# A Markov chain algorithm for Eulerian orientations of planar triangular graphs

Johannes Fehrenbach and Ludger Rüschendorf
University of Freiburg

**Abstract**

On the set of Eulerian orientations of a planar Eulerian graph a natural Markov chain is defined and is shown to converge to the uniform distribution. For the class of planar triangular graphs this chain is proved to be rapidly mixing. The proof uses the path coupling technique of Bubley and Dyer (1997) and the comparison result of Randall and Tetali (1998). For the class of planar triangular graphs our result improves essentially the mixing rate result from Mihail and Winkler (1996) for general Eulerian graphs. As consequence we obtain a faster polynomial randomized approximation scheme for counting the number of Euler orientations.

*Keywords:* Markov chain algorithm, rapidly mixing, path coupling, Eulerian graphs

## 1 Introduction

An undirected, connected graph $G = (V, E)$ is called Euler graph if all vertices have even degree. A Eulerian orientation $X$ of $G$ is an orientation of the edges of $G$ such that for each vertex $v \in V$ the set of edges directed towards $v$ and the set of edges directed out of $v$ have the same cardinality, i.e., with $E^-(v) := \{e = (w, v) \in X \mid w \in V\}$ and $E^+(v) := \{e = (v, w) \in X \mid w \in V\}$ holds: $|E^+(v)| = |E^-(v)|$, for all $v \in V$.

Counting the number of Euler orientations is relevant to some problems in statistical physics. Welsh (1990) observed that the crucial partition function of the *ice-type model* is equal to the number of Eulerian orientations of some underlying Eulerian graphs. It has also been observed that the counting problem for Eulerian orientations corresponds to evaluating the Tutte polynomial, which encodes important information on the graph, at the point $(0, -2)$. It is not difficult to construct a Eulerian orientation in polynomial time. The corresponding exact counting problem of all Eulerian orientations is however $\#P$-complete as was established in Mihail and Winkler (1996). Mihail and Winkler (1996) also proved that counting of Eulerian orientations of $G$ can be reduced to counting the perfect matchings of a related graph $G'$. Thus by the randomized approximation scheme (RAS) of Jerrum and Sinclair (1989) this yields a polynomial RAS for the orientations. The mixing time for this scheme is of the class $O\left((n')^3 m'(n' \log n' + \log \varepsilon^{-1})\right)$, $\varepsilon$ the approximation error, $n' = |V'|$, $m' = |E'|$, $G' = (V', E')$. Since by the construction of $G'$ in Mihail and Winkler (1996) $n' \geq nm$, $m' \geq m^2$ one gets mixing times of considerable high polynomial order.

In this paper, which is based on the dissertation of Fehrenbach (2003), we introduce a natural direct Markov chain on the set of Eulerian orientations and prove that in the case of planar triangular graphs one gets a considerable lower mixing rate order and thus one obtains a faster randomized scheme. A related construction and mixing rate result for sampling Eulerian orientations has been given for bounded Cartesian lattices with specified boundaries in Luby, Randall, and Sinclair (2001).

Our mixing rate result uses the path coupling method of Bubley and Dyer (1997). For an ergodic Markov chain $\mathcal{M}$ with transition matrix $P = (p_{ij})$ on a finite set $\Omega$ a (Markov-)coupling is a stochastic process $(X_t, Y_t)_{t \in \mathbb{N}}$ on $\Omega \times \Omega$ such that for all $x, y, z \in \Omega$ and $t \in \mathbb{N}$

$$\begin{aligned} P(X_{t+1} = x \mid X_t = y, Y_t = z) &= p_{yx} \\ P(Y_{t+1} = x \mid X_t = y, Y_t = z) &= p_{zx} \end{aligned}$$

and $X_t = Y_t$ implies $X_{t+1} = Y_{t+1}$. Then by the coupling lemma

$$\left\| P^{X_t} - P^{Y_t} \right\| \leq P(X_t \neq Y_t), \tag{1.1}$$

where $\|\ \|$ is the variation norm.

Let $(X_t, Y_t)_{t \in \mathbb{N}}$ be a coupling and let $\delta : \Omega \times \Omega \to \mathbb{N}$ be a metric, such that for some $\beta \leq 1$ and all $t$

$$E(\delta(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t) = (x, y)) \leq \beta \delta(x, y). \tag{1.2}$$

Let $\tau(\varepsilon)$ be the mixing time of the Markov chain for the approximation error $\varepsilon$, then

$$\tau(\varepsilon) \leq \frac{\log\left(\delta(\Omega)\varepsilon^{-1}\right)}{1 - \beta} \qquad \text{if } \beta < 1. \tag{1.3}$$

If $\beta = 1$ and if for some $\alpha > 0$, $P\left(\delta(X_{t+1}, Y_{t+1}) \neq \delta(x,y) \mid (X_t, Y_t) = (x,y)\right) \geq \alpha$, for all $t$ and all $x, y \in \Omega$, then

$$\tau(\varepsilon) \leq \frac{e\delta(\Omega)^2}{\alpha} \log \varepsilon^{-1}, \tag{1.4}$$

where $\delta(\Omega) = \max\{\delta(x,y) \mid x, y \in \Omega\}$ is the diameter of $\Omega$ (see Dyer and Greenhill (1998), Aldous (1983)).

