# Comparison of multivariate risks and positive dependence

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**Abstract**

In this paper we extend some recent results on the comparison of multivariate risk vectors w.r.t. supermodular and related orderings. We introduce a dependence notion called 'weakly conditional increasing in sequence order' that allows to conclude that 'more dependent' vectors in this ordering are also comparable w.r.t. the supermodular ordering. At the same time this ordering allows to compare two risks w.r.t. the directionally convex order if the marginals increase convexly. We further state comparison criteria w.r.t. the directionally convex order for some classes of risk vectors which are modelled by functional influence factors. Finally we discuss Fréchet-bounds w.r.t. $\Delta$-monotone functions when multivariate marginals are given. It turns out that comonotone vectors in the case of multivariate marginals no longer yield necessarily the largest risks but even may in some cases be vectors which minimize risk.

*Keywords:* supermodular ordering, Fréchet-bounds, positive dependence, risk vectors

## 1 Introduction

In a large number of recent papers it has been shown that the methods and tools of stochastic ordering and construction of probabilities with given marginals are of relevance for the modelling of multivariate portfolios and bounding functions of dependent risks, like the value at risk, the expected excess of loss and other financial derivatives and risk measures. (See Embrechts et al. (2003) and references in that paper.) A comprehensive survey of this field is given in the recent book of Müller and Stoyan (2002). The stochastic comparison of risks (random vectors) w.r.t. supermodular ordering or the related orderings by directionally convex functions resp. $\Delta$-monotone functions is of particular interest in many applications. One interesting type of question is to identify large function classes which allow to conclude that positive (negative) dependent random vectors are more (less) risky than independent vectors w.r.t. these functions. More generally it is of interest to establish general conditions for the comparison of risk of two 'models' $P$ and $Q$. A natural aim is to state some results of the form: More 'positive dependence' implies 'more risk' in the case of identical marginals and more positive dependence plus convex increase of the marginals also implies more risk. In this paper we state several results in this direction.

For functions $f : \mathbb{R}^n \to \mathbb{R}^1$ define the difference operator $\Delta_i^\varepsilon$, $\varepsilon > 0$, $1 \le i \le n$ by

$$\Delta_i^\varepsilon f(x) = f(x+\varepsilon e_i) - f(x) \tag{1.1}$$

where $e_i$ is the $i$-th unit vector. Then $f$ is called

a) *supermodular* if for all $1 \le i < j \le n$ and $\varepsilon, \delta > 0$

$$\Delta_i^\varepsilon \Delta_j^\delta f(x) \ge 0 \quad \text{for all } x \tag{1.2}$$

b) *directionally convex* if (1.2) holds for all $1 \le i \le j \le n$

c) $\Delta$-*monotone* if for all $J = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ and $\varepsilon_1, \dots, \varepsilon_k > 0$ holds

$$\Delta_{i_1}^{\varepsilon_1} \dots \Delta_{i_k}^{\varepsilon_k} f(x) \ge 0. \tag{1.3}$$

Let $\mathcal{F}^{sm}, \mathcal{F}^{dcx}, \mathcal{F}^{\Delta}$ denote the classes of supermodular, directionally convex and $\Delta$-monotone functions and $\le_{sm}, \le_{dcx}, \le_\Delta$ the induced integral stochastic orders on the class of probability measures. Then $\mathcal{F}^{dcx} \subset \mathcal{F}^{sm}$ and $\mathcal{F}^{\Delta} \subset \mathcal{F}^{sm}$ and for differentiable functions $f$ one obtains:

$$f \in \mathcal{F}^{sm} \text{ iff } \frac{\partial^2}{\partial x_i \partial x_j} f \ge 0 \quad \text{for } i < j \tag{1.4}$$

$$f \in \mathcal{F}^{dcx} \text{ iff (1.4) holds for } i \le j$$

$$f \in \mathcal{F}^{\Delta} \text{ iff } \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}} \ge 0 \text{ for all } k \le n \text{ and } i_1 < \dots < i_k. \tag{1.5}$$

