim \frac{1}{\log n}$, since the arguments in Proposition 4.2 also can be applied to subsequences and thus imply that a necessary condition for a nontrivial convergence result for X_n is

$$0 < \underline{\lim}\gamma(n)\log n \leq \overline{\lim}\gamma(n)\log n < \infty. \tag{4.20}$$

In the following theorem we prove that also a logarithmic normalization as in (4.20) does not allow to establish a nontrivial convergence result for X_n. The argument holds true for the more general class of random trees with logarithmic height.

Proposition 4.3 *Let (T_n) be a sequence of random binary trees with n internal nodes, assume that T_n has logarithmic height, i.e.*

$$0 < \overline{\lim}\frac{Eh(T_n)}{\log n} = \alpha < \infty, \tag{4.21}$$

and let $\gamma(n)$ be a logarithmic normalization sequence satisfying (4.20). *Then the normalized DFS-process $(X_n) = (\gamma(n)\mathfrak{T}_{4n,0}V_n)$ is not convergent in $D[0,1]$ w.r.t. the Skorohod-topology.*

Proof: By Markov's inequality we obtain $P(h(T_n) > 2(\alpha+1)\log n) < \frac{1}{2}$ for $n \geq n_0$. Thus $P(A_n) \geq \frac{1}{2}$ with $A_n = \{h(T_n) \leq \alpha^* \log n\}$, $\alpha^* := 2(\alpha+1)$. For $\omega \in A_n$ there exists a level $\tilde{k} = \tilde{k}(T_n)$ in the tree with at least $\frac{n}{\alpha^* \log n}$ nodes (see Figure 4.2).

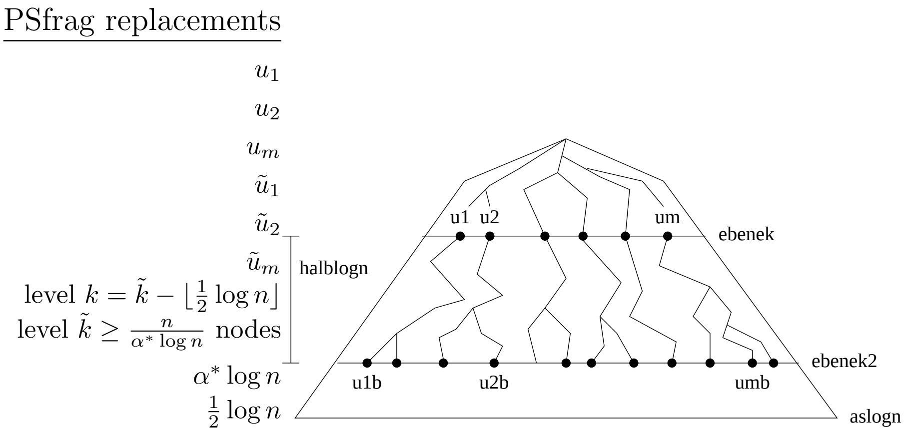

Figure 4.2 A tree with n nodes and height $< \alpha^* \log n$.

At the level $k := \widetilde{k} - \lfloor \frac{1}{2} \log n \rfloor$, therefore, we find at least

$$\frac{n}{\alpha^* \log n} \left(\frac{1}{2}\right)^{\lfloor \frac{1}{2} \log n \rfloor} \geq \frac{n^{1/2}}{\alpha^* \log n} =: m_n \tag{4.22}$$

nodes which have at least one successor in level $\widetilde{k}$. In particular we have $m_n \to \infty$ and $k = k_n > 1$. Thus we may choose nodes $u_1, \ldots, u_{m_n}$ in level k and for any of these nodes a successor $\tilde{u}_1, \ldots, \tilde{u}_{m_n}$ in level $\tilde{k}$. We list the nodes in the order in which they are visited by the DFS-process and thus get time points $0 \leq t_1 < \tilde{t}_1 < \cdots < t_{m_n} < \tilde{t}_{m_n} \leq 1$, where at time t_i the node u_i and at time $\tilde{t}_i$ the node $\tilde{u}_i$ is visited, $1 \leq i \leq m_n$.