The path coupling method is a technique which simplifies the construction of a coupling on all of $\Omega \times \Omega$ that satisfies condition (1.2). It was introduced in Bubley and Dyer (1997). The following formulation is from Dyer and Greenhill (1998).

Let $S \subset \Omega \times \Omega$ be a set of transitions such that for all $x, y \in \Omega$ there exists a path $x = z_0, z_1, \ldots, z_r = y$ for $x$ to $y$ with transitions $(z_i, z_{i+1}) \in S$, $\forall i < r$. If $(x,y) \to (X', Y')$ is an $\mathcal{M}$-coupling for all $(x,y) \in S$, then an extension can be defined via the path in $S$ for any state $(X_t, Y_t) = (x,y)$. One obtains thus a sequence $Z'_0, \ldots, Z'_r$ and a coupling $(X_{t+1}, Y_{t+1})_{t \in \mathbb{N}}$ on $\Omega \times \Omega$ with $X_{t+1} = Z'_0$ and $Y_{t+1} = Z'_r$. For a function $\phi : S \to \mathbb{N}_0$ we define a metric $\delta(x,y) := \min \sum_{i=0}^{r-1} \phi(z_i, z_{i+1})$, the minimum taken over all paths from $x$ to $y$ in $S$. If for some $\beta \leq 1$

$$E(\delta(X', Y') \mid (X, Y) = (x,y)) \leq \beta\delta(x,y) \text{ for all } (x,y) \in S, \tag{1.5}$$

then

$$E(\delta(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t) = (x,y)) \leq \beta\delta(x,y) \text{ for all } (x,y) \in \Omega \times \Omega. \tag{1.6}$$

For the comparison of mixing times $\tau_1, \tau_2$ of two ergodic, reversible Markov chains $\mathcal{M}_1, \mathcal{M}_2$ with the same stationary distribution $\pi$ and transition functions $p_1, p_2$ an effective method has been developed by Randall and Tetali (1998). Let $T_i = \{(x,y) \in \Omega \times \Omega \mid p_i(x,y) > 0\}$, $i = 1, 2$, be the graphs of $\mathcal{M}_1, \mathcal{M}_2$ and let $\Gamma = \{\gamma_{x,y}; (x,y) \in T_2\}$ be a set of *canonical* paths $\gamma_{x,y}$ in $T_1$ from $x$ to $y$ for any pair $(x,y) \in T_2$. For any $(w,z) \in T_1$ let $\Gamma(w,z) = \{(x,y) \in T_2; (w,z) \in \gamma_{x,y}\}$ be the set of all canonical paths in $\Gamma$ which contain the edge $(w,z)$. Finally, we denote by

$$A(\Gamma) := \max_{(w,z) \in T_1} \frac{1}{\pi(\omega) p_1(w,z)} \sum_{(x,y) \in \Gamma(w,z)} \pi(x) p_2(x,y) |\gamma_{x,y}|, \tag{1.7}$$

where $\gamma_{x,y}$ is the length of the path $\gamma_{x,y}$, the comparison measure. Then the following holds for all $\varepsilon \in \left(0, \frac{1}{2}\right)$:

$$\tau_1(\varepsilon) \leq A(\Gamma) \frac{4\tau_2(\varepsilon)}{\log((2\varepsilon)^{-1})} \left(\log \hat{\pi}^{-1} + \log \varepsilon^{-1}\right) \tag{1.8}$$

where $\hat{\pi} = \min_{x\in\Omega} \pi(x)$.

This comparison result will be applied in section 3 to determine bounds for the mixing time of a natural Markov chain by comparison with a chain which is simpler to analyse.

# 2 A Markov chain on Eulerian orientations of planar graphs

Let $G = (V, E)$ be a planar, undirected, connected Euler graph and let $EO(G)$ denote the set of Eulerian orientations. Let $F(G)$ denote the open domains generated by the embedding of $G$ in $\mathbb{R}^2$. The inner domains are bounded while exactly one domain—the outer domain—is unbounded. For $\alpha \in F(G)$ and a Eulerian orientation $X$ we denote by $X^\alpha$ the edges in the boundary of $\alpha$ directed according to $X$. The inversion of an edge $e = (v, w) \in X$ is defined as $\overline{e} := (w, v)$. Similarly for $C \subset X$ we define the inversion $\overline{C} := \{\overline{e}; e \in C\}$. The following construction of a Markov chain $\mathcal{M}_0(G) = (X_t)_{t\in\mathbb{N}}$ on $EO(G)$ is quite natural.

**Markov chain $\boldsymbol{\mathcal{M}_0(G)}$ on $\boldsymbol{EO(G)}$:** To define the transition of $\mathcal{M}_0(G)$ from $X_t = x \in EO(G)$, we use two steps:

1) Let $\Lambda \in F(G)$ be a randomly sampled domain

2) If $\Lambda = \alpha$ and with $C := x^\alpha$ define

$$x' := \begin{cases} (x - C) \cup \overline{C} & \text{if } (x - C) \cup \overline{C} \in EO(G) \\ x & \text{else.} \end{cases} \tag{2.1}$$

Then define

$$X_{t+1} = \begin{cases} x' & \text{with probability } \frac{1}{2} \\ x & \text{else.} \end{cases} \tag{2.2}$$

The corresponding transition matrix is denoted by $P_0$. The inversion of the edges in $x^\alpha$ for $\alpha \in F(G)$ is called $\alpha$-transition.