While comparison of $P$ and $Q$ w.r.t. the largest class $\mathcal{F}^{sm}$ is restricted to the case of identical marginals $P_i = Q_i$, the comparison w.r.t. the smaller class $\mathcal{F}^{\Delta}$ allows stochastically increasing marginals $P_i \le_{st} Q_i$ and the comparison w.r.t. $\mathcal{F}^{dcx}$ allows convexly increasing marginals $P_i \le_{cx} Q_i$. The most easy to apply order is $\le_\Delta$. It is equivalent to the simple upper orthant order defined for $P, Q \in M^1(\mathbb{R}^n)$ – the class of all probability measures on $\mathbb{R}^n$ – by

$$P \le_{uo} Q \quad \text{iff} \quad P([x,\infty]) \le Q([x,\infty]) \tag{1.6}$$

for all $x \in \mathbb{R}^n$ (Rüschendorf (1980), in the following abbreviated as Rü (1980)). A basic result for $\leq_{sm}$ is the *Lorentz theorem* due to Tchen (1980): For any random vector $X$ with df $F$ and one dimensional marginals $F_1, \dots, F_n$ holds:

$$X \leq_{sm} (F_1^{-1}(U), \dots, F_n^{-1}(U)), \tag{1.7}$$

where $U$ is uniformly distributed on $[0,1]$. The so called *comonotone vector* $(F_1^{-1}(U), \dots, F_n^{-1}(U))$ therefore represents in many respects the riskiest portfolio vector. In particular (1.7) implies for the combined portfolio

$$\sum_{i=1}^n X_i \leq_{cx} \sum_{i=1}^n F_i^{-1}(U), \tag{1.8}$$

where $\leq_{cx}$ is the convex ordering which compares expectations of all convex functions. (1.8) was first proved in Meilijson and Nadas (1979) for the increasing convex order $\leq_{icx}$ and in Rü (1983) for $\leq_{cx}$. For various aspects of this kind of comparison see also the recent survey in Rü (2003b).

In contrast to the $\leq_\Delta$ ordering the comparison of $\leq_{sm}$ and $\leq_{dcx}$ is not so easy and there are many open problems. In section 2 we introduce a new dependence ordering $\leq_{wcs}$ called 'weakly conditionally ordered in sequence'. We show that in the case of identical marginals $P \leq_{wcs} Q$ implies supermodular ordering $P \leq_{sm} Q$. This generalizes the very interesting recent result of Christofides and Vaggelatou (2002) stating that weakly associated random vectors $X$ are riskier than independent vectors w.r.t. all supermodular functions. We also derive a corresponding comparison result for the $\leq_{dcx}$ ordering if marginals increase convexly in section 3.

In section 4 we consider various functional models of random vectors and derive $\leq_{dcx}$ ordering results for convexly increasing marginals. In this way we obtain some variations on a result of Müller and Scarsini (2001) for the case of distributions with conditionally increasing copula.

In the final section we consider an extension of sharp Fréchet-bounds to the case of multivariate marginals and obtain some comparison results for $\Delta$-monotone functions. Fixing some multivariate marginals corresponds to the situation where for some joint parts of the portfolio one knows the joint distribution while for others one only knows the marginal distribution. Some of the nice properties of one dimensional marginals no longer are valid. For example, the *comonotone* vectors no longer represent the riskiest portfolio.

## 2 Supermodular ordering

Define for two random vectors $X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_n)$

$$X \leq_{wcs} Y$$

– $X$ is smaller than $Y$ in the *weakly conditional increasing in sequence* order – if for all $t$, $1 \leq i \leq n-1$, and $f$ monotonically nondecreasing

$$\mathrm{Cov}(1(X_i > t), f(X_{(i+1)})) \leq \mathrm{Cov}(1(Y_i > t), f(Y_{(i+1)})). \tag{2.1}$$

Here $X_{(i+1)} = (X_{i+1}, \ldots, X_n)$, $Y_{(i+1)} = (Y_{i+1}, \ldots, Y_n)$.

Let $X^*$ denote the vector with independent components $X_i^* \stackrel{d}{=} X_i$. Then $X$ is called *weakly associated in sequence* (WAS) if

$$X^* \leq_{wcs} X. \tag{2.2}$$

We remark that weak association in sequence as defined in (2.2) is equivalent to

$$P^{X_{(i+1)}|X_i>t} \geq_{st} P^{X_{(i+1)}}, \tag{2.3}$$

where $\leq_{st}$ is the usual stochastic order.

**Remark 2.1** *There would be some formal reason to use in definition (2.1) the corresponding "forward" version with inverted "time" like in CIS. But interpreting the indices as time points our WAS condition says that "future depends positively on the present state" which corresponds to the CIS condition, which says that "present state depends positively on the past".*

*Weak association in sequence (WAS) is a wakening of weak association (WA) as introduced in Christofides and Vaggelatou (2002). With the conditional increasing (CI), the conditional increasing in sequence (CIS) and the positive dependence through stochastic ordering (PDS) the following relations hold:*

*CI $\Rightarrow$ CIS $\Rightarrow$ Association $\Rightarrow$ WA $\Rightarrow$ WAS,*
*and PDS $\Rightarrow$ WAS*

*(see Müller and Stoyan (2002, p 146))*

The next theorem states that more positive dependence w.r.t. the $\leq_{wcs}$ ordering implies more risk with respect to the supermodular ordering.