By the choice of u_i, $\tilde{u}_i$ we obtain for $n \geq n_0$

$$\begin{aligned} X_n(\tilde{t}_i) - X_n(t_i) &= \gamma(n) \log n \Big[\tilde{V}_n(\tilde{t}_i) - \tilde{V}_n(t_i)\Big] \\ &> \frac{1}{2}\gamma(n) \log n > \alpha_1 \cdot \frac{1}{2} =: \varepsilon, \end{aligned}$$

where $\alpha_1 = \underline{\lim}\gamma(n) \log n$ and $\tilde{V}_n = \frac{1}{\log n} \mathfrak{T}_{4n,0} V_n$ is the logarithmically normalized DFS-process. Thus the scaled DFS-process X_n crosses at least m_n times an interval of height $\frac{1}{2}\alpha_1 > 0$. By Lemma 2.1 however for any $K \subset D$ compact there exists some n_ε such that any $f \in K$ crosses at most n_ε times intervals of length ε. Therefore we conclude

$$P(X_n \in K) \leq P(h(T_n) > \alpha^* \log n) < \frac{1}{2}$$

for $n \geq n_0$ and thus the process (X_n) is not tight. □

Remark 4.4 *Condition* (4.21) *on the logarithmic height of* T_n *applies in particular to rBST, to tries, to digital search trees, Patricia tries and others.*

The proof can easily also be extended to trees where the number of successors of any node is bounded above by some fixed number m like e.g. for m-ary search trees.

Depth first search processes converge for conditional Galton–Watson trees to a Brownian excursion process in $D[0,1]$, while – as shown in Proposition 4.3 – they do not converge for rBST in $D[0,1]$. It now turns out however that our limit theorem in L^p does not apply to give convergence of the DFS-process for Catalan trees or for trees of logarithmic Catalan height. Both cases lead to degenerate limit equations. It is an important open problem to establish some extension of the convergence result to the degenerate case. We give a short sketch of the arguments showing this degenerate behaviour in the following remark which might be of use as an extension of the limit theorem to this degenerate case will include some copying of the recursive structure by a limiting process (as in the $\mathbb{R}^p$ case in [NR02]).

Remark 4.5 *a) Catalan trees*
The DFS-process V_n of a Catalan tree satisfies the recursive equation

$$V_n \stackrel{d}{=} b_{L_n}^{(n)} + \mathfrak{T}_{1,1} V_{L_n} + \mathfrak{T}_{1,2L_n+3} \overline{V}_{R_n}, \tag{4.23}$$

where L_n is Catalan-distributed i.e. $P(L_n = k) = \frac{C_k C_{n-1-k}}{C_n}$ with Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ and $b_{L_n}^{(n)}$ defined as in Proposition 4.2. Further, the normalized DFS process $Y_n := \frac{1}{\sqrt{n}} \mathfrak{T}_{4n,0} V_n$ converges to the Brownian excursion process E, i.e.

$$Y_n \xrightarrow{\mathcal{D}} \frac{1}{\sigma} E \text{ in } [0,1] \tag{4.24}$$

w.r.t. sup-metric (here n denotes the number of internal nodes, thus we normalize by $4n$ considering also external nodes). (Y_n) satisfies the recursive equation

$$Y_n \stackrel{d}{=} \frac{1}{\sqrt{n}} \mathfrak{T}_{4n,0} b_{L_n}^{(n)} + \sqrt{\frac{L_n}{n}} \mathfrak{T}_{\frac{n}{L_n},\frac{1}{4n}} Y_{L_n} + \sqrt{\frac{R_n}{n}} \mathfrak{T}_{\frac{n}{R_n},\frac{4L_n+3}{4n}} \overline{Y}_{R_n}. \tag{4.25}$$

The first summand converges to zero. Further, $\frac{1}{n}(L_n, R_n) \xrightarrow{\mathcal{D}} (I, 1-I)$ with $I \sim \frac{1}{2}(\varepsilon_{\{0\}} + \varepsilon_{\{1\}})$ and for all $f \in L^p$,

$$\begin{aligned} \sqrt{\frac{L_n}{n}} \mathfrak{T}_{\frac{n}{L_n},\frac{1}{4n}} f &\xrightarrow{\mathcal{D}} \sqrt{I}\, \mathfrak{T}_{\frac{1}{I},0} f =: \mathfrak{A}_1 f \text{ and} \\ \sqrt{\frac{R_n}{n}} \mathfrak{T}_{\frac{n}{R_n},\frac{4L_n+3}{4n}} f &\xrightarrow{\mathcal{D}} \sqrt{1-I}\, \mathfrak{T}_{\frac{1}{1-I},I} f =: \mathfrak{A}_2 f. \end{aligned} \tag{4.26}$$