Thus the Markov chain randomly chooses domains of the planar graph and inverts with probability $\frac{1}{2}$ the orientation of the boundary if possible.

To determine an initial state $S_G$ of the Markov chain let $G^* = (V^*, E^*)$ be the dual graph with node set $V^*$ the set of domains $F(G)$ and where two domains $\alpha$, $\alpha'$ are connected in $V^*$ if $\alpha$, $\alpha'$ have a common boundary in $G$. For more details on planar graphs see Diestel (1996). $G^*$ is a bipartite graph. Let $V_1^*$, $V_2^*$ denote the partition of $V^*$ with corresponding partition $F_1$, $F_2$ of

$F(G)$. W.l.g. let the outer domain of $G$ be in $F_2$. To define the initial state $S_G$ of the Markov chain, the edges of $G$ are oriented in such a way that for any $\alpha \in F_1$ the edges in $S_g^\alpha$ are oriented clockwise. This defines in fact a Eulerian orientation since any node $v \in V$ in the boundary of $\alpha$ is final node of two edges in the boundary of $\alpha$ which are oriented clockwise. Thus one edge points into $v$ while the other edge points out of $v$. For each edge towards $v$ thus there exists an edge out of $v$ and thus $|E^+(v)| = |E^-(v)|$ for all $v \in V$. Since the domains with $v$ in their boundary belong cyclically to $F_1$ or $F_2$, the edges of $v$ in $S_G$ are cyclically oriented out of $v$ or towards $v$ (see Figure 1). We use this Euler orientation $S_G$ as initial state of the chain. The analysis of the mixing time however will be independent of the initial state.

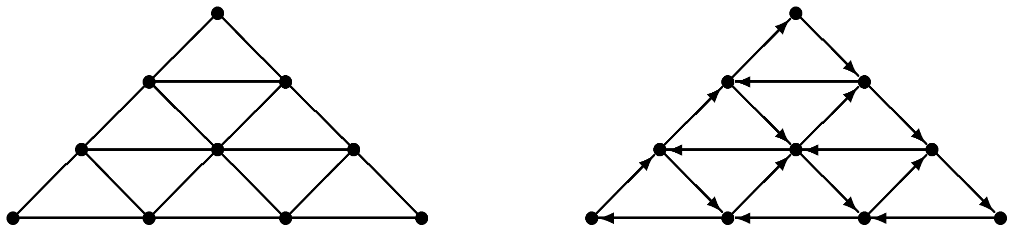


**Figure 1** The domains with clockwise orientation define $F_1$. $F_2$ consists of the domains which are oriented counterclockwise.

The following property of the initial state $S_G$ will be used in the following.

**Lemma 2.1** *If $C \subset S_G$ is a simple clockwise oriented circle in $G$, then $C^*$ is a minimal cut in $G^*$, which decomposes $V^*$ in two components $Z_1^*$ and $Z_2^*$. If the outer domain of $G$ belongs to $Z_2^*$, then all nodes in $Z_1^*$ which are final nodes of an edge in $C^*$ are in $V_1^*$.*

**Proof:** The property that $C^*$ is a minimal cut is stated in Diestel (1996, Proposition 3.10). For $v \in V$ a node in the circle $C$ let $k$ be the number of edges $e$ on $v$ with dual edge $e^* \in Z_1^*$. $k$ is even since by construction of $S_G$ the edges on $v$ cyclically are out of $v$ or towards $v$. Therefore, for the domains $\alpha$, $\beta$ on both edges on $v$ in $C$ with $\alpha^*$, $\beta^* \in Z_1^*$ holds $\alpha^*$, $\beta^* \in V_1^*$ (see Figure 2). □

Similarly, if $C$ as in Lemma 2.1 is counterclockwise oriented then the endnodes of edges in $Z_1^*$ are in $V_2^*$.

**Theorem 2.2** *Let $G = (V, E)$ be a planar undirected, connected Euler graph. Then the Markov chain $\mathcal{M}_0(G)$ defined in (2.2) is ergodic. The stationary distribution $\pi$ of $\mathcal{M}_0(G)$ is the uniform distribution on $EO(G)$.*

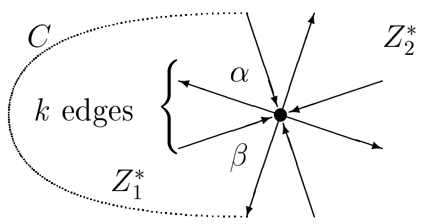


**Figure 2** orientation $S_G$. The edges on $v$ are cyclically *in* or *out* $v$. Thus $k$ is even.

**Proof:** By the construction of $\mathcal{M}_0(G)$ the transition matrix is symmetric $p_0(x, y) = p_0(y, x)$ for $x, y \in EO(G)$. Thus $\mathcal{M}_0(G)$ is reversible with respect to the uniform distribution on $E(G)$. By (2.2) it is also aperiodic. The main part of the proof is to establish irreducibility. We will prove by induction on $|F(G)|$ that there is a path of any $x \in EO(G)$ to the initial state $S_G$. This implies by symmetry irreducibility of $\mathcal{M}_0(G)$.