**Theorem 2.2** *Let $X$, $Y$ be $n$-dimensional random vectors with identical marginal distributions $P_i = Q_i$, $1 \leq i \leq n$. Then $X \leq_{wcs} Y$ implies that $X \leq_{sm} Y$.*

**Proof:** Let $X^*$, $Y^*$ be random vectors with independent components and $X_i^* \stackrel{d}{=} P_i$, $Y_i^* \stackrel{d}{=} Q_i$, $1 \leq i \leq n$. Then it is sufficient to compare expectations of functions $f \in \mathcal{F}^{sm}$ which are bounded and twice differentiable (see e.g. the approximation argument in Christofides and Vaggelatou (2002, Prop. 1)). Denote $g(t, x_{(2)}) := \frac{\partial f}{\partial x_1}(t, x_{(2)})$ and note that $g(t, \cdot)$ is increasing since $f \in$

$\mathcal{F}^{sm}$. Then we obtain using the simple representation formula of Christofides and Vaggelatou (2002)

$$\begin{aligned} E(f(X) - f(X_1^*, X_{(2)})) &= \int \mathrm{Cov}(1(X_1 > t), g(t, X_{(2)}))dt \\ &\leq \int \mathrm{Cov}(1(Y_1 > t), g(t, Y_{(2)}))dt \\ &= E(f(Y) - f(Y_1^*, Y_{(2)})), \end{aligned} \tag{2.4}$$

where the inequality follows from $X \leq_{wcs} Y$ and the assumption of identical marginals $X_1 \stackrel{d}{=} Y_1$. This implies that $Ef(X) \leq Ef(Y) + A_{n-1}$, where $A_{n-1} := Ef(X_1^*, X_{(2)}) - Ef(Y_1^*, Y_{(2)})$. The function $\int f(x_1, \cdot)\, dP_1(x_1)$ is a supermodular function in $(n-1)$ arguments. Therefore, using that $X_1^* \stackrel{d}{=} Y_1^*$, we obtain from induction $A_{n-1} \leq 0$ and thus $Ef(X) \leq Ef(Y)$. $\square$

As corollary we obtain

**Corollary 2.3** *If $X$ is weakly associated in sequence then $X^* \leq_{sm} X$ i.e. $X$ has positive supermodular dependence.*

**Remark 2.4** *a) Corollary 2.3 is due to Christofides and Vaggelatou (2002) under the assumption of weak association. Under the stronger CIS-condition this conclusion is due to Meester and Shantikumar (1993).*

*b) Since $X \leq_{sm} Y$ implies that $\varphi(X) \leq_{icx} \varphi(Y)$ for all monotonically nondecreasing, supermodular functions $\varphi$ one obtains as consequence of Theorem 2.2 a convex comparison result for a general class of risk-functionals of the random vectors. For $X$ weakly associated in sequence WAS holds: $\varphi(X)$ has a higher risk than $\varphi(X^*)$ for any monotonically nondecreasing supermodular function $\varphi$.*
*Thus the positive dependence notion WAS implies a higher risk for a general class of functionals. Denuit, Dhaene, and Ribas (2001) have established this conclusion for the special case of the combined portfolio $X_1 + \ldots + X_n$ for an associated random vector $X$. Christofides and Vaggelatou (2002) have stated this interesting conclusion for weakly associated random vectors $X$. A version of this result for a large class of directionally convex functions is in the previous version Rü (2003b) of this paper.*

*c) In the case $n = 2$ holds (for identical marginals) $X \leq_{wcs} Y$ if and only if*

$$X \leq_{uo} Y \qquad \textit{(upper orthant ordering).} \tag{2.5}$$

*In this case the result of Theorem 2.2 goes back to the classical paper of Cambanis, Simons, and Stout (1976).*

# 3 Directionally convex order

For the comparison of risk vectors it is of interest to compare vectors $X$ and $Y$ in the case where the marginals increase convexly ($\leq_{cx}$) or w.r.t. the increasing convex order ($\leq_{icx}$). The most general and suitable class of functions for this purpose are the directional convex functions $\mathcal{F}^{dcx}$ resp. the increasing, directional convex functions $\mathcal{F}^{idcx}$.