The contraction condition (3) for the limiting equation (3.2) *is not satisfied*

$$E(\| \mathfrak{A}_1 \|_{\mathrm{op}} + \| \mathfrak{A}_2 \|_{\mathrm{op}}) = \frac{1}{2}(1+0) + \frac{1}{2}(0+1) = 1.$$

The fixpoint equation (3.2) *degenerates to*

$$Y \stackrel{d}{=} IY + (1-I)\overline{Y}. \tag{4.27}$$

which does not have a unique solution, but any process $Y \in M_1(L_p)$ solves (4.27)*. Thus Theorem 3.1 does not apply to this case.*

b) Trees with logarithmic height

Let V_n be the DFS process of a family of trees T_n with logarithmic height i.e.

$$0 < \overline{\lim}\ \frac{Eh(T_n)}{\log n} < \infty. \tag{4.28}$$

To obtain convergence we have to use a logarithmic scaling (see Section 2.). In the case of an rBST the scaled DFS-process $Y_n := \frac{1}{\log n}\mathfrak{T}_{4n,0}V_n$ satisfies the recursive equation (see (4.16)*).*

$$\begin{aligned} Z_0 &\stackrel{d}{=} 0 \\ Z_n &\stackrel{d}{=} \tfrac{1}{\log n}\mathfrak{T}_{4n,0}b^{(n)}_{L_n} + \tfrac{\log L_n}{\log n}\mathfrak{T}_{\frac{n}{L_n},\frac{1}{4n}}Z_{L_n} + \tfrac{\log R_n}{\log n}\mathfrak{T}_{\frac{n}{R_n},\frac{4nL_n+3}{4n}}\overline{Z}_{R_n}. \end{aligned} \tag{4.29}$$

We have $\frac{1}{\log n}\mathfrak{T}_{4n,0}b^{(n)}_{L_n} \longrightarrow 0$ as $b^{(n)}_{L_n} \leq 1$. For any sequence $(k_n) \subset \mathbb{R}$ with $\sup \frac{k_n}{n} < \infty$ holds $\frac{\log k_n}{\log n} = \frac{\log \frac{k_n}{n} + \log n}{\log n} \to 1$. With a.s. convergent versions $\left(\frac{L_n}{n}, \frac{R_n}{n}\right) \to (U, 1-U)$ we thus obtain for all $f \in L^p$

$$\tfrac{\log L_n}{\log n}\mathfrak{T}_{\frac{n}{L_n},\frac{1}{4n}}f \to \mathfrak{T}_{\frac{1}{U},0}f \tag{4.30}$$

With $\mathfrak{A}^{(n)}_1 f := \frac{\log L_n}{\log n}\mathfrak{T}_{\frac{L_n}{n},\frac{1}{4n}}f$ holds $\|\mathfrak{A}^{(n)}_1\|_{\mathrm{op}} = \frac{\log L_n}{\log n}\left(\frac{L_n}{n}\right)^{1/p}$ and thus

$$E\|\mathfrak{A}^{(n)}_1\|_{\mathrm{op}} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{\log k}{\log n}\left(\frac{k}{n}\right)^{1/p} \geq \frac{1}{n}\sum_{k=0}^{n-1}\frac{\log k}{\log n}\ \frac{k}{n}.$$

For $\varepsilon > 0$ holds $\frac{\log k}{\log n} > 1 - \varepsilon$ iff $k > n^{1-\varepsilon}$ and thus

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{\log k}{\log n}\ \frac{k}{n} \geq \frac{1-\varepsilon}{n^2}\sum_{k=\lceil n^{1-\varepsilon}\rceil}^{n-1} k \approx \frac{1-\varepsilon}{n^2}\frac{1}{2}n(n-1)(1-n^{-\varepsilon}) \to \frac{1-\varepsilon}{2}.$$

As a result we obtain $\overline{\lim} E(\|\mathfrak{A}_1^{(n)}\|_{\text{op}} + \|\mathfrak{A}_2^{(n)}\|) \geq 1$ *and thus the contraction condition (3) for the limiting equation in* (3.1) *is not satisfied. The limiting equation degenerates to*

$$Z \stackrel{d}{=} \mathfrak{T}_{\frac{1}{U},0} Y + \mathfrak{T}_{\frac{1}{1-U},U} \overline{Z} \tag{4.31}$$

which again has any $Z \in M_1(L^p)$ *as solution. Thus also in this case Theorem 3.1 is not applicable.*

c) Convergent DFS-processes

The DFS processes of Catalan trees and of rBST's do not fit the conditions of Theorem 3.1. The reason however seems to be not the frame of convergence in L^p.

