On approximative solutions of multistopping problems

Andreas Faller and Ludger Rüschendorf
University of Freiburg

Abstract

We consider multistopping problems for discrete time sequences as well as for continuous time Poisson processes which serve as limiting models for the discrete time problem. The choice of m-stopping times is allowed and the aim is to maximize the expected value of the best of the m stops. The optimal m-stopping curves of the Poisson process are determined by differential equations of first order and allow the construction of approximative solutions of the discrete time m-stopping problem.

Keywords: optimal stopping, best choice problem, extreme values, Poisson process

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1 Introduction

In this paper we consider multistopping problems for discrete time sequences $X_1, \dots, X_n$. In comparison to the usual stopping problem there are m choices of stopping times $1 \leq T_1 < \cdots < T_m \leq n$ allowed. The aim is to determine these stopping times in such a way that

$$E[\max_{1 \leq i \leq m} X_{T_i}] = E[X_{T_1} \vee \cdots \vee X_{T_m}] = \sup . \tag{1.1}$$

Thus the gain of a stopping sequence $(T_i)_{i \leq m}$ is the expected maximal value of the m choices X_{T_i}.

In order to solve approximatively problem (1.1) we introduce at first the m-stopping problem for continuous time Poisson processes which by point process convergence serve as approximative limit model for the discrete time model. This limiting Poisson process has in a typical case infinite intensity along a lower boundary of its support. The solution of the m-stopping problem in the Poisson process case can be described by an increasing sequence of stopping curves with their related threshold stopping times. These curves solve usual one-stopping problems for transformed Poisson processes and are characterized by differential equations of first order. We then establish convergence of the discrete time m-stopping problem to the stopping problem in the limiting case given some regularity conditions. For some classes of intensity functions the limiting problem can be solved in explicit form. We discuss in extensive form the m-stopping of iid sequences with discount and observation costs.

Multistopping problems were introduced in Haggstrom (1967) who derived some structural results corresponding roughly to Theorem 2.3. Compare also some extensions in Nikolaev (1999). The two stopping problem has been considered in the case of Poissonian streams in Saario and Sakaguchi (1992). In this paper differential equations were derived corresponding to the one-stopping problems as in Karlin (1962), Siegmund (1967), and Sakaguchi (1976). In [KR][1] (2002) a particular class of 2-stopping problems was treated based on the approximative approach in [KR] (2000a). In this paper we extend this approach to a general framework. Based on the recent paper in [FR] (2009) we obtain as a result in particular a fairly complete treatment of optimal multistopping problems for the stopping of iid sequences with discount and observation costs. For several details and proofs in this paper we refer to the dissertation of Faller (2009) on which this paper is based.

2 m-stopping problems for finite sequences

Given a discrete time sequence $(X_i, \mathcal{F}_i)_{1\le i\le n}$ in a probability space $(\Omega, \mathcal{A}, P)$ with filtration $\mathcal{F} = (\mathcal{F}_i)_{0\le i\le n}$ the m-stopping problem $(1 \le m \le n)$ is to find stopping times $1 \le T_1 < T_2 < \cdots < T_m \le n$ w.r.t. the filtration $(\mathcal{F}_i)_{1\le i\le n}$ such that

$$E[\max_{1\le i\le m} X_{T_i}] = E[X_{T_1} \vee \cdots \vee X_{T_m}] = \sup . \tag{2.1}$$

In case $m = 1$ (2.1) is identical to the usual (one-)stopping problem. A well known recursive solution of this problem (see Chow et al. (1971, Theorem 3.2)) is given by threshold curves $W_i = W_F(X_{i+1}, \dots, X_n)$ of the optimal stopping time defined by

$$\begin{aligned} W_n &:= -\infty, \\ W_i &:= E[X_{i+1} \vee W_{i+1} \mid \mathcal{F}_i] \quad \text{for } i = n-1, \dots, 0. \end{aligned} \tag{2.2}$$

We need a version of this classical result where the beginning time point is given by a stopping time (for details see [F] (2009)).

[1]Kühne and Rüschendorf is abbreviated within this paper with [KR], Faller with [F], and Faller and Rüschendorf with [FR].

Theorem 2.1 (Recursive solution of one-stopping problems) *a) For any time point $0 \le k \le n-1$ the $\mathcal{F}$-stopping time*

$$T(k) := \min\{k < i \le n : X_i > W_i\}$$

is optimal in the sense that for any $\mathcal{F}$-stopping time $T > k$ we have

$$E[X_{T(k)} \mid \mathcal{F}_k] = W_k \ge E[X_T \mid \mathcal{F}_k] \quad P\text{-a.s.} \tag{2.3}$$

b) For any $\mathcal{F}$-stopping time S the $\mathcal{F}$-stopping time

$$T(S) = \min\{S < i \le n : X_i > W_i\}$$

is optimal in the sense that for any $\mathcal{F}$-stopping time T with $S < T$ on $\{S < n\}$ and $S = T$ on $\{S = n\}$ we have

$$E[X_{T(S)} \mid \mathcal{F}_S] = W_S \ge E[X_T \mid \mathcal{F}_S] \quad P\text{-a.s.} \tag{2.4}$$

Remark 2.2 *For m stopping problems also the following variant of Theorem 2.1 will be needed:*

Let $Y_1, \dots, Y_n : (\Omega, \mathcal{A}, P) \to E$ random variables taking values in a measurable space E and $\mathcal{F} := (\mathcal{F}_i)_{0 \le i \le n}$ a filtration in $\mathcal{A}$ such that $\sigma(Y_i) \subset \mathcal{F}_i$ for all $1 \le i \le n$. Let S be an $\mathcal{F}$-stopping time, let $Z : (\Omega, \mathcal{A}, P) \to \overline{\mathbb{R}}$ be $\mathcal{F}_S$-measurable and $h : E \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be measurable and $Eh(Y_i, Z)^+ < \infty$. Also define recursively for $z \in \overline{\mathbb{R}}$

$$\begin{aligned} W_n(z) &:= h(Y_n, z), \\ W_i(z) &:= E[h(Y_{i+1}, z) \vee W_{i+1}(z) \mid \mathcal{F}_i] \quad \text{for } i = n-1, \dots, 0. \end{aligned}$$

Then the $\mathcal{F}$ stopping time

$$T(S, Z) := \min\{S < i \le n : h(Y_i, Z) > W_i(Z_i)\}$$

where $Z_i := Z\chi_{\{S \le i\}}$ is optimal in the sense that for any further $\mathcal{F}$-stopping time T with $S < T$ on $\{S < n\}$ and $S = T$ on $\{S = n\}$ we have

$$E\big[h(Y_{T(S,Z)}, Z) | \mathcal{F}_S\big] = W_S(Z_S) \ge E\big[h(Y_T, Z) | \mathcal{F}_S\big] \quad P\text{-a.s.}$$

Similar as in the one-stopping problems the idea of solving (2.1) is simple. The ℓ-th stopping time T_ℓ should be i if the $(m-\ell)$-stopping value past i with guarantee value X_i is in expectation larger than the $(m-\ell+1)$-stopping value past i and with guarantee value reached before time i. This idea leads to the following construction. Define $W_i^0(x) := x$ for $x \in \overline{\mathbb{R}}$ and inductively for $1 \le m \le n$, $x \in \mathbb{R}$ define thresholds $W_k^m(x)$ by

$$\begin{aligned} W_{n-m+1}^m(x) &:= x, \\ W_i^m(x) &:= E[W_{i+1}^{m-1}(X_{i+1}) \vee W_{i+1}^m(x) \mid \mathcal{F}_i] \quad \text{for } i = n-m, \dots, 0. \end{aligned} \tag{2.5}$$

The related threshold stopping times are defined recursively for $k \le n - m$ by

$$\begin{aligned} T_1^m(k,x) &:= \min\{k < i \le n-m+1 : W_i^{m-1}(X_i) > W_i^m(x)\}, \\ T_\ell^m(k,x) &:= \min\{T_{\ell-1}^m(k,x) < i \le n-m+\ell : W_i^{m-l}(X_i) > W_i^{m-l+1}(x \vee M_{\ell-1,i})\} \end{aligned} \tag{2.6}$$

for $2 \le \ell \le m$ and $M_{j,i} := X_{T_j^m(k,x)} \chi_{\{T_j^m(k,x) \le i\}}$.

(2.6) corresponds to a sequence of m one-stopping problems for (more complicated) transformed sequences of random variables. The following result extends the classical recursive characterization of optimal stopping times for one-stopping problems in Theorem 2.1 to the case $m \ge 1$. Related structural results can be found in the papers of Haggstrom (1967), Saario and Sakaguchi (1992), Nikolaev (1999), and [KR] (2002).

Theorem 2.3 (Recursive characterization of m-stopping problems) *The $\mathcal{F}$-stopping times $(T_\ell^m(k,x))_{1 \le \ell \le m}$ are optimal in the sense that for all $\mathcal{F}$-stopping times $(T_\ell)_{1 \le \ell \le m}$ with $k < T_1 < \cdots < T_m \le n$ we have*

$$\begin{aligned} E[x \vee X_{T_1^m(k,x)} \vee \ldots \vee X_{T_m^m(k,x)} \mid \mathcal{F}_k] &= E[W^{m-1}_{T_1^m(k,x)}(x \vee X_{T_1^m(k,x)}) \mid \mathcal{F}_k] = W_k^m(x) \\ &\ge E[x \vee X_{T_1} \vee \ldots \vee X_{T_m} \mid \mathcal{F}_k] \quad P\text{-a.s.} \end{aligned}$$

The proof of Theorem 2.1 follows by induction in m similarly as in the case $m = 1$. For details see [F] (2009, Satz 2.1) or [KR] (2002, Proposition 2.1). In general the recursive characterization of optimal m-stopping times and values is difficult to evaluate. Our aim is to prove that one can construct optimal stopping times and values approximatively by considering related limiting stopping problems for Poisson processes in continuous time.

3 m-stopping of Poisson processes

We consider optimal m-stopping of a Poisson process $N = \sum_k \delta_{(\tau_k, Y_k)}$ in the plane restricted to some set

$$M_f = \{(t,x) \in [0,1] \times \overline{\mathbb{R}}; x > f(t)\}$$

where $f : [0,1] \to \mathbb{R} \cup \{-\infty\}$ is a continuous lower boundary function of N. The intensity of N may be infinite along the lower boundary f. As in [KR] (2000a) resp. [FR] (2009) who consider the case $m = 1$ we assume that the intensity measure μ of N is a Radon measure on M_f with the topology on M_f induced by the usual topology on $[0,1] \times \overline{\mathbb{R}}$. Thus any compact set $A \subset M_f$ has only finitely many points. By convergence in distribution '$N_n \xrightarrow{d} N$ on M_f' we mean convergence in distribution of the restricted point processes.

We generally assume the boundedness condition

$$\text{(B)} \qquad E[(\sup_k Y_k)^+] < \infty. \tag{3.1}$$

Let $\mathcal{A}_t = \sigma(N(\cdot \cap [0,t] \times \overline{\mathbb{R}} \cap M_f))$, $t \in [0,1]$, denote the relevant filtration of the point process N. A stopping time for N or N-stopping time is a mapping $T : \Omega \to [0,1]$ with $\{T \le t\} \in \mathcal{A}_t$ for each $t \in [0,1]$. Denote by

$$\overline{Y}_T := \sup\{Y_k : 1 \le k \le N(M_f), T = \tau_k\}, \ \ \sup \emptyset := -\infty,$$

the reward w.r.t. stopping time T.

Let $v : \overline{M}_f \to \overline{\mathbb{R}}$ be a continuous transformation of the points of N such that

$$\left.\begin{array}{l} v(t,x) \le ax^+ + b \ \ \forall\,(t,x) \in M_f, \text{with real constants } a,b \ge 0, \\ v(t,\cdot) \text{ is for each } t \text{ a monotonically nondecreasing function,} \\ v(\cdot,x) \text{ is for each } x \text{ a monotonically nonincreasing function.} \end{array}\right\} \tag{3.2}$$

Define $c := f(1)$ and for any guarantee value $x \in [c,\infty)$ and $t \in [0,1)$ the optimal stopping curve $\hat{u}$ of the transformed Poisson process by

$$\begin{array}{l} \hat{u}(t,x) := \sup\{E[v(T,\overline{Y}_T \vee x)] : T > t \text{ is an } N\text{-stopping time}\}, \\ \hat{u}(1,x) := v(1,x). \end{array} \tag{3.3}$$

For the basic notions of stopping of point processes see [KR] (2000a) resp. [FR] (2009). The following proposition is the analogue of Theorem 2.1 for continuous time Poisson processes. It is essential for the solution of the m-stopping problem of N.