1) $|\boldsymbol{F(G)}| = \mathbf{2}$.
   Then $G$ has two Euler orientations $S_G$ and $\bar{S}_G$. A transition from $\bar{S}_G$ to $S_G$ has by (2.1) positive probability.

2) **Induction step:** $|\boldsymbol{F(G)}| = \boldsymbol{\ell} > \mathbf{2}$.
   Let $x \in EO(G)$. Let in the first case $x = \bar{S}_G$. Choosing in the first construction step of $\mathcal{M}_0(G)$ iteratively points from the set $F_1$ one gets a finite path from $x$ to $S_G$. The result is independent of the sequence since the inversion of the orientation of edges on a domain in $F_1$ does not influence edges on other domains in $F_1$. No edge is in the boundary of two domains in $F_1$ (see also Figure 1).

   If $x \neq \bar{S}_G$ then we consider $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ with $\widetilde{E} = \{e \in E : e_{S_G} \neq e_x\}$ and $\widetilde{V} = \{v \in V : v \text{ is final node of an edge in } \widetilde{E}\}$; $e_{S_G}$, $e_x$ denote the orientations of $e$ by $S_G$ resp. $x$. We first assume that $\widetilde{G}$ is connected and define $\widetilde{S}_G = \{e_{S_G} = (v, w) \in S_G : \{v, w\} \in \widetilde{E}\}$, $\widetilde{X} = \{e_x = (v, w) \in x : \{v, w\} \in \widetilde{E}\}$ the corresponding orientations of $\widetilde{G}$. Then $\widetilde{G}$ is a Euler graph and $\widetilde{S}_G$, $\widetilde{x} \in EO(\widetilde{G})$. Since $x \neq \widetilde{S}_G$ we have $|F(G)| > |F(\widetilde{G})|$ and, therefore, from the assumption of the induction there is a path $(C_i)_{0 \leq i \leq k}$ from $\widetilde{X} = C_0$ to $\widetilde{S}_G = C_k$ w.r.t. $\mathcal{M}_0(\widetilde{G})$. As consequence one gets also a path from $x$ to $S_G$ in $\mathcal{M}_0(G)$.

   Let the transition from $C_0$ to $C_1$ be an $\alpha$-transition for some $\alpha \in F(\widetilde{G})$. If $\alpha \in F(G)$ then this transition of $\mathcal{M}_0(\widetilde{G})$ is also a transition of $\mathcal{M}_0(G)$. If $\alpha \notin F(G)$ then the edges of $\widetilde{x}^\alpha$ are a simple directed circle in $\widetilde{x}$, which corresponds to a minimal cut in $G^*$. We denote the component which does not contain the outer domain of $G^*$ by $T^*$. The edges dual to those in $T^*$

are in $X$ and in $S_G$ identically oriented. Therefore, all nodes in $T^*$ which are final nodes of edges in $\widetilde{x}^\alpha$ are by Lemma 2.1 in the same partition of $G^*$ say in $V_1^*$. Since in $S_G$ the edges in $S_G^\beta$ are directed circles for any $\beta \in F(G)$, also the edges in $x^\gamma$ are directed circles in $x$ for nodes $\gamma^*$ in $T^* \cap V_2^*$. Thus starting from $x$ for all these $\gamma$ the $\gamma$-transitions can be made and as a result the edges at all $\alpha \in F(G)$ with $\alpha^* \in T^* \cap V_2^*$ form circles which get an inverse orientation. Finally the orientation of all edges $e \in E$ with $e^* \in T^*$ has two times been inverted, while the edges in $\widetilde{x}^\alpha$ have one time been inverted. Thus the transition of $C_0$ to $C_1$ in $\mathcal{M}_0(\widetilde{G})$ has been transferred to a sequence of transitions in $\mathcal{M}_0(G)$. Iterating this procedure for all the other transitions we finally obtain a path from $x$ to $S_G$.

If $\widetilde{G}$ has several components, then this procedure can be applied to all of these producing finally a path in $\mathcal{M}_0(G)$ and establishes irreducibility. Thus the Markov chain $\mathcal{M}_0(G)$ is ergodic and the uniform distribution of $EO(G)$ is the unique stationary distribution of $\mathcal{M}_0(G)$. $\square$

# 3 Rapid mixing of $\mathcal{M}_0(G)$ for triangular graphs

The aim of this section is to prove the rapid mixing property of the Markov chain $\mathcal{M}_0(G)$ of section 2 for the case of planar, triangular graphs. A graph $G$ is called a triangular graph if the boundary of any inner domain contains exactly three nodes (cp. the example in Figure 1). Let $\tau_0(\varepsilon)$ denote the mixing time of $\mathcal{M}_0(G)$ for approximation error $\varepsilon > 0$.