Our limit theorem applies to give convergence of the DFS-process in a scenario of random tree models T_n *with height of order* $\sqrt{n}$ *(as in the case of Catalan trees) and with DFS-process* (V_n) *satisfying the recursive equation* (4.10). *If* $\frac{1}{n}(L_n, R_n) \to (I, 1-I)$ *for some rv* I *on* $(0,1)$ *and* (4.26) *holds, then with* $\mathfrak{A}_1^* f = \sqrt{I} \mathfrak{T}_{\frac{1}{I},0} f$ *we have by* (2.10)

$$E\|\mathfrak{A}_1^*\|_{\text{op}} \leq c_1 := E \frac{I^{1/2}}{(1/I)^{1/p}} = E I^{1/2+1/p} \tag{4.32}$$

Thus the essential contraction condition (4) of Theorem 3.1 is fulfilled if

$$c_1 + c_2 = E(I^{1/2+1/p} + (1-I)^{1/2+1/p}) < 1 \tag{4.33}$$

For $1 \leq p < 2$ *holds* $r := \frac{1}{2} + \frac{1}{p} > 1$ *and thus* $c_1 + c_2 < 2\left(\frac{1}{2}\right)^r < 1$ *and the contraction condition is fulfilled. Thus our limit theorem applies to these kind of examples and yields convergence of the DFS processes. It is however not obvious whether there are some natural examples of random trees of this type.*

4.3 Random fractal processes

The limit theorem proved in this paper generalizes some existence and uniqueness results in [HR00] on random fractal processes. To show this connection we remind on some of the basic constructions there. Let I be a closed bounded interval in $\mathbb{R}^1$ and let $f : I \to \mathbb{R}^n$ be a function. Let $I = I_1 \cup \cdots \cup I_N$ be a partition of I into disjoint subintervals. Further let $\phi_i : I \to I_i$ be increasing Lipschitz maps with $p_i = \operatorname{Lip} \phi_i$ and define for $g_i : I_i \to \mathbb{R}^n$ the composition $\bigsqcup_i g_i : I \to \mathbb{R}^n$ by $\bigsqcup_i g_i(x) = g_j(x)$ for $x \in I_j$.

A scaling law $\mathbb{S}$ on $\mathbb{R}^n$ is an N-tuple $(S_1, \dots, S_N)$ of Lipschitz maps $S_i : \mathbb{R}^n \to \mathbb{R}^n$ with Lipschitz constants r_i. A function f is called selfsimilar w.r.t. the scaling law $\mathbb{S}$ if

$$\mathbb{S}f = f \tag{4.34}$$

where $\mathbb{S}f = \bigsqcup_i S_i \circ f \circ \phi_i^{-1} = \sum_{i=1}^N \mathfrak{A}_i f$, where $\mathfrak{A}_i f = S_i \circ f \circ \phi_i^{-1}$ on I_i and zero else.

Thus selfsimilarity in this deterministic case is described via continuous homomorphisms $\mathfrak{A}_i$ on L^p. Our limit theorem in Section 3 of this paper then implies convergence of the iterative sequence $S^n f$ to some selfsimilar function f^* under the conditions of Theorem 5 in [HR00]. Note that for this result we do not need the subgroup sizes $I_r^{(n)}$ of our general recursion.

Random fractal (selfsimilar) processes are defined via random scaling laws $\mathbb{S} = (S_1, \dots, S_N)$ which are random variables with values in the class of scaling laws. A random process $f = f(t, w)$ is called fractal process or selfsimilar process w.r.t. a random scaling law $\mathbb{S}$ if

$$f \stackrel{d}{=} \mathbb{S}f := \bigsqcup_i S_i \circ f^{(i)} \circ \phi_i^{-1}, \tag{4.35}$$

where $(f^{(i)})_{1 \le i \le N}$ are iid copies of f. Similarly as in (4.34) we can imbed the operator $\mathbb{S}f$ in (4.35) as a particular case of our general recursive sequence in (3.1) and obtain as application of our limit theorem the existence of random fractal processes under the conditions of Theorem 6 in [HR00].

It is of interest that in [HR00] also Brownian bridges are constructed and characterized by fractal properties, where however an additional parameter standing for the volatility has to be introduced. A related characterization of the Brownian excursion process seems to be possible and should be of interest in connection with the limiting results for the DFS-process of the conditional Galton–Watson process based on the recursive structure as in Section 4.2.

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