Proposition 3.1 (Optimal stopping times $> S$) *Let N satisfy (B) and v condition* (3.2) *and assume the following separation condition for the optimal stopping boundary $\hat{u}$:*

($\hat{S}$)
$$\hat{u}(t,c) > \hat{f}(t) := v(t,f(t)), \quad \forall\, t \in [0,1). \tag{3.4}$$

Then

a) $\hat{u}$ is continuous on $[0,1] \times [c,\infty]$ and for all $(t,x) \in [0,1] \times [c,\infty]$

$$\hat{u}(t,x) = E[v(T(t,x),\overline{Y}_{T(t,x)} \vee x)] = E[v(T(t,x),\overline{Y}_{T(t,x)} \vee c) \vee v(1,x)] \tag{3.5}$$

with the optimal stopping time

$$T(t,x) := \inf\{\tau_k > t : v(\tau_k,Y_k) > \hat{u}(\tau_k,x)\}, \ \ \inf \emptyset := 1.$$

$\hat{u}(\cdot,x)$ is for $x \in [c,\infty]$ the optimal stopping curve of the transformed Poisson process $\hat{N} := \sum_k \delta_{(\tau_k,\, v(\tau_k,Y_k))}$ in $M_{\hat{f}}$ for the guarantee value $v(1,x)$.

b) Let S be an N-stopping time, let $Z \ge c$ be real $\mathcal{A}_S$-measurable with $EZ^+ < \infty$ and $\mathcal{T}(S)$ the set of all N-stopping times T with $T > S$ on $\{S < 1\}$ and $T = 1$ on $\{S = 1\}$. Then $T(S,Z) \in \mathcal{T}(S)$ is optimal in the sense that

$$E[v(T(S,Z),\overline{Y}_{T(S,Z)} \vee Z) \mid \mathcal{A}_S] = \hat{u}(S,Z) \ge E[v(T,\overline{Y}_T \vee Z) \mid \mathcal{A}_S] \quad P\text{-a.s.} \tag{3.6}$$

for all $T \in \mathcal{T}(S)$.

Proof:

a) The statement in a) is proved by discretization as in the proof of Theorem 2.5 a) in [KR] (2000a). Since $\hat{f}$ is continuous and $\hat{u}(\cdot, c)$ is right continuous there exists a monotonically nonincreasing, continuous function $\hat{f}_2 : [0,1] \to [\hat{c}, \infty)$, $\hat{c} := \hat{f}(1) = v(1, c)$ such that $\hat{f} < \hat{f}_2 < \hat{u}(\cdot, c)$ on $[0, 1)$. Thus for $t < 1$ the sets $[0,t] \times \overline{\mathbb{R}} \cap M_{\hat{f}_2}$ are compact in $M_{\hat{f}}$.

For $x \in [c, \infty)$, $n \in \mathbb{N}$ and $1 \le i \le 2$ define

$$M^n_{\frac{i}{2^n}}(x) := \sup_{\tau_k \in \left(\frac{i-1}{2}, \frac{i}{2}\right]} v(\tau_k, Y_k \vee x).$$

Consider the filtration $\mathcal{A}^n = (\mathcal{A}_{\frac{i}{2^n}})_{1 \le i \le 2^n}$. Then $M^n_{\frac{i}{2^n}}(x)$ is $\mathcal{A}_{\frac{i}{2^n}}$ measurable and $\mathcal{A}_{\frac{i}{2^n}}$, $\sigma(M^n_{\frac{i+1}{2^n}}(x))$ are independent. We define $w_n : [0,1] \times [c, \infty) \to \overline{\mathbb{R}}$ by

$$\begin{aligned} &w_n(t,x) := \sup\{E[M^n_T(x)] : T > t \text{ an } \mathcal{A}^n\text{-stopping time}\} \quad \text{for } t \in [0,1), \\ &w_n(1,x) := v(1,x). \end{aligned} \tag{3.7}$$

Then for $t \in [0,1)$ by Theorem 2.1 we have

$$w_n(t,x) = E[M^n_{T_n(t,x)}(x)] = V^n_{\lfloor 2^n t \rfloor}(x),$$

with the optimal $\mathcal{A}^n$-stopping time

$$T_n(t,x) := \min\left\{t < \tfrac{i}{2^n} \le 1 : M^n_{\frac{i}{2^n}}(x) > w_n(\tfrac{i}{2^n}, x)\right\}, \quad \min \emptyset := 1,$$

and

$$\begin{aligned} &V^n_{2^n}(x) := v(1,x), \\ &V^n_i(x) := E[M^n_{\frac{i+1}{2^n}}(x) \vee V^n_{i+1}(x)], \quad i = 2^n - 1, \ldots, 0. \end{aligned} \tag{3.8}$$

The function $w_n(\cdot, x)$ is monotonically nonincreasing and constant on the intervals $[0, \frac{1}{2^n}), [\frac{1}{2^n}, \frac{2}{2^n}), \ldots, [\frac{2^n-1}{2^n}, 1)$. We also have

$$\begin{array}{lll} \textit{(1)} & w_n(t,x) \ge \hat{u}(t,x) & \forall\, t \in [0,1], \\ \textit{(2)} & w_n(t,x) \ge w_{n+1}(t,x) & \forall\, t \in [0,1]. \end{array}$$

For the proof of *(1)* note that for any stopping time $T > t$, $T_n := \frac{\lceil T 2^n \rceil}{2^n}$ is an $\mathcal{A}^n$-stopping time with $T_n > t$ and $T_n - \frac{1}{2^n} < T \le T_n$. Therefore

$$M^n_{T_n}(x) = \sup_{\tau_k \in \left(T_n - \frac{1}{2^n}, T_n\right]} v(\tau_k, Y_k \vee x) \ge v(T, \overline{Y}_T \vee x). \tag{3.9}$$

This implies $w_n(t,x) \ge \sup\{E[v(T, \overline{Y}_T \vee x)] : T > t \ N\text{-stopping time}\} = \hat{u}(t,x)$.

The proof of *(2)* is similar. If $T > t$ is an $\mathcal{A}^{n+1}$-stopping time, then $T' := \frac{\lceil T2^n \rceil}{2^n}$ is an $\mathcal{A}^n$-stopping time with $T' > t$ and $T' - \frac{1}{2^n} < T \leq T'$. Thus as above we obtain $w_n(t,x) \geq w_{n+1}(t,x)$.

(1) and *(2)* imply the existence of a monotonically nonincreasing function $w(\cdot,x) : [0,1] \to \mathbb{R} \cup \{-\infty\}$ with $w(\cdot,x) \geq \hat{u}(\cdot,x)$ and $w_n(\cdot,x) \downarrow w(\cdot,x)$ pointwise. It can be shown by our assumptions on v and N that w is continuous (see [F] (2009)).

For $\omega \in \Omega$ with $\hat{N}(\omega, K) < \infty$ for all compact $K \subset M_f$ and for $(t,x) \in [0,1] \times [c,\infty]$ and $t_n \downarrow t$ we have the convergence

$$M^n_{T_n(t_n,x)}(x) \to v(T(t,x), \overline{Y}_{T(t,x)} \vee x) \tag{3.10}$$

with the stopping time

$$\begin{aligned} T(t,x) &:= \inf\{\tau_k > t : v(\tau_k, Y_k \vee x) > w(\tau_k, x)\} \\ &\overset{(*)}{=} \inf\{\tau_k > t : v(\tau_k, Y_k) > w(\tau_k, x)\}, \quad \inf \emptyset := 1. \end{aligned} \tag{3.11}$$

For the proof note that monotone convergence of $w_n(\cdot,x)$ and continuity of the limit ω implies uniform convergence from above. Thus for $x \in [c,\infty)$ points of N on the graph of $w(\cdot,x)$ are ignored by all stopping times $T_n(t,x)$ and $T(t,x)$. The second equality $(*)$ holds since $w(t,x) \geq \hat{u}(t,x) \geq v(t,x)$ and since by assumption $v(t,\cdot)$ is strictly monotonically increasing. This implies by Fatou's Lemma the following sequence of inequalities:

$$\begin{aligned} \hat{u}(t,x) \leq w(t,x) = \lim_{n\to\infty} w_n(t,x) &= \lim_{n\to\infty} E[M^n_{T_n(t,x)}(x)] \\ &\leq E[v(T(t,x), \overline{Y}_{T(t,x)} \vee x)] \leq \hat{u}(t,x). \end{aligned}$$

Thus $\hat{u}(\cdot,x) = w(\cdot,x)$ is continuous and $\hat{u}(t,x) = E[v(T(t,x), \overline{Y}_{T(t,x)} \vee x)]$. As $w(t,x) \geq v(t,x)$ implies that $\overline{Y}_{T(t,x)} > x$ for $T(t,x) < 1$, we have $\hat{u}(t,x) = E[v(T(t,x), \overline{Y}_{T(t,x)} \vee c) \vee v(1,x)]$, which means that $\hat{u}(\cdot,x)$ is the optimal stopping curve of the Poisson process $\hat{N}$ with guarantee value $v(1,x)$.

b) To prove optimality of the stopping time $T(S,Z)$ set $S_n := \frac{\lceil S2^n \rceil}{2^n}$. Then S_n is an $\mathcal{A}^n$-stopping time and by (3.10) holds

$$M^n_{T_n(S_n,Z)}(Z) \to v(T(S,Z), \overline{Y}_{T(S,Z)} \vee Z) \quad P\text{-a.s.} \tag{3.12}$$

Let $\mathcal{T}(S_n)$ be the set of all $\mathcal{A}^n$-stopping times T_n with $T_n > S_n$ on $\{S_n < 1\}$ and $T_n = S_n$ on $\{S_n = 1\}$. Let $T \in \mathcal{T}(S)$. By discretization $T > S$ in general does not imply $\frac{\lceil T2^n \rceil}{2^n} > \frac{\lceil S2^n \rceil}{2^n}$. Thus we modify the discretization and define $T_n := \frac{\lceil T2^n \rceil}{2^n} \chi_{\{\frac{\lceil T2^n \rceil}{2^n} > S_n\}} + 1\chi_{\{\frac{\lceil T2^n \rceil}{2^n} = S_n\}} \in \mathcal{T}(S_n)$. Then analogously to (3.9)

$$v(T, \overline{Y}_T \vee Z) \leq M^n_{T_n}(Z)\chi_{\{\frac{\lceil T2^n \rceil}{2^n} > S_n\}} + v(T, \overline{Y}_T \vee Z)\chi_{\{\frac{\lceil T2^n \rceil}{2^n} = S_n\}}.$$

This implies the inequalitites

$$\begin{aligned}
&E\big[v(T,\overline{Y}_T\vee Z)\mid \mathcal{A}_{S_n}\big]\\
&\quad\le E\big[M^n_{T_n}(Z)\mid \mathcal{A}_{S_n}\big]\chi_{\{\frac{\lceil T2^n\rceil}{2^n}>S_n\}}+E\big[v(T,\overline{Y}_T\vee Z)\mid \mathcal{A}_{S_n}\big]\chi_{\{\frac{\lceil T2^n\rceil}{2^n}=S_n\}}\\
&\quad\overset{(*)}{\le} \underbrace{E\big[M^n_{T_n(S_n,Z)}(Z)\mid \mathcal{A}_{S_n}\big]}_{=w_n(S_n,Z)}\chi_{\{\frac{\lceil T2^n\rceil}{2^n}>S_n\}}+E\big[v(T,\overline{Y}_T\vee Z)\mid \mathcal{A}_{S_n}\big]\chi_{\{\frac{\lceil T2^n\rceil}{2^n}=S_n\}}.
\end{aligned}$$

$(*)$ holds by Remark 2.2. Since we have $M^n_{\frac{i}{2^n}}(Z)=h(Y_i,Z)$, where $Y_i:=N(\cdot\cap(\frac{i-1}{2^n},\frac{i}{2^n}]\times\overline{\mathbb{R}}\cap M_f)$ and with $h:\mathcal{N}_R(M_f)\times[c,\infty)\to\overline{\mathbb{R}}$, $h(\sum_k\delta_{(t_k,y_k)},x):=\sup_k v(t_k,y_k\vee x)$.