In order to state a comparison result we need a condition which ensures increase in the dependence structure and a condition for the increase in the convex ordering of the marginals. Both conditions are contained in the wcs-ordering and yield in the following statement.

**Theorem 3.1** *Let $X$, $Y$ be random vectors with marginals $P_i, Q_i$ such that $P_i \leq_{cx} Q_i$, $1 \leq i \leq n$. Assume that $X \leq_{wcs} Y$, then $X \leq_{dcx} Y$.*

**Proof:** Let $f \in \mathcal{F}^{dcx}$ be twice differentiable and let $g := \frac{\partial}{\partial x_1} f$, then as in (2.4) by the wcs-ordering and using monotonicity of $g(t, \cdot)$ we obtain

$$\begin{aligned} Ef(X) - Ef(X_1^*, X_{(2)}) &= \int \operatorname{Cov}(1(X_1 > t), g(t, X_{(2)}))dt \\ &\leq \int \operatorname{Cov}(1(Y_1 > t), g(t, Y_{(2)}))dt \\ &= Ef(Y) - Ef(Y_1^*, Y_{(2)}). \end{aligned}$$

Since $X_1^* \leq_{cx} Y_1^*$ and $f(\cdot, y_{(2)})$ is convex this implies

$$\begin{aligned} Ef(X) &\leq Ef(Y) + Ef(X_1^*, X_{(2)}) - Ef(Y_1^*, Y_{(2)}) \\ &\leq Ef(Y) + Ef(Y_1^*, X_{(2)}) - Ef(Y_1^*, Y_{(2)}) \\ &= Ef(Y) + A_{n-1} \end{aligned} \tag{3.1}$$

with $A_{n-1} := Ef(Y_1^*, X_{(2)}) - Ef(Y_1^*, Y_{(2)})$. Now using that $f(y_1^*, \cdot) \in \mathcal{F}^{dcx}$ we conclude by induction, that $A_{n-1} \leq 0$ and thus the result follows. □

**Remark 3.2** *For the conclusion that $X$ is smaller than $Y$ w.r.t. the increasing directionally convex order $X \leq_{idcx} Y$ one can weaken the wcs-ordering to the $\leq_{iwcs}$ ordering of $X$ and $Y$ defined as in* (2.1) *but restricted to $f \geq 0$, monotonically nondecreasing. Thus similar to Theorem 3.1 we obtain:*

$$P_i \leq_{icx} Q_i, 1 \leq i \leq n \text{ and } X \leq_{iwcs} Y \text{ implies that } X \leq_{idcx} Y. \tag{3.2}$$

The wcs-ordering condition simplifies if only one marginal increases convexly.

**Corollary 3.3** *Let $X = (X_1, X_{(2)})$, $Y = (Y_1, X_{(2)})$ and assume*

$$\operatorname{Cov}(1(X_1 > t), f(X_{(2)})) \geq \operatorname{Cov}(1(Y_1 > t), f(X_{(2)})) \tag{3.3}$$

*for all nondecreasing bounded $f$.*

*If $X_1 \leq_{cx} Y_1$, then $X \leq_{dcx} Y$.*

**Proof:** In the proof of Theorem 3.1 one can omit the induction step after formula (3.1) if only one marginal increases convexly. □

In the case $n = 2$ Theorem 3.1 yields a very simple sufficient condition for comparing risks in terms of the survival functions $\bar{F}(u,v) = P(X_1 \geq u, X_2 \geq v)$, $\bar{G}(u,v) = P(Y_1 \geq u, Y_2 \geq v)$. We formulate also a variant for increasing convex risks.

**Corollary 3.4** *Let $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$. Assume that for the survival functions $\bar{F}, \bar{G}$ of $X$ and $Y$ holds*

$$\bar{F}(u,v) - \bar{F}_1(u)\bar{F}_2(v) \leq \bar{G}(u,v) - \bar{G}_1(u)\bar{G}_2(v). \tag{3.4}$$

*a) If $X_i \leq_{cx} Y_i$, $i = 1, 2$, then $X \leq_{dcx} Y$.*

*b) If $X_i \leq_{icx} Y_i$, $i = 1, 2$, then $X \leq_{idcx} Y$.*

*Here $\leq_{idcx}$ denotes the ordering w.r.t. the class $\mathcal{F}^{idcx}$ of increasing directionally convex functions.*

**Proof:** The wcs-ordering condition is in the case $n = 2$ identical to the dependence ordering in (3.4) of the survival functions. □

# 4 Some functional models

In general the wcs-ordering condition is not easy to verify. In this section we, therefore, state comparison criteria for some functional models which allow to combine some conditioning arguments with known ordering results for $\leq_{sm}$ resp. $\leq_{dcx}$. Some related comparison results have been given in Shaked and Tong (1985), Bäuerle (1997), and Bäuerle and Müller (1998).