**Theorem 3.1** *Let $G = (V, E)$ be a Eulerian, planar triangular graph. Then the mixing time $\tau_0(\varepsilon)$ of the Markov chain $\mathcal{M}_0(G)$ is polynomially bounded and*

$$\tau_0(\varepsilon) \leq 3072n^3(\log \varepsilon^{-1})(3n + \log \varepsilon^{-1}). \tag{3.1}$$

To establish the bound in (3.1) we introduce a modified Markov chain for planar Eulerian triangle graphs with extended transitions which is easier to analyse (see also the proof of Theorem 3.5)

**Lemma 3.2** *Let $G = (V, E)$ be a planar, Eulerian triangular graph and $x \in EO(G)$. If $\alpha \in F(G)$ and $x^\alpha$ is not a directed circle, then $\alpha$ has at most one neighbour domain $\beta \in F(G)$ such that $x^\alpha \oplus x^\beta$ is a directed circle.*

**Proof:** If $x^\alpha$ is not a directed circle, then there is an edge $e \in x^\alpha$ such that $(x^\alpha - \{e\}) \cup \{\bar{e}\}$ is a directed circle. This edge $e$ is not contained in any simple directed circle in $x$ together with any of the other two edges form $x^\alpha$.

Therefore, only the neighbour domain $\beta$, separated from $\alpha$ by the edge $e$, can possibly satisfy that $x^\alpha \oplus x^\beta$ is a circle, see Figure 3 for the corresponding part of $G$.

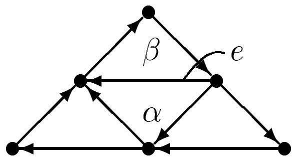


**Figure 3** $\square$

If for a Eulerian orientation $x$ it is not possible to invert the edges of a domain $\alpha$ then there can be one (but at most one) neighbour domain $\beta$ such that $x^\alpha \oplus x^\beta$ is a directed circle. The following modification $\mathcal{M}_0'(G)$ of $\mathcal{M}_0(G)$ allows the Markov chain to invert circles of this kind.

**Definition 3.3** *Let $G = (V, E)$ be a planar Eulerian triangular graph. The Markov chain $\mathcal{M}_0'(G) = (X_t)_{t \in \mathbb{N}}$ on $EO(G)$ is defined by the following transitions of $\mathcal{M}_0'(G)$ from $X_t = x \in EO(G)$:*

*1) Let $\Lambda \in F(G)$ be a randomly sampled domain*

*2) For $\Lambda = \alpha$ and if $C := x^\alpha$ is a directed circle then define $x' := (x - C) \cup \bar{C}$. If $C$ is not a directed circle and if $\beta \in F(G)$ is a neighbour domain of $\alpha$ such that $C' = x^\alpha \oplus x^\beta$ is a directed circle, then set*

$$X' := \begin{cases} (x - C') \cup \bar{C}' & \text{with probability } \frac{1}{6} \\ x & \text{else} \end{cases} \tag{3.2}$$

*If no neighbour $\beta$ of this type exists then set $X' := x$.*

*3) Define $X_{t+1} := \begin{cases} X' & \text{with probability } \frac{1}{2} \\ x & \text{else.} \end{cases}$*

*Let $P_0' = (p_0'(\cdot,\cdot))$ denote the transition matrix of $\mathcal{M}'(G)$. If $X' \neq x$ in the construction above then we speak of transitions of type 1 and type 2 corresponding to the use of simple circles or combined circles in step 2).*

**Theorem 3.4** *Let $G = (V, E)$ be a Eulerian, planar, triangular graph. Then the Markov chain $\mathcal{M}_0'(G)$ is ergodic and the stationary distribution $\pi$ of $\mathcal{M}_0'(G)$ is the uniform distribution on $EO(G)$.*

**Proof:** Obviously $\mathcal{M}_0'(G)$ is irreducible and aperiodic since $\mathcal{M}_0(G)$ has these properties. To prove reversibility, let $x, y \in EO(G)$, $x \neq y$ with $p_o'(x, y) > 0$. Then $p_0'(x, y) = p_0'(y, x)$ by the proof of Theorem 2.2 for those pairs $x, y$ with

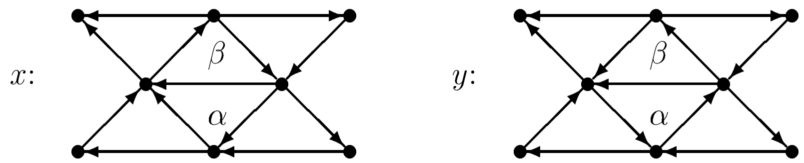


**Figure 4**

$p_0(x,y) > 0$. If $p_0(x,y) = 0$, then for some neighbour domains $\alpha, \beta$ holds, that $x^\alpha$ is not a directed circle and $y = (x - C) \cup \bar{C}$ with $C = x^\alpha \oplus x^\beta$. We have that $p_0'(x,y) = \frac{1}{|F(G)|-1} \cdot \frac{1}{12}$. Thus we have a situation as in Figure 4. One sees directly that $x^\beta$ and $y^\alpha$ are directed circles while $x^\alpha$ and $y^\beta$ are not directed circles. If in one step transition of the Markov chain $\mathcal{M}_0'(G)$ one draws in step 1) $\beta$ then in step 2) one obtains $C' = y^\beta \oplus y^\alpha$ and thus $Y' = x$ with probability $\frac{1}{6}$. Together we obtain $p_0'(y,x) = \frac{1}{|F(G)|-1} \cdot \frac{1}{12} = p_0'(x,y)$. □

The mixing time $\tau_0'$ of $\mathcal{M}_0'(G)$ can be efficiently bounded for this Markov chain in the case of triangular graphs.