As $\mathcal{A}_S\subset\mathcal{A}_{S_n}$ we conclude

$$\begin{aligned}
&E\big[v(T,\overline{Y}_T\vee Z)\mid \mathcal{A}_S\big]\\
&\quad\le E\big[M^n_{T_n(S_n,Z)}(Z)\chi_{\{\frac{\lceil T2^n\rceil}{2^n}>S_n\}}\mid \mathcal{A}_S\big]+E\big[v(T,\overline{Y}_T\vee Z)\chi_{\{\frac{\lceil T2^n\rceil}{2^n}=S_n\}}\mid \mathcal{A}_S\big]\\
&\quad= w_n(S_n,Z)E\big[\chi_{\{\frac{\lceil T2^n\rceil}{2^n}>S_n\}}\mid \mathcal{A}_S\big]+E\big[v(T,\overline{Y}_T\vee Z)\chi_{\{\frac{\lceil T2^n\rceil}{2^n}=S_n\}}\mid \mathcal{A}_S\big],
\end{aligned}$$

and by the Lemma of Fatou we have by (3.12)

$$E\big[v(T,\overline{Y}_T\vee Z)\mid \mathcal{A}_S\big]\le E\big[v(T(S,Z),\overline{Y}_{T(S,Z)}\vee Z)\mid \mathcal{A}_S\big]=\hat{u}(S,Z).$$

As $T>S$ was choosen arbitrary this implies b). □

In the sequel we need the following differentiability condition to be fulfilled

(D) Assume that there is a version of the density g of μ on M_f such that the intensity function

$$G(t,y)=\int_y^\infty g(t,z)dz$$

is continuous on $M_f\cap[0,1]\times\mathbb{R}$. Furthermore we assume that $\lim_{y\to\infty} yG(t,y)=0$ for all $t\in[0,1]$.

The following proposition determines the intensity function of transformed Poisson processes.

Proposition 3.2 (Intensity function of transformed Poisson processes) *Let $N=\sum\delta_{(\tau_k,Y_k)}$ be a Poisson process with intensity function G satisfying the boundedness condition (B). Let $v:\overline{M}_f\to\overline{\mathbb{R}}$, $v=v(t,x)$ be a C^1-function monotonically nonincreasing in t and monotonically nondecreasing in x with $v(t,\infty)=\infty$ for all $t\in[0,1]$. Define $R(t,x):=(t,v(t,x))$ and $f_v(t):=v(t,f(t))$. Then $R(M_f)=M_{f_v}$, $R^{-1}(t,y)=(t,\xi(t,y))$ with a C^1-function $\xi:M_{f_v}\to\overline{\mathbb{R}}$.*

$\widehat{N}:=\sum_k\delta_{(\tau_k,v(\tau_k,Y_k))}$ is a Poisson process on M_{f_v} with intensity measure $\widehat{\mu}=\mu\circ R^{-1}$ and intensity fuction $\widehat{G}(t,y):=G(t,\xi(t,y))$, $(t,y)\in M_{f_v}$.

Proof: By Resnick (1987, Prop. 3.7) $\widehat{N}$ is a Poisson process with intensity measure $\widehat{\mu} = \mu \circ R^{-1}$. The transformation formula implies that the density $\widehat{g}$ of $\widehat{\mu}$ is given by

$$\begin{aligned}\hat{g}(t,y) &= g(R^{-1}(t,y)) \left|\det J(R^{-1})(t,y)\right| \\ &= g(t,\xi(t,y)) \frac{\partial}{\partial y}\xi(t,y) = -\frac{\partial}{\partial y} G(t,\xi(t,y)). \qquad \square\end{aligned}$$

After this preparation we now consider the m-stopping problem for Poisson processes. The aim is to solve

$$E[\overline{Y}_{T_1} \vee \ldots \vee \overline{Y}_{T_m}] = \sup \tag{3.13}$$

where the supremum is over all N-stopping times[2] $0 \leq T_1 < \cdots < T_m \leq 1$.

This problem has been considered for Poisson processes on $[0,1] \times (c,\infty)$ already in Saario and Sakaguchi (1992) in the special case of intensity functions of the form

$$G(t,y) = \lambda(1 - F(y)) \tag{3.14}$$

with $\lambda > 0$ and F a continuous distribution function with $F(c) = 0$. (3.14) models the case of iid random variables arriving at Poisson distributed arrival times. Saario and Sakaguchi (1995) derive for this case differential equations for the optimal stopping curves. Explicit solutions are however not given in any case. In the following we extend these results to the case of general intensities. We subsequently also identify classes of examples of intensity functions which allow essentially explicit solutions.

In order to guarantee the existence of optimal m-stopping times we restrict ourselves in the following to the case where the lower boundary is constant, $f \equiv c$. Define optimal m-stopping curves for guarantee value $x \in [c,\infty)$, $m \in \mathbb{N}$, and $t \in [0,1)$ by[2]

$$\begin{aligned} u^m(t,x) &:= \sup\Big\{ E[\overline{Y}_{T_1} \vee \ldots \vee \overline{Y}_{T_m} \vee x] : \\ &\qquad\qquad t < T_1 < \ldots < T_m \leq 1 \ \ N\text{-stopping times}\Big\}, \\ u^m(1,x) &:= x. \end{aligned} \tag{3.15}$$

Further let $u^0(t,x) := x$ for $(t,x) \in [0,1] \times [c,\infty]$ and $u^m(t) := u^m(t,c)$ for $t \in [0,1]$.

$u^m(\cdot,x)$ is called *optimal m-stopping curve* of N for guarantee value x. Define the inverse function $\xi^m : \overline{M}_{u^m} \to \overline{R}$ by

$$\xi^m(t,u^m(t,x)) = x \quad \text{for} \quad (t,x) \in [0,1] \times [c,\infty]. \tag{3.16}$$

[2] $T_1 < \ldots < T_m \leq 1$ signifies that $T_{i-1} < T_i$ for each i on $\{T_{i-1} < 1\}$ and $T_i = 1$ on $\{T_{i-1} = 1\}$.

Further define $\gamma^m : [0,1] \times [c,\infty] \to \overline{\mathbb{R}}$ by

$$\gamma^m(t,x) := \xi^{m-1}(t, u^m(t,x)) \tag{3.17}$$

as well as

$$\gamma^m(t) := \gamma^m(t,c) = \xi^{m-1}(t, u^m(t)). \tag{3.18}$$

Then $\gamma^m(t,x) > x$ iff $u^m(t,x) > u^{m-1}(t,x)$ and further

$$y > \gamma^m(t,x) \Leftrightarrow u^{m-1}(t,y) > u^m(t,x).$$

The optimal m-stopping for Poisson processes can be reduced by the previous structural results to m 1-stopping problem for transformed Poisson processes. The transformations are given by the optimal stopping curves u^m or equivalently by the inverses γ^m – both sequences of curves are defined recursively. Thus we consider the transformed Poisson processes

$$N^m := \sum_k \delta_{(\tau_k, u^{m-1}(\tau_k, Y_k))} \qquad \text{on } M_{u^{m-1}}. \tag{3.19}$$

Define the (optimal) stopping times $T^m_\ell(t,k)$ with guarantee value x by

$$\begin{aligned} T^m_1(t,x) &:= \inf\left\{\tau_k > t : Y_k > \gamma^m(\tau_k, x)\right\}, \\ T^m_\ell(t,x) &:= \inf\left\{\tau_k > T^m_{\ell-1}(t,x) : Y_k > \gamma^{m-\ell+1}(\tau_k, \overline{Y}_{T^m_{\ell-1}(t,x)} \vee x)\right\}. \end{aligned} \tag{3.20}$$

Then we have the following solution to the m-stopping problem for Poisson processes.

Theorem 3.3 (Optimal m-stopping of Poisson processes) *Let $f \equiv c$ and N satisfy the boundedness condition (B) and the separation condition (S), i.e. $u^1(t) > c$ for $t \in [0,1)$. Let $t_0(x) := \inf\{t \in [0,1] : \mu((t,1] \times (x,\infty]) = 0\}$.*

a) Then for $m \in \mathbb{N}$, $(t,x) \in [0,1) \times [c,\infty)$ holds

$$u^m(t,x) = E\left[\overline{Y}_{T^m_1(t,x)} \vee \ldots \vee \overline{Y}_{T^m_m(t,x)} \vee x\right] = E\left[u^{m-1}(T^m_1(t,x), \overline{Y}_{T^m_1(t,x)} \vee x)\right],$$

with optimal stopping times $(T^m_\ell(t,x))_{1\le\ell\le m}$ defined in (3.20).

b) For $(t,x) \in A := \{(t,x) \in (0,1] \times [c,\infty) : t < t_0(x)\}$ holds $u^m(t,x) > u^{m-1}(t,x)$ while $u^m(t,x) = u^{m-1}(t,x) = x$ else. In particular $u^m(t) > u^{m-1}(t)$ for $t \in [0,1)$ and $u^m(\cdot,x)$ is the optimal stopping curve of the transformed Poisson process N^m.

c) Under the differentiability condition (D) $u^m(\cdot,x)$ solves the differential equation

$$\begin{aligned} \frac{\partial}{\partial t} u^m(t,x) &= -\int_{u^m(t,x)}^\infty G(t, \xi^{m-1}(t,y))\, dy, \qquad t \in [0,1), \\ u^m(1,x) &= x. \end{aligned} \tag{3.21}$$

d) For $x > -\infty$ (3.21) has a unique solution. If $c = -\infty$ and if

$$\liminf_{s\uparrow 1} \frac{u(s)}{b(s)} < \infty, \tag{3.22}$$

where $b(s) := E[\sup_{\tau_k > s} Y_k]$, then also in this case $u^m = u^m(\cdot, -\infty)$ for $m \geq 2$ is uniquely determined by (3.21).

Proof: The proof is by induction in m. Our induction hypothesis is that the statement of Theorem 3.3 holds and moreover that for any n-stopping time S and any $\mathcal{A}_S$-measurable $Z \geq c$ with $EZ^+ < \infty$ we have P-a.s.

$$E[Z \vee \overline{Y}_{T_1^m(S,Z)} \vee \ldots \vee \overline{Y}_{T_m^m(S,Z)} \mid \mathcal{A}_S] = u^m(S,Z) \geq E[Z \vee \overline{Y}_{T_1} \vee \ldots \vee \overline{Y}_{T_m} \mid \mathcal{A}_S]$$

for all N-stopping times $S < T_1 < \ldots < T_m \leq 1$. Further,

$$A = \{(t,x) \in [0,1] \times [c,\infty) : u^m(t,x) > u^{m-1}(t,x)\}. \tag{3.23}$$

For the one-stopping problem $m = 1$ the statement of Theorem 3.3 is contained in [FR] (2009). Proposition 3.1 with $v(t,x) := x$ implies the first part of the induction hypothesis while the second part follows from [FR] (2009, Lemma 2.1 (c)).

For the induction step $m \to m+1$ we obtain for all stopping times $S < T_1 < T_2 < \cdots < T_{m+1} \leq 1$ and $Z \geq c$ $\mathcal{A}_S$-measurable by the induction hypothesis (note that $\mathcal{A}_S \subset \mathcal{A}_{T_1}$):

$$\begin{aligned}
&E[(Z \vee \overline{Y}_{T_1}) \vee \overline{Y}_{T_2} \vee \ldots \vee \overline{Y}_{T_{m+1}} \mid \mathcal{A}_S] \\
&\quad \leq E[(Z \vee \overline{Y}_{T_1}) \vee \overline{Y}_{T_1^m(T_1, Z\vee \overline{Y}_{T_1})} \vee \ldots \vee \overline{Y}_{T_m^m(T_1, Z\vee \overline{Y}_{T_1})} \mid \mathcal{A}_S] \\
&\quad = E[u^m(T_1, Z \vee \overline{Y}_{T_1}) \mid \mathcal{A}_S].
\end{aligned} \tag{3.24}$$

This expression is maximized by Proposition 3.1 by $T_1 = T_1^{m+1}(S,Z)$ where

$$T_1^{m+1}(t,x) := \inf\{\tau_k > t : u^m(\tau_k, Y_k) > \hat{u}(\tau_k, x)\}, \quad \inf \emptyset := 1.$$

The maximizing value is given by $\hat{u}(S,Z)$.

For the proof we need to show that $\hat{u}(t,c) > u^m(t)$ for $t \in [0,1)$. We shall do this and at the same time show (3.23) for $m+1$.

Note that for $x \in [c,\infty)$

$$\begin{aligned}
\hat{u}(t,x) &\overset{\text{Def.}}{=} \sup\left\{E[u^m(T, \overline{Y}T \vee x)] : T > t \ N\text{-stopping time}\right\} \\
&\geq E[u^m(T_1^m(t,x), \overline{Y}_{T_1^m(t,x)} \vee x)] \\
&\overset{(*)}{\geq} E[u^{m-1}(T_1^m(t,x), \overline{Y}_{T_1^m(t,x)} \vee x)] \\
&= u^m(t,x), \qquad \text{by induction hypothesis.}
\end{aligned}$$

By (3.23) we have strict inequality in $(*)$ if and only if $P((T_1^m(t,x), \overline{Y}_{T_1^m(t,x)}) \in A) > 0$. Using Lemma 2.4 in [FR] (2009) we see that this is equivalent to $\mu(A \cap M_{\gamma^m(\cdot,x)} \cap (t,1] \times \mathbb{R}) > 0$. This in turn is equivalent to

$$A \cap M_{\gamma^m(\cdot,x)} \cap (t,1] \times \mathbb{R} \neq \emptyset \tag{3.25}$$

(since $\gamma^m(\cdot,x)$ is monotonically nonincreasing and by definition of A). We are going to show that this is fulfilled for all points $(t,x) \in A$.

So let $(t,x) \in A$ and thus by induction hypothesis $u^m(t,x) > u^{m-1}(t,x)$ or equivalently $\gamma^m(t,x) > x$. Under the assumption that $M_{\gamma^m(\cdot,x)} \cap (t,1] \times \mathbb{R} \subset A^c$ we obtain that also $(t, \gamma^m(t,x)) \in A^c$ since A^c is closed. This implies that

$$u^m(t, \gamma^m(t,x)) = u^{m-1}(t, \gamma^m(t,x)) = u^m(t,x).$$

Since $u^m(t,\cdot)$ is strictly increasing it follows that $\gamma^m(t,x) = x$, which is a contradiction. Thus (3.25) holds true.

With the choice $S := t$, $Z := x$ further we obtain

$$\hat{u}(t,x) = E[u^m(T_1^{m+1}(t,x), \overline{Y}_{T_1^{m+1}(t,x)} \vee x)] = u^{m+1}(t,x).$$

Finally, in (3.24) holds

$$T_l^m\big(T_1^{m+1}(S,Z), Z \vee \overline{Y}_{T_1^{m+1}(S,Z)}\big) = T_{l+1}^{m+1}(S,Z).$$

By Proposition 3.1 $u^{m+1}(\cdot,x)$ is the optimal stopping curve of the Poisson process $N^{m+1} = \sum_k \delta_{(\tau_k, u^m(\tau_k, Y_k))}$ on M_{u^m} at the guarantee value x. We already proved that the separation condition is fulfilled for the stopping of N^{m+1} and by Proposition 3.2 N^{m+1} has the intensity function $G^{m+1}(t,y) := G(t, \xi^m(t,y))$. The existence and uniqueness results for the differential equation (3.21) therefore follow with our assumption from the corresponding result in [FR] (2009) for the case $m = 1$. □

4 Explicit calculation of optimal m-stopping curves

For the case of one-stopping problems some classes of intensity functions $G(t,y)$ have been introduced in [FR] (2009) which allow to determine optimal stopping curves in explicit form. Solving the optimality equations in (3.21) for the sequence of optimal stopping curves for the m-stopping problem is in general much more demanding. However for some of the classes considered in [FR] (2009) explicit solutions can be given also in the m-stopping case.

We consider intensity functions $G(t,y)$ of the form

$$G(t,y) = H\left(\frac{y}{v(t)}\right) \frac{|v'(t)|}{v(t)} \tag{4.1}$$

$$\text{or} \quad G(t,y) = H(y - v(t))|v'(t)| \tag{4.2}$$

as in [FR] (2009) with $v(1) = 0$ or $v(1) = \infty$ in case (4.1) and $v(1) = -\infty$ in case (4.2). For the general motivation of these classes and these conditions we refer to [FR] (2009). In particular we will see that the main application considered in this paper to m-stopping of iid seqences is covered by these classes.

We first state the results in the three cases mentioned and then give the proof.

Case 1: G satisfies (4.1) with v monotonically nonincreasing, $v(1) = 0$. Here $c = 0$. $H : (0, \infty] \to [0, \infty)$ is monotonically nonincreasing continuous, $\int_0^\infty H(x)dx > 0$ and we assume that $v : [0,1] \to [0, \infty)$ is a C^1-function with $v > 0$ on $[0, 1)$.

We define

$$R^1(x) := x - \int_x^\infty H(y)\,dy, \quad x \in (0, \infty). \tag{4.3}$$

and assume that there exists some $r > 0$ with $R^1(r) = 0$. Define $r_0 := 0$, $\Phi^0(x) := x$. Then for $m \geq 1$ by induction holds:

The function $R^m : (r_{m-1}, \infty) \to \mathbb{R}$ given by

$$R^m(x) := x - \int_x^\infty H(\Phi^{m-1}(y))\,dy \tag{4.4}$$

has exactly one zero $r_m \in (r_{m-1}, \infty)$ and the optimal m-stopping curves are given for $(t, x) \in [0, 1) \times [0, \infty]$ by

$$u^m(t,x) = \phi^m\left(\frac{x}{v(t)}\right) v(t), \tag{4.5}$$

where $\phi^m : [0, \infty] \to [r_m, \infty]$ is the inverse function of $\Phi^m : [r_m, \infty] \to [0, \infty]$,

$$\Phi^m(x) := x \exp\left(-\int_x^\infty \left(\frac{1}{R^m(y)} - \frac{1}{y}\right) dy\right).$$

The system of functions (R^m, Φ^m) resp. (u^m, ϕ^m) is by (4.5) recursively defined. In particular it holds that

$$u^m(t) = r_m v(t). \tag{4.6}$$

Case 2: G satisfies (4.1) with v monotonically nondecreasing, $v(1) = \infty$. Here $c = -\infty$. $H : (-\infty, \infty] \to [0, \infty)$ is monotonically nonincreasing continuous, $\int_{-\infty}^0 H(x)\,dx > 0$, $\int_0^\infty H(x)\,dx = 0$ and $\int_y^0 \frac{H(x)}{-x}\,dx < \infty$ for $y < 0$. Further, we assume that $v : [0,1] \to [0, \infty]$ is a C^1-function with $v < \infty$ on $[0, 1)$.

We define

$$R^1(x) := x + \int_x^\infty H(y)\,dy, \quad x \in (-\infty, \infty).$$

and assume that there exists some $r < 0$ with $R^1(r) = 0$. Define $r_0 := -\infty$, $\Phi^0(x) := x$. Then for $m \geq 1$ by induction holds:

The function $R^m : (r_{m-1}, 0) \to \mathbb{R}$ defined by

$$R^m(x) := x + \int_x^0 H(\Phi^{m-1}(y))\, dy$$

has exactly one zero $r_m \in (r_{m-1}, 0)$ and the optimal m-stopping curves are given for $(t, x) \in [0, 1) \times \overline{\mathbb{R}}$ by

$$u^m(t, x) = \begin{cases} x, & \text{if } x \geq 0, \\ \phi^m\big(\frac{x}{v(t)}\big) v(t), & \text{if } x < 0, \end{cases} \tag{4.7}$$

where $\phi^m : [-\infty, 0] \to [r_m, 0]$ is the inverse of $\Phi^m : [r_m, 0] \to [-\infty, 0]$,

$$\Phi^m(x) := x \exp\left(\int_x^0 \left(\frac{1}{y} - \frac{1}{R^m(y)} \right) dy \right).$$

In particular, $u^m(t) = r_m v(t)$.

Case 3: G satisfies (4.2) with v monotonically nonincreasing $v(1) = -\infty$. Then $c = -\infty$. $H : (-\infty, \infty] \to [0, \infty)$ is monotonically nonincreasing continuous, $\int_{-\infty}^{\infty} H(x)\, dx > 0$ and $\int_z^\infty \int_y^\infty H(x)\, dx\, dy < \infty$ for $z \in \mathbb{R}$. Further, we assume that $v : [0, 1] \to [-\infty, \infty)$ is a C^1-function with $v > -\infty$ on $[0, 1)$.

We define

$$R^1(x) := 1 - \int_x^\infty H(y)\, dy, \quad x \in \mathbb{R}.$$

and assume that there exists some $r \in \mathbb{R}$ such that $R^1(r) = 0$. Define $r_0 := -\infty$, $\Phi^0(x) := x$. Then for $m \geq 1$ by induction holds:

The function $R^m : (r_{m-1}, \infty) \to \mathbb{R}$ defined by

$$R^m(x) := 1 - \int_x^\infty H(\Phi^{m-1}(y))\, dy$$

has exactly one zero $r_m \in (r_{m-1}, \infty)$. The optimal m-stopping curves are given for $(t, x) \in [0, 1) \times \overline{\mathbb{R}}$ by

$$u^m(t, x) = \phi^m(x - v(t)) + v(t), \tag{4.8}$$

where $\phi^m : \overline{\mathbb{R}} \to [r_m, \infty]$ is the inverse of $\Phi^m : [r_m, \infty] \to \overline{\mathbb{R}}$,

$$\Phi^m(x) := x - \int_x^\infty \left(\frac{1}{R^m(y)} - 1 \right) dy.$$

We have $u^m(t) = r_m + v(t)$.

Proof: We only give the proof of case 2. The proof of both other cases is similar. The proof is by induction in m where we additionally include that $R^m \geq R^{m-1}$ and thus $\Phi^m \geq \Phi^{m-1}$.

In the case $m = 1$ the statement has been shown in [FR] (2009) (with $r_0 := -\infty$, $\Phi^0(x) := x$, $R^0(x) := x$).

Induction step $m \to m+1$: $u^{m+1}(\cdot, x)$ is the optimal stopping curve of N^{m+1} at the guarantee value x. N^{m+1} has the intensity function

$$G^{m+1}(t, y) = H\left(\Phi^m\left(\frac{y}{v(t)}\right)\right)\frac{v'(t)}{v(t)} \qquad \text{for } (t, y) \in M_{u^m}$$

Thus G^{m+1} again is of type (4.1) and we have to check the conditions of Case 2 in [FR] (2009), who deal with optimal one-stopping w.r.t. this type of intensity functions. First we note that R^{m+1} has a zero in $(r_m, 0)$ since $\Phi^m(x) \geq \Phi^{m-1}(x)$ and thus $R^{m+1} \geq R^m$. Further by substitution we have

$$\int_y^0 \frac{H(\Phi^m(x))}{-x}\, dx \overset{\text{Subst.}}{=} \int_{\Phi^m(y)}^0 \frac{H(z)}{-z}\frac{-z}{\phi^m(z)}\, (\phi^m)'(z)\, dz < \infty,$$

as $\lim\limits_{z\to 0} \frac{-z}{\phi^m(z)} = 1$ and $\lim\limits_{z\to 0}(\phi^m)'(z) = 1$. Thus the conditions hold true and the result follows. □

For intensity functions G not of the form as in (4.1), (4.2) the optimality differential equations in Theorem 3.3 typically can only be solved numerically. In some cases however one can derive bounds for the optimal stopping curves $u^m(t, x)$.