**Theorem 3.5** *Let $G = (V,E)$ be a Eulerian, planar triangular graph. Then the mixing time $\tau_0'$ of $\mathcal{M}_0'(G)$ is polynomially bounded in $n = |V|$:*

$$\tau_0'(\varepsilon) \leq \lceil 32en^3 \rceil \lceil \log \varepsilon^{-1} \rceil \text{ for all } \varepsilon \in (0,1). \tag{3.3}$$

**Proof:** We apply the path coupling method with

$$S := \{(x,y) \in \Omega \times \Omega : \exists \text{ inner domain } \alpha \text{ such that } y = (x - x^\alpha) \cup \overline{x^\alpha}\},$$

where $\Omega = EO(G)$. The irreducibility of $\mathcal{M}_0(G)$ implies that for any $(x,y) \in \Omega$ there exists a path $(z_i)_{0 \leq i \leq r}$ with $(z_i, z_{i+1}) \in S$ for all $i < r$ and $z_0 = x$, $z_r = y$. Define $\phi(x,y) = 1$ for all $(x,y) \in S$. Then $\delta(x,y)$ is the length of the shortest path from $x$ to $y$ in $S$. We have to determine the coupling on $S$.

For $(x_1, y_1) \in S$ with $y_1 = (x_1 - x_1^\alpha) \cup \overline{x_1^\alpha}$ for some inner domain $\alpha \in F(G)$ let $\beta, \gamma, \lambda$ denote the neighbour domains of $\alpha$. To construct the coupling $(X_2, Y_2)$ starting from $(X_1, Y_1) = (x_1, y_1)$ choose in step 1 for both pairs the same inner domain $\kappa \in F(G)$. After construction of $x_2$ the coupling transition step of $y_1$ is given as follows:

**1. case: $\boldsymbol{\kappa \notin \{\alpha, \beta, \gamma, \lambda\}}$.**
In this case holds $x_1^\kappa = y_1^\kappa$ as well as $x_1^\kappa \oplus x_1^{\kappa'} = y_1^\kappa \oplus y_1^{\kappa'}$ for all neighbours $\kappa'$ of $x$. Therefore, in step 2 for the transition of $x_1$ w.r.t. $\mathcal{M}_0'(G)$ we have the same situation as for $y_1$ and thus may choose a transition from $y_1$ to $y_2$ parallel to that of $x_1$ to $x_2$. $y_2$ differs from $x_2$ only in the orientation of the edges in $x_1^\alpha$. Thus we obtain

$$(x_2, y_2) \in S \text{ and } \delta(x_2, y_2) = \delta(x_1, y_1) = 1. \tag{3.4}$$

**2. case: $\boldsymbol{\kappa = \alpha}$.**
Then by step 2) we obtain $X_1' = y_1$ and $Y_1' = x_1$. Defining

$$(X_2, Y_2) = \begin{cases} (x_1, x_1) & \text{with probability } \frac{1}{2} \\ (y_1, y_1) & \text{with probability } \frac{1}{2} \end{cases}$$

we obtain a.s.

$$\delta(X_2, Y_2) = \delta(x_1, y_1) - 1 = 0. \tag{3.5}$$

There are two extreme situations to consider.

**3. case: $\boldsymbol{\kappa \in \{\beta, \gamma, \lambda\}}$.**
Without loss of generality let $\kappa = \beta$. The first possible extreme situation is this: $x_1^\beta$ is no directed circle, but for some neighbour $\beta'$ of $\beta$, $\beta' \neq \alpha$, $C = x_1^\beta \oplus x_1^{\beta'}$ is an oriented circle. Also assume that $y_1^\beta$ is no directed circle but $C'' = y_1^\beta \oplus y_1^{\beta''}$ for some neighbour $\beta''$ of $\beta$, $\beta'' \notin \{\alpha, \beta'\}$ is a directed circle. Then define

$$(X_2, Y_2) = \begin{cases} (x_1', y_1') & \text{with probability } \frac{1}{12}, \\ (x_1, y_1) & \text{with probability } \frac{11}{12}, \end{cases}$$

where $x_1' = (X_1 - C') \cup \bar{C}'$, $y_1' = (Y_1 - C'') \cup \bar{C}''$. Then for this worst case we obtain

$$\delta(X_2, Y_2) = \begin{cases} \delta(x_1, y_1) + 4 & \text{with probability } \frac{1}{12}, \\ \delta(x_1, y_1) & \text{with probability } \frac{11}{12}. \end{cases} \tag{3.6}$$

Since $\delta(x_1, y_1) = 1$ we obtain from (3.4)–(3.6):

$$\begin{aligned} & E(\delta(X_2, Y_2) \mid (X_1, Y_1) = (x_1, y_1)) - \delta(x_1, y_1) \\ \leq \; & \frac{1}{|F(G)| - 1} \left( (|F(G)| - 5) \cdot 0 + 1 \cdot (-1) + 3 \cdot \frac{1}{12} \cdot 4 \right) \\ = \; & \frac{1}{|F(G)| - 1} (-1 + 1) = 0 \end{aligned} \tag{3.7}$$

and thus $E(\delta(X_2, Y_2) | (X_1, Y_1) = (x_1, y_1)) \leq \delta(x_1, y_1)$. Note that this situation allows to choose the transition of type 2 in Definition 3.3 with probability at most $\frac{1}{12}$.