Example 4.1 *We consider intensity functions on* $[0, 1) \times \overline{\mathbb{R}}$ *of the form*

$$G_{c,d}(t, y) = \begin{cases} 0, & \text{if } \frac{y}{v(t)} \geq d, \\ \frac{1}{t}\left(-\frac{y}{v(t)} + d\right)^\alpha, & \text{if } \frac{y}{v(t)} < d, \end{cases} \tag{4.9}$$

with $v(t) := t^{c-\frac{1}{\alpha}}$, *where* $\alpha > 0$ *and* $c, d \in \mathbb{R}$ *with* $c \neq \frac{1}{\alpha}$ *as considered in [FR] (2009, Example 3.5). We treat at first the case* $d = 0$. *Then*

$$G_{c,0}(t, y) = \left\{\begin{array}{ll} 0, & \text{if } y \geq 0, \\ t^{-\alpha c}(-y)^\alpha, & \text{if } y < 0, \end{array}\right\} = \tilde{H}\left(\frac{y}{\tilde{v}(t)}\right)\frac{\tilde{v}'(t)}{\tilde{v}(t)}$$

with $\tilde{H}(x) := (\alpha+1)(-x)^\alpha$ *for* $x < 0$ *and* $\tilde{H}(x) := 0$ *for* $x \geq 0$, *and*

$$\tilde{v}(t) := -u_{c,0}(t) = \left(\tfrac{\alpha}{\alpha+1}\tfrac{1}{1-c\alpha}(1 - t^{1-c\alpha})\right)^{-\frac{1}{\alpha}}.$$

$G_{c,0}$ *thus satisfies* (4.1) *with these functions, too. Note that* $\tilde{H}$ *in this case is independent of c.* $\tilde{H}$, $\tilde{v}$ *satisfy the conditions of Case 2. With* Φ^{m-1}, r_m *as introduced in Case 2 we obtain for* $m \in \mathbb{N}$

$$\gamma^m_{c,0}(t) = \xi^{m-1}_{c,0}(t, u^m_{c,0}(t)) = \Phi^{m-1}(r_m)\tilde{v}(t), \qquad t \in [0, 1]. \tag{4.10}$$

For general $d \in \mathbb{R}$ we next derive as in Example 3.5 in [FR] (2009) the optimal m-stopping curve $u^m_{c,d}$ of the Poisson process with intensity function $G_{c,d}$ in case $m \geq 1$. In the cases $c > \frac{1}{\alpha}$, $d > 0$ and $c < \frac{1}{\alpha}$, $d < 0$, we have for all $t \in [0,1]$, $x \in \overline{\mathbb{R}}$

$$u^m_{c,0}(t, x - dv(t)) + dv(t) \;\leq\; u^m_{c,d}(t,x) \;\leq\; u^m_{c,0}(t, x-d) + d. \tag{4.11}$$

In both further cases $c > \frac{1}{\alpha}$, $d < 0$ and $c < \frac{1}{\alpha}$, $d > 0$, we have

$$u^m_{c,0}(t, x-d) + d \;\leq\; u^m_{c,d}(t,x) \;\leq\; u^m_{c,0}(t, x - dv(t)) + dv(t). \tag{4.12}$$

In particular in all four cases we obtain

$$\lim_{t \uparrow 1} u^m_{c,d}(t) - u^m_{c,0}(t) = d.$$

Furthermore, it can be shown that

$$\lim_{t \uparrow 1} \gamma^m_{c,d}(t) - \gamma^m_{c,0}(t) = d. \tag{4.13}$$

We give the proof in case $c > \frac{1}{\alpha}$, $d > 0$ or $c < \frac{1}{\alpha}$, $d < 0$. The other cases are similarly. The proof of (4.11) is by induction in m. For the case $m = 1$ compare [FR] (2009, Example 3.5).

For the induction step $m \to m+1$ we assume that (4.11) holds for m and any t, x. This is easily seen to be equivalent to

$$\xi^m_{c,0}(t, y-d) + d \;\leq\; \xi^m_{c,d}(t,y) \;\leq\; \xi^m_{c,0}(t, y - dv(t)) + dv(t) \tag{4.14}$$

for any t, y. This implies by definition and the case $m = 1$

$$\begin{aligned} G^{m+1}_{c,d}(t,y) &\overset{Def.}{=} G_{c,d}(t, \xi^m_{c,d}(t,y)) \\ &\leq\; G_{c,d}(t, \xi^m_{c,0}(t, y-d) + d) \leq G_{c,0}(t, \xi^m_{c,0}(t, y-d)) = G^{m+1}_{c,0}(t, y-d) \end{aligned}$$

and

$$\begin{aligned} G^{m+1}_{c,d}(t,y) &\geq\; G_{c,d}(t, \xi^m_{c,0}(t, y - dv(t)) + dv(t)) = G_{c,0}(t, \xi^m_{c,0}(t, y - dv(t))) \\ &\overset{Def.}{=} G^{m+1}_{c,0}(t, y - dv(t)). \end{aligned}$$

From [FR] (2009) (Proposition 2.8 and Remark 3.4) we conclude

$$u^{m+1}_{c,0}(t, x - dv(t)) + dv(t) \;\leq\; u^{m+1}_{c,d}(t,x) \;\leq\; u^{m+1}_{c,0}(t, x-d) + d,$$

which is (4.11) for $m+1$.

We next prove (4.13). By the calculation of the optimal stopping curves in (4.7), Case 2 we obtain that

$$u^m_{c,0}(t) = r_m \tilde{v}(t),$$

$$\xi_{c,0}^{m-1}(t,y) = \Phi^{m-1}\left(\frac{y}{\tilde{v}(t)}\right)\tilde{v}(t).$$

By (4.14) *for* $m-1$ *and* (4.11) *for* $x=-\infty$ *we obtain*

$$\begin{aligned}
\gamma_{c,d}^{m}(t)-\gamma_{c,0}^{m}(t) &= \xi_{c,d}^{m-1}(t,u_{c,d}^{m}(t))-\xi_{c,0}^{m-1}(t,u_{c,0}^{m}(t))\\
&\leq \xi_{c,0}^{m-1}(t,u_{c,d}^{m}(t)-dv(t))+dv(t)-\xi_{c,0}^{m-1}(t,u_{c,0}^{m}(t))\\
&\leq \xi_{c,0}^{m-1}(t,u_{c,0}^{m}(t)+d-dv(t))-\xi_{c,0}^{m-1}(t,u_{c,0}^{m}(t))+dv(t)\\
&= \Phi^{m-1}\left(r_m+d\frac{1-v(t)}{\tilde{v}(t)}\right)\tilde{v}(t)-\Phi^{m-1}\left(r_m\right)\tilde{v}(t)+dv(t)\\
&= \frac{\Phi^{m-1}\left(r_m+d\frac{1-v(t)}{\tilde{v}(t)}\right)-\Phi^{m-1}\left(r_m\right)}{d\,\frac{1-v(t)}{\tilde{v}(t)}}\,d(1-v(t))+dv(t)\\
&\xrightarrow{t\uparrow 1} \left(\Phi^{m-1}\right)'(r_m)\cdot 0+d \;=\; d.
\end{aligned}$$

Similarly, we obtain the estimate from below.

5 Approximation of m-stopping problems

In this section an extension of the approximation results in [KR] (2004, Theorem 2.1) and [FR] (2009, Theorem 4.1) for optimal one-stopping problems for dependent sequences is given to the class of m-stopping problems. For the special case of iid sequences with distribution function F in the domain of the Gumbel extreme value distribution Λ a corresponding approximation result was given in the case $m=2$ in [KR] (2002). The following result concerns the dependent case and needs a new technique of proof which is based on discretization. The main result of this section states that under some conditions convergence of the finite imbedded point processes N_n to a Poisson process N implies approximation of the stopping behaviour.

We use the same general assumptions as in Section 4 of [FR] (2009) as well as the notation in Section 2 for the Poisson process N. In particular $\gamma^1,\dots,\gamma^m$ are the functions defined in (3.17). Further the lower boundary curve f of N is given by $f\equiv c$, N is a Poisson process on $[0,1]\times(\overline{\mathbb{R}}\setminus\{c\})$ and $\mathcal{F}^n$ are the canonical filtrations.

The first result is an extension of Proposition 2.4 in [KR] (2000a) on the convergence of threshold stopping times to the case $m\geq 1$.

Proposition 5.1 (Convergence of multiple threshold stopping times) *Let* $(t,x)\in[0,1]\times[c,\infty)$ *be fixed and let* $v_n^m:[0,1]\to\overline{\mathbb{R}}$ *be functions such that* $v_n^m\to\gamma^m(\cdot,x)$ *uniformly on any interval* $[0,s]$ *with* $s<1$. *Define the corresponding threshold stopping times*

$$\hat{T}_1^{n,m}(t,x) \;:= \min\left\{tn<i\leq n-m+1 : X_i^n > v_n^m\left(\tfrac{i}{n}\right)\right\},$$

$$\hat{T}_\ell^{n,m}(t,x) := \min\left\{\hat{T}_{\ell-1}^{n,m}(t,x) < i \le n-m+\ell : X_i^n > \gamma^{m-\ell+1}\left(\tfrac{i}{n}, X^n_{\hat{T}_{\ell-1}^{n,m}(t,x)} \vee x\right)\right\}$$

for $2 \le \ell \le m$. If $N_n \xrightarrow{d} N$ on M_c, we obtain convergence

$$\left(\tfrac{\hat{T}_\ell^{n,m}(t,x)}{n}, X^n_{\hat{T}_\ell^{n,m}(t,x)} \vee x\right)_{1\le\ell\le m} \xrightarrow{d} (T_\ell^m(t,x), \overline{Y}_{T_\ell^m(t,x)} \vee x)_{1\le\ell\le m}. \tag{5.1}$$

Proof: By the Skorohod theorem we can assume w.l.g. that $N_n(\cdot \cap M_c) \to N(\cdot \cap M_c)$ P-a.s. By our general differentiability assumption (D) on the intensity of the Poisson process N a.s. no point of N lies on a given graph of a function.

We will establish P-a.s. convergence in (5.1). Let $(t_p)_{p\in\mathbb{N}}$ be a sequence of points in $[0,1)$ with $t_p \uparrow 1$. Choose $\omega \in \Omega$ such that N has no points on the following graphs, in A^c (see Theorem 3.3 for notation), on $\{t_p\} \times (c,\infty]$ or on $\{1\} \times (c,\infty]$ and such that $N_n(\omega) \to N(\omega)$.

For the proof it will be important that $\gamma^k(t,x) > x$ for $(t,x) \in A$ as shown in Theorem 3.3. In particular $\gamma^k(t,x) \ge \gamma^k(t,c) > c$ for all $t \in [0,1)$. The proof is now given by induction in $\ell = 1,\dots,m$. The induction hypothesis is

$$\left(\tfrac{\hat{T}_\ell^{n,m}(t,x)}{n}, X^n_{\hat{T}_\ell^{n,m}(t,x)} \vee x\right) \to (T_\ell^m(t,x), \overline{Y}_{T_\ell^m(t,x)} \vee x).$$

$\boldsymbol{\ell = 1}$**:** 1. Case: N has no points in $(t,1]\times\overline{\mathbb{R}}\cap M_{\gamma^m(\cdot,x)}$, i.e. $N((t,1]\times\overline{\mathbb{R}}\cap M_{\gamma^m(\cdot,x)}) = 0$.

Then $(T_1^m(t,x), \overline{Y}_{T_1^m}(t,x) \vee x) = (1,x)$. Let $p \in \mathbb{N}$ be a fixed number. Since N has no points on the graph of $\gamma^m(\cdot,x)$ there exists $\varepsilon > 0$ with $N((t,t_p] \times \overline{\mathbb{R}} \cap \overline{M}_{\gamma^m(\cdot,x)-\varepsilon}) = 0$. Thus by our assumptions v_m^n converges uniformly on $[0,t_p]$ and thus $N_n((t,t_p] \times \overline{\mathbb{R}} \cap M_{v_m^n}) = 0$, i.e. $\frac{\hat{T}_1^{n,m}(t,x)}{n} \ge t_p$ for $n \ge n_0$. In conseqeunce $\lim \frac{\hat{T}_1^{n,m}(t,x)}{n} = 1$. Since $N(\{1\} \times (c,\infty]) = 0$ we obtain $X^n_{\hat{T}_1^{m,n}(t,x)} \vee x \to x$. This also is true for the further stopping times and thus finishes the proof.

2. Case: N has points in $(t,t_p]\times\overline{\mathbb{R}}\cap M_{\gamma^m(\cdot,x)}$ for some t_p. Let the number of these points be s. Since N has no points on the graph of $\gamma^m(\cdot,x)$ there exists $\varepsilon > 0$ such that N has no points in $\bigcup_{s\in(t,t_p]}\{s\} \times [\gamma^m(s,x)-\varepsilon, \gamma^m(s,x]+\varepsilon]$. By Proposition 3.13 in Resnick (1987) there exist for $n \ge n_0$ representations of the form

$$N_n(\cdot \cap (t,t_p] \times \overline{\mathbb{R}} \cap M_{\gamma^m(\cdot,x)-\varepsilon}) = \sum_{r=1}^s \delta_{(\frac{i_r}{n}, X^n_{i_r})}$$

with $i_1 < i_2 < \ldots < i_s$ and

$$N(\cdot \cap (t,t_p] \times \overline{\mathbb{R}} \cap M_{\gamma^m(\cdot,x)-\varepsilon}) = \sum_{r=1}^s \delta_{(\tau_{k_r}, Y_{k_r})}$$

with $\tau_{k_1} < \tau_{k_2} < \ldots < \tau_{k_s}$, such that the points converge. By uniforme convergence of v_n^m on $(t,t_p]$ it follows for $n \ge n_1$ that

$$\left(\tfrac{\hat{T}_1^{n,m}(t,x)}{n}, X^n_{\hat{T}_1^n}\right) = \left(\tfrac{i_1}{n}, X^n_{i_1}\right) \to (\tau_{k_1}, Y_{k_1}) = (T_1^m(t,x), \overline{Y}_{T_1^m(t,x)}).$$

The induction step $\ell-1 \to \ell$ uses similar arguments but is somewhat technical. For details we refer to [F] (2009, p. 60–61). $\square$

Let now $W_k^{n,m}(x)$ be the stopping thresholds for the m stopping of $X_1^n, \dots, X_n^n$ and the filtration $\mathcal{F}^n$ (see Section 2). The optimal *m-stopping curves* w.r.t. $\mathcal{F}^n$ are defined as follows. For $t \in [0, \frac{n-m+1}{n})$ and $x \in \overline{\mathbb{R}}$ let

$$u_n^m(t,x) := W_{\lfloor tn \rfloor}^{n,m}(x)$$

and $u_n^m(t,x) := W_{n-m+1}^{n,m}(x)$ for $t \in [\frac{n-m+1}{n}, 1]$.

More explicitly we have for $t \in [0, \frac{n-m+1}{n})$

$$\begin{aligned} u_n^m(t,x) &= \operatorname{ess\,sup}\left\{ E[X_{T_1}^n \vee \ldots \vee X_{T_m}^n \vee x \mid \mathcal{F}_{\lfloor tn \rfloor}^n] : \begin{array}{c} tn < T_1 < \cdots < T_m \leq n \\ \mathcal{F}^n\text{-stopping times} \end{array} \right\} \\ &= E[X^n_{T_1^{n,m}(t,x)} \vee \ldots \vee X^n_{T_m^{n,m}(t,x)} \vee x \mid \mathcal{F}^n_{\lfloor tn \rfloor}] \qquad P\text{-a.s.} \end{aligned} \tag{5.2}$$

The corresponding optimal m-stopping times are given by

$$\begin{aligned} T_1^{n,m}(t,x) &:= \min\left\{ tn < i \leq n-m+1 : u_n^{m-1}(\tfrac{i}{n}, X_i^n) > u_n^m(\tfrac{i}{n}, x) \right\}, \\ T_\ell^{n,m}(t,x) &:= \min\left\{ T_{\ell-1}^{n,m}(t,x) < i \leq n-m+\ell : \right. \\ &\qquad\qquad \left. u_n^{m-\ell}(\tfrac{i}{n}, X_i^n) > u_n^{m-\ell+1}(\tfrac{i}{n}, M_{\ell-1,i}^{n,m} \vee x) \right\} \end{aligned} \tag{5.3}$$

for $2 \leq \ell \leq m$, where $M_{j,i}^{n,m} := X^n_{T_j^{n,m}(t,x)} \chi_{\{T_j^{n,m}(t,x) \leq i\}}$.

$u_n^m(\cdot, x)$ is right continuous and piecewise curve in the space of random variables. We have the iterative representation (see Theorem 2.3)

$$u_n^m(t,x) = E\left[u_n^{m-1}\left(\frac{T_1^{n,m}(t,x)}{n}, X^n_{T_1^{n,m}(t,x)} \vee x \right) \Bigm| \mathcal{F}^n_{\lfloor tn \rfloor} \right] \quad P\text{-a.s.}$$

Further, u_n^m are monotone in the sense that for $0 \leq s \leq t \leq 1$

$$u_n^m(s,x) \;\geq\; E\left[u_n^m(t,x) \mid \mathcal{F}^n_{\lfloor sn \rfloor} \right] \quad P\text{-a.s.}$$

In the opposite direction we obtain for $0 \leq s \leq t \leq 1$

$$u_n^m(s,x) \leq E\big[\max_{s < \frac{i}{n} \leq t} u_n^{m-1}(\tfrac{i}{n}, X_i^n) \vee u_n^m(t,x) \mid \mathcal{F}^n_{\lfloor sn \rfloor} \big] \quad P\text{-a.s.} \tag{5.4}$$

This follows inductively from the recursive definition of the thresholds $W_\ell^m(x)$. We also need the following further conditions (for more details and motivation see [FR] (2009)):

(A) *Asymptotic independence condition*
For $0 \leq s < t \leq 1$

$$P\Big(\max_{s < \frac{i}{n} \leq t} X_i^n \leq x \mid \mathcal{F}^n_{\lfloor sn \rfloor} \Big) \xrightarrow{P} P\Big(\sup_{s < \tau_k \leq t} Y_k \leq x \Big) \quad \forall x \in (c, \infty)$$

(U) *Uniform integrability condition*
M_n^+, with $M_n := \max_{1\le i\le n} X_i^n$, is uniformly integrable and $E[\limsup_{n\to\infty} M_n^+] < \infty$.

(L) *Uniform integrability from below*
For some sequence $(v_n)_{n\in\mathbb{N}}$ of monotonically nonincreasing functions $v_n : [0,1] \to \mathbb{R}\cup\{-\infty\}$ with $v_n \to u$ pointwise, for all $t \in [0,1)$ and the corresponding threshold stopping times

$$\hat{T}_n(t) := \min\{tn < i \le n : X_i^n > v_n(\tfrac{i}{n})\}$$

holds

$$\lim_{s\uparrow 1} \limsup_{n\to\infty} E[X^n_{\hat{T}_n(t)} \chi_{\{\hat{T}_n(t) > sn\}}] = 0. \tag{5.5}$$

An extension of (L) is

(L^m) For $m \in \mathbb{N}$ there exists some sequence of monotonically nonincreasing functions $v_n^m : [0,1] \to \overline{\mathbb{R}}$ such that $v_n^m \to \gamma^m(\cdot, -\infty)$ pointwise and further the corresponding threshold stopping times

$$\hat{T}_1^{n,m}(t) := \min\{tn < i \le n - m + 1 : X_i^n > v_n^m(\tfrac{i}{n})\}$$

satisfy

$$\lim_{s\uparrow 1} \limsup_{n\to\infty} E[X^n_{\hat{T}_1^{n,m}(t)} \chi_{\{\hat{T}_1^{n,m}(t) > sn\}}] = 0.$$

Condition (L^m) in combination with (U) implies uniform integrability of $(X^n_{\hat{T}_1^{n,m}(t)})_{n\in\mathbb{N}}$. Denote

$$T_\ell^{n,m} := T_\ell^{n,m}(0,c) \quad \text{and} \quad T_\ell^m := T_\ell^m(0,c).$$

Theorem 5.2 (Approximation of m-stopping problems) *Assume that $N_n \xrightarrow{d} N$ on $[0,1]\times(\overline{\mathbb{R}}\setminus\{c\})$ and also assume conditions (A) and (U). In case $c = -\infty$ also assume the uniform integrability condition (L^m).*

a) For all $(t,x) \in [0,1]\times[c,\infty)$ holds

$$u_n^m(t,x) \xrightarrow{P} u^m(t,x).$$

If $c \in \mathbb{R}$ assume $X_n^n \xrightarrow{L^1} c$. Then we have in particluar

$$E\big[X^n_{T_1^{n,m}} \vee \cdots \vee X^n_{T_m^{n,m}}\big] \to u^m(0). \tag{5.6}$$

b) *In case* $(X_i^n)_{1\le i\le n}$ *are independent random variables and if for* $c\in\mathbb{R}$ *we assume that* $\mu(M_{\gamma^m})=\infty$ *or* $X_{n-i}^n \xrightarrow{P} c$ *for* $i=0,\dots,m-1$, *then we obtain*

$$\left(\frac{T_\ell^{n,m}}{n}, X^n_{T_\ell^{n,m}}\right)_{1\le\ell\le m} \xrightarrow{d} (T_\ell^m, \overline{Y}_{T_\ell^m}\vee c)_{1\le\ell\le m}.$$

c) *If* $c\in\mathbb{R}$ *and* $X_n^n \xrightarrow{L^1} c$, *then*

$$\hat{T}_1^{n,m} := \min\left\{1\le i\le n-m+1 : X_i^n > \gamma^m(\tfrac{i}{n}, c)\right\},$$
$$\hat{T}_\ell^{n,m} := \min\left\{\hat{T}_{\ell-1}^{n,m} < i\le n-m+\ell : X_i^n > \gamma^{m-\ell+1}(\tfrac{i}{n}, X^n_{\hat{T}_{\ell-1}^{n,m}}\vee c)\right\},\quad 2\le\ell\le m,$$

defines an asymptotically optimal sequence of m*-stopping times, i.e. convergence as in* (5.6) *holds for these stopping times. In case* $c=-\infty$

$$\hat{T}_1^{n,m} := \min\left\{1\le i\le n-m+1 : X_i^n > v_n^m(\tfrac{i}{n})\right\},$$
$$\hat{T}_\ell^{n,m} := \min\left\{\hat{T}_{\ell-1}^{n,m} < i\le n-m+\ell : X_i^n > \gamma^{m-\ell+1}(\tfrac{i}{n}, X^n_{\hat{T}_{\ell-1}^{n,m}})\right\},\quad 2\le\ell\le m,$$

are asymptotically optimal stopping times, where v_n^m *are the threshold functions from condition* (L^m).

Proof: Since we use point process convergence on $[0,1]\times(\overline{\mathbb{R}}\setminus\{c\})$ and canonical filtrations we can apply the Skorohod theorem and thus we assume P-a.s. convergence of the point processes.

a) Consider at first the case $c\in\mathbb{R}$. Let $t\in[0,1)$ be a fixed element. We introduce at first discrete majorizing stopping problems. For $m\ge 1$ and $k>m$ define the discrete time points

$$a_i^k := \left(1-\frac{i}{k}\right)t + \frac{i}{k}1,\quad 0\le i\le k,$$

and discrete time random variables

$$X_i^{n,k} := \max_{\frac{j}{n}\in(a_{i-1}^k, a_i^k]} X_j^n \vee c \quad \text{for } 1\le i\le k,$$

and consider the filtration $\mathcal{F}^{n,k} := (\mathcal{F}_i^{n,k})_{0\le i\le k}$ with $\mathcal{F}_i^{n,k} := \mathcal{F}^n_{\lfloor a_i^k n\rfloor}$. The corresponding m-stopping curves are given inductively for $m\ge1$ by backwards induction for $i=k,\dots,0$ by

$$ {}^mW_{k-m+1}^{n,k}(x) := x,$$
$$ {}^mW_i^{n,k}(x) := E[\,{}^{m-1}W_{i+1}^{n,k}(X_{i+1}^{n,k})\vee {}^mW_{i+1}^{n,k}(x)\mid \mathcal{F}_i^{n,k}] \quad \text{for } i=k-m,\dots,0.$$

These stopping problems majorize the original m-stopping problem.

$$ {}^mW_0^{n,k}(x)$$

$$
\begin{aligned}
&= \operatorname{ess\,sup}\{E[X^{n,k}_{T'_1} \vee \ldots \vee X^{n,k}_{T'_m} \vee x : \mathcal{F}^{n,k}_0] : \\
&\qquad\qquad 0 < T'_1 < \ldots < T'_m \le k \ \mathcal{F}^{n,k}\text{-stopping times}\} \\
&\overset{(*)}{=} \operatorname{ess\,sup}\{E[X^{n,k}_{T'_1} \vee \ldots \vee X^{n,k}_{T'_m} \vee x : \mathcal{F}^{n,k}_0] : \\
&\qquad\qquad 0 < T'_1 \le \ldots \le T'_m \le k \ \mathcal{F}^{n,k}\text{-stopping times}\} \\
&\ge \operatorname{ess\,sup}\{E[X^{n}_{T_1} \vee \ldots \vee X^{n}_{T_m} \vee x \mid \mathcal{F}^{n}_{\lfloor tn \rfloor}] : \\
&\qquad\qquad tn < T_1 < \ldots < T_m \le n \ \mathcal{F}^{n}\text{-stopping times}\} \\
&= u^m_n(t,x) \quad P\text{-a.s.},
\end{aligned}
$$

since for all $\mathcal{F}^n$-stopping times $tn < T_1 < \ldots < T_m \le n$ it holds that $T'_i := \lceil \frac{1}{1-t}(\frac{T_i}{n} - t)k \rceil > 0$ are $\mathcal{F}^{n,k}$-stopping times with $a^k_{T'_i - 1} < \frac{T_i}{n} \le a^k_{T'_i}$, thus $X^{n,k}_{T'_i} \ge X^n_{T_i}$. For the proof of $(*)$ define for $\mathcal{F}^{n,k}$-stopping times $0 < T'_1 \le \ldots \le T'_m \le k$ the $\mathcal{F}^{n,k}$-stopping times $0 < T^*_1 < \ldots < T^*_m \le k$ by

$$
\begin{aligned}
T^*_1 &:= T'_1 \wedge (k - m + 1), \\
T^*_\ell &:= \Big((T'_\ell + 1)\chi_{\{T^*_{\ell-1} = T'_\ell\}} + T'_\ell \chi_{\{T^*_{\ell-1} < T'_\ell\}} \Big) \wedge (k - m + \ell), \quad \ell = 2, \ldots, m.
\end{aligned}
$$