The second extreme situation to consider is this: $x_1^\beta$ is no directed circle but $C = x_1^\beta \oplus x_1^\alpha$ is a directed circle. Simultaneously $y_1^\beta$ is a directed circle. Then the transitions of type 2 in $x_1$ and of type 1 in $y_1$ yield the same result $x_1^{'} = y_1^{'}$;

the first transition occurs with probability $\frac{1}{12}$, the second with probability $\frac{1}{2}$. As a result we obtain in this second situation

$$\delta(X_2, Y_2) = \begin{cases} \delta(x_1, y_1) - 1 & \text{with probability } \frac{1}{12} \\ \delta(x_1, y_1) + 1 & \text{with probability } \frac{1}{2} - \frac{1}{12} \\ \delta(x_1, y_1) & \text{with probability } \frac{1}{2} \end{cases}$$

and, therefore,

$$\begin{aligned} & E(\delta(X_2, Y_2) \mid (X_1, Y_1) = (x_1, y_1)) - \delta(x_1, y_1) \\ = \; & \frac{1}{|F(G)| - 1} \Bigg( (|F(G)| - 5) \cdot 0 + 1 \cdot (-1) \\ & \qquad + 3 \cdot \left( \frac{1}{12}(-1) + \left( \frac{1}{2} - \frac{1}{12} \right) \cdot (+1) + \frac{1}{2} \cdot 0 \right) \Bigg) \\ = \; & \frac{1}{|F(G)| - 1}(-1 + 1) = 0. \end{aligned}$$

In this second situation the transition of type 2 has to be chosen with probability at least $\frac{1}{12}$. So both of these situations lead to the transition probability $\frac{1}{12}$. This second situation is also the reason that the original Markov chain has to be modified by allowing transitions of type 2. For the Markov chain $\mathcal{M}_0(G)$ the contraction condition is not fulfilled in this situation.

As a result we obtain that the assumption of the path coupling theorem in (1.5) is fulfilled with $\beta = 1$ and thus we obtain a coupling on $\Omega \times \Omega$.

In the proof of Theorem 2.2 it was shown that any $x$ in $\Omega$ is connected with $S_G$ by a path of length $\leq |F(G)| - 1$. Thus we obtain $\delta(\Omega) = \max_{x,y \in \Omega} \delta(x, y) \leq 2(|F(G)| - 1)$.

From case 2 we obtain for $(x_1, y_1) \in S$

$$\alpha := P(\delta(X_2, Y_2) \neq \delta(x_1, y_1) \mid (X_1, Y_1) = (x_1, y_1)) \geq (|F(G)| - 1)^{-1}. \tag{3.8}$$

This estimate extends by an induction argument to any pair $(x_1, y_1) \in \Omega^2$. Thus (1.4) implies

$$\tau_0'(\varepsilon) \leq [4e(|F(G)| - 1)^3] \lceil \log \varepsilon^{-1} \rceil. \tag{3.9}$$

If $a$ denotes the number of edges in the outer domain of $G$, then using that $G$ is triangular we obtain with $n = |V|$, $m = |E|$, $\ell = |F(G)|$

$$3(\ell - 1) + a = 2m. \tag{3.10}$$

The Euler polyeder formula for planar graphs gives

$$n - m + \ell = 2. \tag{3.11}$$

Thus we obtain $n + \ell - \frac{3}{2}(\ell - 1) - \frac{a}{2} = 2$. This implies

$$\ell < 2n \tag{3.12}$$

and again by Euler's polyeder formula

$$m < 3n. \tag{3.13}$$

From (3.9) we conclude

$$\tau_0'(\varepsilon) \leq [4e\ell^3]\lceil \log \varepsilon^{-1} \rceil \leq [32en^3]\lceil \log \varepsilon^{-1} \rceil. \qquad \square$$

From the polynomial bound for the mixing time $\tau_0'$ of $\mathcal{M}_0'(G)$ we next derive the mixing time bound for $\tau_0$ by means of the comparison method of Randall and Tetali (1998).

**Proof of Theorem 3.1:** In order to compare the mixing time $\tau_0(\varepsilon)$ with $\tau_0'(\varepsilon)$ from $\mathcal{M}_0'(G)$ we have to construct a set $\Gamma$ of canonical paths $\gamma_{x,y}$ in $\mathcal{M}_0(G)$ for each transition $(x, y)$ in $\mathcal{M}_0'(G)$, $\Gamma = \{\gamma_{x,y}; (x, y) \text{ a transition in } \mathcal{M}_0'(G)\}$.

**1. case:** $(x, y)$ is a type 1 transition.
Then for some $\alpha \in F(G)$, $y = (x - C) \cup \bar{C}$ with $C = x^\alpha$. This is also a transition in $\mathcal{M}_0(G)$ and thus we define $\gamma_{x,y} = (x, y)$.

**2. case:** $(x, y)$ is a type 2 transition.
Then $y = (x - C) \cup \bar{C}$ with $C = x^\alpha \oplus x^\beta$ for some inner neighbour domains $\alpha, \beta \in F(G)$. $C$ is a directed circle while $x^\alpha$ is not a directed circle. Therefore, $x^\beta$ is also a directed circle. Therefore, the type 2 transition $(x, y)$ can be replaced by two type 1 transitions, first in the domain $\beta$ and then in the domain $\alpha$. These two transitions define the path $\gamma_{x,y}$. Together we obtain the set of canonical paths $\Gamma$.