We will prove convergence as $n \to \infty$ to the stopping problem of

$$
Y^k_i := \sup_{\tau_l \in (a^k_{i-1}, a^k_i]} Y_l \vee c \quad \text{for } 1 \le i \le k,
$$

with filtrations $\mathcal{A}^k := (\mathcal{A}^k_i)_{1 \le i \le k}$, $\mathcal{A}^k_i := \mathcal{A}_{a^k_i}$ and optimal thresholds

$$
\begin{aligned}
{}^m u^k_{k-m+1}(x) &:= x, \\
{}^m u^k_i(x) &:= E[\, {}^{m-1}u^k_{i+1}(Y^k_{i+1}) \vee {}^m u^k_{i+1}(x)] \quad \text{for } i = k - m, \ldots, 0.
\end{aligned}
$$

By definition for $i \le k - m$ holds

$$
\begin{aligned}
{}^m u^k_i(x) &= V(\, {}^{m-1}u^k_{i+1}(Y^k_{i+1}) \vee x, \ldots, {}^{m-1}u^k_{k-m+1}(Y^k_{k-m+1}) \vee x) \\
&= \sup\{E[\, {}^{m-1}u^k_T(Y^k_T) \vee x] : i < T \le k - m + 1 \ \mathcal{A}^k\text{-stopping times}\} \\
&= {}^m u^k(a^k_i, x),
\end{aligned}
$$

where ${}^m u^k(\cdot, x)$ are the optimal stopping curves of the processes

$$
{}^m N^k := \sum_{i=1}^{k-m+1} \delta_{(a^k_i, \, {}^{m-1}u^k_i(Y^k_i))} = \sum_{i=1}^{k-m+1} \delta_{(a^k_i, \, {}^{m-1}u^k(a^k_i, Y^k_i))}
$$

at guarantee value x.

At first we establish that for any i the random variable Y^k_{i+1} is independent of the σ-algebra $\mathcal{F}^k_i := \sigma(\bigcup_{n \in \mathbb{N}} \mathcal{F}^{n,k}_i)$.

For the proof note that by condition (A)

$$
P(X^{n,k}_{i+1} \in \cdot \mid \mathcal{F}^{n,k}_i) \xrightarrow{P} P(Y^k_{i+1} \in \cdot).
$$

Thus we obtain by the continuous mapping theorem that for any continuous $f : \overline{\mathbb{R}} \to [0,1]$ we have

$$P(f(X_{i+1}^{n,k}) \in \cdot \mid \mathcal{F}_i^{n,k}) \xrightarrow{P} P(f(Y_{i+1}^k) \in \cdot).$$

This implies using uniform integrability that

$$E[f(X_{i+1}^{n,k}) \mid \mathcal{F}_i^{n,k}] \xrightarrow{L^1} E[f(Y_{i+1}^k)].$$

On the other hand by point process convergence it holds that $X_{i+1}^{n,k} \to Y_{i+1}^k P$-a.s. and thus also $f(X_{i+1}^{n,k}) \xrightarrow{L^1} f(Y_{i+1}^k)$. This implies L^1-convergence of conditional expectations:

$$E[f(X_{i+1}^{n,k}) \mid \mathcal{F}_i^{n,k}] \xrightarrow{L^1} E[f(Y_{i+1}^k) \mid \mathcal{F}_i^k].$$

In consequence we obtain $E[f(Y_{i+1}^k)] = E[f(Y_{i+1}^k) \mid \mathcal{F}_i^k]$ P-a.s. for all continuous functions $f : \overline{\mathbb{R}} \to [0,1]$, and thus independence of $\mathcal{F}_i^k$ and $\sigma(Y_{i+1}^k)$.

The next point to extablish now is proved by induction in m. The induction hypothesis is

1.) For all $k > m$, $x \in [c, \infty)$ and $i = k - m + 1, \dots, 0$

$$^m W_i^{n,k}(x) \xrightarrow{P} {}^m u_i^k(x), \quad n \to \infty.$$

2.) For all $s \in [t, 1]$ and all $x \in [c, \infty)$ we further have

$$^m u^k(s, x) \to u^m(s, x), \quad k \to \infty.$$

We do the induction step for $m - 1 \to m$: Assertion 1.) we shall prove by backwards induction on i: For $i = k - m + 1$ the assertion is trivial. We now consider the induction step from $i + 1$ to i: From the induction hypothesis we know that

$$^{m-1} W_{i+1}^{n,k}(x) \xrightarrow{P} {}^{m-1} u_{i+1}^k(x), \quad n \to \infty,$$

for all $x \in [c, \infty)$. From this, the monotonicity of $^{m-1} W_{i+1}^{n,k}(x)$ in x and the continuity of $^{m-1} u_{i+1}^k(x)$ in x we can conclude that

$$^{m-1} W_{i+1}^{n,k}(X_{i+1}^{n,k}) \xrightarrow{P} {}^{m-1} u_{i+1}^k(Y_{i+1}^k), \quad n \to \infty.$$

For details see [F] (2009). By the induction hypothesis for i we also know that

$$^m W_{i+1}^{n,k}(x) \xrightarrow{P} {}^m u_{i+1}^k(x), \quad n \to \infty,$$

for $x \in [c, \infty)$, implying

$$^{m-1} W_{i+1}^{n,k}(X_{i+1}^{n,k}) \vee {}^m W_{i+1}^{n,k}(x) \xrightarrow{L^1} {}^{m-1} u_{i+1}^k(Y_{i+1}^k) \vee {}^m u_{i+1}^k(x), \quad n \to \infty.$$

From this we get

$$E[\,{}^{m-1}W^{n,k}_{i+1}(X^{n,k}_{i+1}) \vee {}^{m}W^{n,k}_{i+1}(x) \,|\, \mathcal{F}^{n,k}_i] \xrightarrow{L^1} E[\,{}^{m-1}u^k_{i+1}(Y^k_{i+1}) \vee {}^{m}u^k_{i+1}(x) \,|\, \mathcal{F}^k_i]$$

as $n \to \infty$. The expression on the left-hand side equals ${}^{m}W^{n,k}_i(x)$, and since $\sigma(Y^k_{i+1})$ and $\mathcal{F}^k_i$ are independent as shown above, the right-hand side equals ${}^{m}u^k_i(x)$. This completes the induction on i and the proof of assertion 1.).

For the proof of assertion 2.) observe that the process $\sum_{i=1}^k \delta_{(a^k_i, Y^k_i)}$ converges on $[t,1] \times (c,\infty]$ to $N = \sum_j \delta_{(\tau_j, Y_j)}$. Further, by induction hypothesis we have uniform convergence of ${}^{m-1}u^k(s,x)$ to $u^{m-1}(s,x)$ as $k \to \infty$. From this we obtain convergence of the transformed point processes

$${}^{m}N^k \;=\; \sum_{i=1}^k \delta_{(a^k_i,\, {}^{m-1}u^{\,k}(a^k_i, Y^k_i))} \;\xrightarrow{d}\; N^m \;=\; \sum_j \delta_{(\tau_j, u^{m-1}(\tau_j, Y_j))}, \quad k \to \infty$$

on $M_{u^{m-1}} \cap [t,1] \times \overline{\mathbb{R}}$ and thus convergence of the optimal stopping curves of these processes, which proves 2.).

Based on 1.) and 2.) we obtain the estimate

$$\begin{aligned} &P\big(u^m_n(t,x) \geq u^m(t,x) + \varepsilon\big) \\ &\quad \leq P\big(\,{}^{m}W^{n,k}_0(x) \geq \underbrace{{}^{m}u^k(t,x)}_{{}^{m}u^k_0(x)} + \tfrac{\varepsilon}{2}\big) + P\big(u^m(t,x) \leq {}^{m}u^k(t,x) - \tfrac{\varepsilon}{2}\big). \end{aligned}$$

The right-hand side converges for $n \to \infty$ and $k \to \infty$ to 0. Thus we have shown

$$\lim_{n\to\infty} P(u^m_n(t,x) \geq u^m(t,x) + \varepsilon) = 0.$$

To obtain convergence in probability we next establish that $\liminf_{n\to\infty} Eu^m_n(t,x) \geq u^m(t,x)$. This however is implied by the inequality

$$Eu^m_n(t,x) \geq E[X^n_{T_1} \vee \ldots \vee X^n_{T_m} \vee x]$$

holding true for all $\mathcal{F}^n$-stopping times $tn < T_1 < \ldots < T_m \leq n$, and in particluar for

$$\begin{aligned} \hat{T}^{n,m}_1(t,x) \; &:= \min\{tn < i \leq n - m + 1 : X^n_i > \gamma^m(\tfrac{i}{n}, x)\}, \\ \hat{T}^{n,m}_\ell(t,x) \; &:= \min\{\hat{T}^{n,m}_{\ell-1}(t,x) < i \leq n - m + \ell : X^n_i > \gamma^{m-\ell+1}(\tfrac{i}{n}, X^n_{\hat{T}^{n,m}_{\ell-1}(t,x)} \vee x)\} \end{aligned}$$

for $2 \leq \ell \leq m$. Proposition 5.1 then implies the above statement.

For $c = -\infty$ we obtain similarly the convergence $u^m_n(t,x) \xrightarrow{P} u^m(t,x)$ for $x > -\infty$. Then the convergence of $u^m_n(t,-\infty) \xrightarrow{P} u^m(t)$ results as follows:

$$u^m_n(t,-\infty) \;\leq\; u^m_n(t,x) \xrightarrow{P} u^m(t,x) \downarrow u^m(t) \quad \text{ as } x \downarrow -\infty.$$

This implies that $\lim_{n\to\infty} P(u_n^m(t,-\infty) \geq u^m(t)+\varepsilon) = 0$ for all $\varepsilon > 0$. Let $\hat{T}_1^{n,m}(t)$ be the stopping times from condition (L^m) and let

$$\hat{T}_\ell^{n,m}(t) := \min\{\hat{T}_{\ell-1}^{n,m}(t) < i \leq n-m+\ell : X_i^n > \gamma^{m-\ell+1}(\tfrac{i}{n}, X^n_{\hat{T}_{\ell-1}^{n,m}(t)})\}$$

for $2 \leq \ell \leq m$. Then we obtain by Proposition 5.1 and uniform integrability of $(X^n_{\hat{T}_1^{n,m}(t)})_{n\in\mathbb{N}}$ that

$$\begin{aligned} Eu_n^m(t,-\infty) &\geq E[X^n_{\hat{T}_1^{n,m}(t)} \vee \ldots \vee X^n_{\hat{T}_m^{n,m}(t)}] \\ &\xrightarrow{n\to\infty} E[\overline{Y}_{T_1^m(t,-\infty)} \vee \ldots \vee \overline{Y}_{T_m^m(t,-\infty)}] = u^m(t). \end{aligned}$$

Thus $\liminf_{n\to\infty} Eu_n^m(t,-\infty) \geq u^m(t)$. As consequence we obtain $u_n^m(t,-\infty) \xrightarrow{P} u^m(t)$ which was to be shown.

b) For the proof ob b) see [F] (2009).

c) For $c = -\infty$ we obtain the statement using uniform integrability and Proposition 5.1. For $c \in \mathbb{R}$ holds

$$\begin{aligned} &E\left[X^n_{\hat{T}_1^{n,m}} \vee \ldots \vee X^n_{\hat{T}_m^{n,m}}\right] \\ &\quad = E\left[X^n_{\hat{T}_1^{n,m}} \vee \ldots \vee X^n_{\hat{T}_m^{n,m}} \vee c\right] - \int\limits_{\{X^n_{\hat{T}_1^{n,m}} \vee \ldots \vee X^n_{\hat{T}_m^{n,m}} < c\}} \left(c - X^n_{\hat{T}_1^{n,m}} \vee \ldots \vee X^n_{\hat{T}_m^{n,m}}\right) dP. \end{aligned}$$

The first term converges by Proposition 5.1 to the stated limit. The modulus of the second term can be estimated from above by

$$\int\limits_{\{X^n_{\hat{T}_m^{n,m}} < c\}} \left(c - X^n_{\hat{T}_m^{n,m}}\right) dP \leq \int\limits_{\{X_n^n < c\}} \left(c - X_n^n\right) dP \leq E|X_n^n - c| \to 0. \qquad \square$$

Remark 5.1 *The reason for restricting in (b) to independent sequences is the necessity to give estimates of $u_n(t,x)$ from above (up the case $m=1$ in [F] (2009)). In the dependent case this amounts to* (5.4)*. For $m \geq 2$ in contrast to the case $m = 1$ one has to consider the terms $\max_{s<\frac{i}{n}\leq t} u_n^{m-1}(\frac{i}{n}, X_i^n)$. It seems however difficult to establish the necessary point process convergence of $\sum_{i=1}^n \delta_{(\frac{i}{n}, u_n^{m-1}(\frac{i}{n}, X_i))}$.*

6 Optimal m-stopping of iid sequences with discount and observation costs

As application we study in this section the optimal m-stopping of iid sequences with discount and observation costs. In the case $m = 1$ this problem has been considered in various degree of generality in Kennedy and Kertz (1990, 1991), [KR] (2000b), and [FR] (2009).