We have to bound the comparison measure $A(\Gamma)$. Let $(w, z) \in T_1$ be a transition of $\mathcal{M}_0(G)$, i.e., $z = (w - C) \cup \bar{C}$, $C = w^\alpha$ for some inner domain $\alpha \in F(G)$. If $\beta$ is a neighbour domain to $\alpha$, then $w^\beta$ and $z^\beta$ can not be both directed circles since one common edge of these transitions is inverted. Therefore, the transition $(w, z)$ is contained for each neighbour $\beta$ of $\alpha$ in at most one path $\gamma_{x,y} \in \Gamma$ and the transition $(x, y)$ in $\mathcal{M}_0'(G)$ is then of type 2. Thus we obtain $|\Gamma(w, z)| \leq 4$ and, therefore,

$$\begin{aligned} A(\Gamma) &\leq \max_{(w,z)\in T_1} \sum_{(x,y)\in\Gamma(w,z)} \frac{p_0'(x,y)}{p_0(w,z)} |\gamma_{x,y}| \\ &\leq \max_{(w,z)\in T_1} 2|\Gamma(w,z)| \leq 8. \end{aligned}$$

Since the stationary distribution $\pi$ of $\mathcal{M}_0(G)$ is the uniform distribution on $EO(G)$ we obtain $\widehat{\pi}^{-1} \leq 2^m$ with $m = |E|$. Also by (3.13) $m < 3n$ and thus $\widehat{\pi}^{-1} \leq 2^{3n}$. This implies by the comparison result in (1.8):

$$\begin{aligned}
\tau_0(\varepsilon) &\leq \frac{4\log((\widehat{\pi}\varepsilon)^{-1})}{\log((2\varepsilon)^{-1})} 8\tau_0'(\varepsilon) \\
&\leq 32\frac{\log(2^{3n}\varepsilon^{-1})}{\log((2\varepsilon)^{-1})}[32en^3]\lceil\log\varepsilon^{-1}\rceil \\
&\leq 3072n^3\lceil\log\varepsilon^{-1}\rceil(3n+\log\varepsilon^{-1})
\end{aligned}$$

for all $\varepsilon \in (0, \frac{1}{2})$. $\square$

**Remarks 3.6** *a) The mixing rates $\tau_0'$, $\tau_0$ are given in Theorems 3.1 and 3.5 only in terms of $n = |V|$ the number of nodes. $\tau_0'$ is of the order $n^3$ while $\tau_0$ is of the order $n^4$. The effective mixing time of the chains could be much lower since the estimation by the path coupling method is uniform and thus too much concentrated on the worst case. In comparison our rates are much better than the rates obtained by the general method of Mihail and Winkler (1996) which are of an order $\geq n^{10}$. In consequence the standard associated randomized approximation scheme (RAS) of our specialized Markov chain yields improved estimates for the number of Euler orientations.*

*b) It is an open problem whether the chain $\mathcal{M}_0(G)$ in section 2 is rapidly mixing for all (or a large class of) planar graphs. The rapid mixing property of this chain has been established also for the Euler orientations in Cartesian grids in the dissertation of Fehrenbach (2003).*

# References

Aldous, D. (1983). *Random walks on finite groups and rapidly mixing Markov chains*, Volume 986 of *Lecture Notes in Mathematics*, pp. 243–297. Springer.

Bubley, R. and M. Dyer (1997). Path coupling: a technique for proving rapid mixing in Markov chains. In *Proceedings of the 38th Annual IEEE Symposium on foundations of Computer Science (FOCS)*, pp. 223–231. IEEE Computer Society Press.

Diestel, R. (1996). *Graphentheorie*. Springer.

Dyer, M. and C. Greenhill (1998). A more rapidly mixing Markov chain for graph colorings. *Random Structure and Algorithms 13*, 285–317.

Fehrenbach, J. (2003). *Design und Analyse stochastischer Algorithmen auf kombinatorischen Strukturen.* PhD thesis, University of Freiburg.

Jerrum, M. and A. Sinclair (1989). Approximating the permanent. *SIAM Journal on Computing 18*, 1149–1178.

Luby, M., D. Randall, and A. Sinclair (2001). Markov chain algorithms for planar lattice structures. *SIAM Journal on Computing 31*, 167–192.

Mihail, M. and P. Winkler (1996). On the number of Eulerian orientations of a graph. *Algorithmica 16*, 402–414.

Randall, D. and P. Tetali (1998). *Analyzing Glauber dynamics by comparison of Markov chains*, Volume 1380 of *Lecture Notes in Computer Science*, pp. 292–304. Springer.

Welsh, D. (1990). The computational complexity of some classical problems from statistical physics. In G. Grimmett and J. Hammersley (Eds.), *Disorder in Physical Systems*, pp. 307–321. Oxford University Press.

Welsh, D. (1998). *Complexity: Knots, Colourings, and Counting*, Volume 186 of *LMS Lecture Note Series*. Cambridge University Press.

Johannes Fehrenbach
Department of Mathematics
University of Freiburg
Eckerstr. 1
79104 Freiburg
Germany

Ludger Rüschendorf
Department of Mathematics
University of Freiburg
Eckerstr. 1
79104 Freiburg
Germany

ruschen@stochastik.uni-freiburg.de