Let $(Z_i)_{i\in\mathbb{N}}$ be an iid sequence with d.f. F in the domain of attraction of an extreme value distribution G, thus for some constants $a_n > 0$, $b_n \in \mathbb{R}$

$$n(1 - F(a_n x + b_n)) \to -\log G(x), \quad x \in \mathbb{R}. \tag{6.1}$$

Consider $X_i = c_i Z_i + d_i$ the sequence with discount and observation factors, $c_i > 0$, $d_i \in \mathbb{R}$ and both sequences monotonically nondecreasing or nonincreasing. For convergence of the corresponding imbedded point processes

$$\hat{N}_n = \sum_{i=1}^{n} \delta_{\left(\frac{i}{n}, \frac{X_i - \hat{b}_n}{\hat{a}_n}\right)} \tag{6.2}$$

the following choices of $\hat{a}_n$, $\hat{b}_n$ turn out to be appropriate:

$$\begin{aligned} &\hat{a}_n := c_n a_n,\ \hat{b}_n := 0 && \text{for } F \in D(\Phi_\alpha) \text{ or } F \in D(\Psi_\alpha), \\ &\hat{a}_n := c_n a_n,\ \hat{b}_n := c_n b_n + d_n && \text{for } F \in D(\Lambda), \end{aligned} \tag{6.3}$$

where Φ_α, Ψ_α, Λ are the Fréchet, Weibull, and Gumbel distributions and a_n, b_n are the corresponding normalizations in (6.1). We give further conditions on c_i, d_i to establish point process convergence in (6.2). Related conditions are given in de Haan and Verkaade (1987) in the treatment of iid sequences with trends resp. in [KR] (2000b).

In the following c denotes some general constant and not as before the guarantee value. The guarantee value of N is in case Φ_α given by 0 and in cases Ψ_α, Λ given generally by $-\infty$. This application shows in particular the importance of treating the case with lower boundary $-\infty$ as in sections 2 and 3 of this paper resp. in [FR] (2009). We state the optimality results for all three cases.

We first consider the case of Fréchet limits.

Theorem 6.1 *Let $F \in D(\Phi_\alpha)$ with $\alpha > 1$ and $F(0) = 0$ (i.e. $Z_i > 0$ P-a.s.). We assume that $b_n = 0$ and also convergence*

$$\frac{d_n}{c_n a_n} \to d, \quad \frac{c_{\lfloor tn \rfloor}}{c_n} \to t^c \quad \forall t \in [0,1]$$

with constants c, $d \in \mathbb{R}$, as well that c_n does not converge to 0.

a) If $c > -\frac{1}{\alpha}$ and if the function $R : (d,\infty) \to \mathbb{R}$,

$$R(x) \ := \ x + \frac{\alpha}{\alpha - 1}\,\frac{1}{1 + c\alpha}\,(x - d)^{-\alpha+1}, \qquad x \in (d,\infty) \tag{6.4}$$

has no zero point, then

$$\frac{E[X_{T_1^{n,m}} \vee \ldots \vee X_{T_m^{n,m}}]}{\hat{a}_n} \to u^m(0) > 0, \tag{6.5}$$

where $u^m(t)$ is the m-stopping curve of the Poisson process $\hat{N}$ with intensity function

$$\hat{G}(t,y) = t^{c\alpha}(y - dt^{c+\frac{1}{\alpha}})^{-\alpha} = H\left(\frac{y}{v(t)}\right)\frac{v'(t)}{v(t)} \qquad on\ M_{\hat{f}}.$$

Here $v(t) := t^{c+\frac{1}{\alpha}}$, $H(x) := \frac{\alpha}{\alpha c+1}(x-d)^{-\alpha}$ and $\hat{f}(t) := dt^{c+\frac{1}{\alpha}}$.

b) Let $\gamma^1, \dots, \gamma^m$ be the functions defined in (3.17) *for $\hat{N}$. Then*

$$\hat{T}_1^{n,m} := \min\{\, 1 \le i \le n-m+1 : X_i > \hat{a}_n \gamma^m(\tfrac{i}{n}, d)\},$$
$$\hat{T}_\ell^{n,m} := \min\{\, \hat{T}_{\ell-1}^{n,m} < i \le n-m+\ell : X_i > \hat{a}_n \gamma^{m-\ell+1}(\tfrac{i}{n}, \left(\tfrac{1}{\hat{a}_n} X_{\hat{T}_{\ell-1}^{n,m}}\right) \vee d)\}$$

for $2 \le \ell \le m$ are asmptotically optimal sequences of m-stopping times, i.e. the limit in (6.5) *is attained also for these sequences.*

The next result concerns the Weibull limit case.

Theorem 6.2 *Let $F \in D(\Psi_\alpha)$ with $\alpha > 0$ and $F(0) = 1$ (i.e. $Z_i \le 0$ P-a.s.). Further let $a_n \downarrow 0$ and $b_n = 0$, and*

$$\frac{d_n}{c_n a_n} \to d, \qquad \frac{c_{\lfloor tn \rfloor}}{c_n} \to t^c, \qquad \forall t \in [0,1]$$

for constants c, $d \in \mathbb{R}$. If $d_n > 0$, then assume that either $(d_n)_{n\in\mathbb{N}}$ is monotonically nondecreasing or $c_n a_n$ does not converge to 0.

a) If $c < \frac{1}{\alpha}$ and $d \le 0$, then it holds

$$\frac{E[X_{T_1^{n,m}} \vee \dots \vee X_{T_m^{n,m}}]}{\hat{a}_n} \to u_{c,d}^m(0) < 0. \tag{6.6}$$

b) If $c > \frac{1}{\alpha}$ and the function $R : \mathbb{R} \to \mathbb{R}$

$$R(x) \ := \ \begin{cases} x, & falls\ x \ge d, \\ x \ - \ \frac{\alpha}{\alpha+1}\,\frac{1}{1-c\alpha}\,(-x+d)^{\alpha+1}, & falls\ x < d, \end{cases} \tag{6.7}$$

has no zero point then (6.6) *holds with $u_{c,d}^m(0) > 0$. Here $u_{c,d}^m(t)$ is the m-stopping curve of the Poisson process $\hat{N} = \hat{N}_{c,d}$ with intensity function $\hat{G} = G_{c,d}$ defined in* (4.9). *$\gamma_{c,d}^m$ are the corresponding functions for $\hat{N}_{c.d}$ defined in* (3.17) *and* (3.18).

c) Let (w_n) be an increasing sequence $w_n < 0$ such that $n(1 - F(w_n)) \to \frac{\alpha+1}{\alpha}$ (e.g. $w_n = -\left(\frac{\alpha+1}{\alpha}\right)^{\frac{1}{\alpha}} a_n$). Define functions v_n^m by

$$v_n^m(t) := \frac{\gamma_{c,0}^m(t)}{u_{0,0}(t)} \frac{w_{\lfloor (1-t)n \rfloor}}{a_n} + \gamma_{c,d}^m(t) - \gamma_{c,0}^m(t),$$

where by (4.10) $\gamma^m_{c,0}(t) = -\Phi^{m-1}(r_m)u_{c,0}(t)$. The m-stopping times defined by

$$\hat{T}^{n,m}_1 := \min\{\, 1 \le i \le n-m+1 : X_i > \hat{a}_n v^m_n(\tfrac{i}{n})\},$$
$$\hat{T}^{n,m}_\ell := \min\{\, \hat{T}^{n,m}_{\ell-1} < i \le n-m+\ell : X_i > \hat{a}_n \gamma^{m-\ell+1}_{c,d}(\tfrac{i}{n}, \tfrac{1}{\hat{a}_n} X_{\hat{T}^{n,m}_{\ell-1}})\}$$

for $2 \le \ell \le m$, are asymptotically optimal, i.e. convergence as in (6.6) does also hold for them.

The final result concerns the Gumbel case.

Theorem 6.3 *Let $F \in D(\Lambda)$ and assume*

$$\frac{b_n}{a_n}\left(1 - \frac{c_{\lfloor tn \rfloor}}{c_n}\right) \to c\log(t), \qquad \frac{d_n - d_{\lfloor tn \rfloor}}{c_n a_n} \to d\log(t) \quad \forall t \in [0,1]$$

for some constants c, $d \in \mathbb{R}$. Assume also that $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ monotonically nondecreasing.

a) If $c + d < 1$, then

$$\frac{E[X_{T^{n,m}_1} \vee \ldots \vee X_{T^{n,m}_m}] - \hat{b}_n}{\hat{a}_n} \to u^m(0), \tag{6.8}$$

where $u^m(t)$ is the m-stopping curve of the Poisson process $\hat{N}$ with intensity function

$$\hat{G}(t,y) = e^{-y} t^{-(c+d)} \qquad \text{on } [0,1]\times\mathbb{R}.$$

b) Let $\gamma^1, \ldots, \gamma^m$ be the functions for $\hat{N}$ defined in (3.17) and (3.18), let $(w_n)_{n\in\mathbb{N}}$ be an increasing sequence with $\lim_{n\to\infty} n(1-F(w_n)) = 1$ (e.g. $w_n := b_n$). Let v^m_n be defined as

$$v^m_n(t) := \frac{w_{\lfloor (1-t)n \rfloor} - b_n}{a_n} + \gamma^m(t) - \log(1-t).$$

Then

$$\hat{T}^{n,m}_1 := \min\{\, 1 \le i \le n-m+1 : X_i > \hat{a}_n v^m_n(\tfrac{i}{n}) + \hat{b}_n\},$$
$$\hat{T}^{n,m}_\ell := \min\{\, \hat{T}^{n,m}_{\ell-1} < i \le n-m+\ell : X_i > \hat{a}_n \gamma^{m-\ell+1}\left(\tfrac{i}{n}, \tfrac{X_{\hat{T}^{n,m}_{\ell-1}} - \hat{b}_n}{\hat{a}_n}\right) + \hat{b}_n\}$$

define an asymptotic optimal sequence of m-stopping times, i.e. convergence as in (6.8) holds for them.

Proof: The proof can be given similarly to the proof of Theorems 5.1–5.3 in [FR] (2009) in the case $m = 1$ using the approximation Theorem 5.2. To establish the uniform integrability condition in case $F \in D(\Psi_\alpha)$ essential use is made of the limit relation (see (4.13) in Example 4.1)

$$\lim_{t \uparrow 1} \gamma^m_{c,d}(t) - \gamma^m_{c,0}(t) = d.$$

In case $F \in D(\Lambda)$ we make essential use of $\lim_{t\uparrow 1} \gamma^m(t) - \log(1-t) = \Phi^{m-1}(r_m)$ (using $\gamma^m(t) = \Phi^{m-1}(r_m) + \log\left(\frac{1}{1-(c+d)}(1 - t^{1-(c+d)})\right)$ with constants r_m and functions Φ^m as defined in the third case in (4.8). □

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Andreas Faller
Mathematical Stochastics
Eckerstr. 1
79104 Freiburg
Germany

Ludger Rüschendorf
Mathematical Stochastics
Eckerstr. 1
79104 Freiburg
Germany
ruschen@stochastik.uni-freiburg.de