A basic ordering result for $\leq_{dcx}$ is the following *Ky Fan–Lorentz theorem*: Let $F_i, G_i$, $1 \leq i \leq n$ be one-dimensional df's with $F_i \leq_{cx} G_i$, $1 \leq i \leq n$, then

$$(F_1^{-1}(U), \dots, F_n^{-1}(U)) \leq_{dcx} (G_1^{-1}(U), \dots, G_n^{-1}(U)), \tag{4.1}$$

where $U$ is uniformly distributed on $(0, 1)$; i.e., the comonotone vectors are comparable w.r.t. the directionally convex order.

(4.1) was first stated in Rü (1983), the proof being based there on some functional inequalities in Ky Fan and Lorentz (1954). Müller and Scarsini (2001) proved an extension of the comparison result in (4.1) for distributions with marginals $F_i, G_i$ ($F \in \mathcal{F}(F_1, \dots, F_n)$, $G \in \mathcal{F}(G_1, \dots, G_n)$) convexly increasing $F_i \leq_{cx} G_i$ if both $F$ and $G$ have the same conditionally increasing copula.

A consequence of the Ky Fan–Lorentz theorem is the following comparison result.

**Theorem 4.1** *Let $V_1, \Theta$ be independent random vectors, $V_1$ real, and let $X_i = h_i(V_1, \Theta)$, $Y_i = g_i(V_1, \Theta)$, $1 \le i \le n$, where $h_i(\cdot, \vartheta)$ and $g_i(\cdot, \vartheta)$ are monotonically nondecreasing. If for all $\vartheta$*

$$h_i(V_1, \vartheta) \le_{cx} g_i(V_1, \vartheta), \qquad 1 \le i \le n \tag{4.2}$$

*then $X \le_{dcx} Y$.*

**Proof:** By assumption, conditionally given $\Theta = \vartheta$, $X|\vartheta$ and $Y|\vartheta$ are comonotone vectors. Therefore, by the Ky Fan–Lorentz theorem componentwise convex ordering of the marginals as in (4.2) implies $X|\vartheta \le_{dcx} Y|\vartheta$. Therefore, by mixing we obtain $X \le_{dcx} Y$. □

One can interpret the representation $X_i = h_i(V_1, \Theta)$, $Y_i = g_i(V_1, \Theta)$ as a model with functional dependence of an internal factor $V_1$ and an external factor $\Theta$ common to both. Both models depend stochastically increasing on $V_1$ while for any external factor $\vartheta$ the second model has more risk than the first model.

A functional type representation of $X$, $Y$ as in Theorem 4.1 can be obtained by the regression construction resp. the standard construction which is a general and useful construction method for random vectors with df $F \in \mathcal{F}_n$ – the class of all $n$-dimensional df's. It is defined in the following way. Let $V = (V_1, \ldots, V_n)$ be a sequence of independent rv's, uniformly distributed on $[0, 1]$ and let $F_{i|1\ldots i-1}(x_i|x_1, \ldots, x_{i-1})$ denote the conditional df's of $X_i$ given $X_j = x_j$, $1 \le j \le i-1$, where $X$ is a random vector with df $F$. Define $\tau_F^{-1} : [0, 1]^n \to \mathbb{R}^n$ recursively by:

$$\begin{aligned}
\tau_F^{-1}(u) &= z = (z_1, \ldots, z_n), \text{ where} \\
& z_1 = F_1^{-1}(u_1), \\
& z_2 = \inf\{y : F_{2|1}(y|z_1) \ge u_2\} = F_{2|1}^{-1}(u_2|z_1) \\
& \quad \vdots \\
& z_n = F_{n|1\ldots n-1}^{-1}(u_n|z_1, \ldots, z_{n-1}). \text{ Then}
\end{aligned} \tag{4.3}$$

$$Z = \tau_F^{-1}(V) \tag{4.4}$$

is a random vector with df $F$, the *regression representation*.

This construction was introduced in O'Brien (1975), Arjas and Lehtonen (1978), and in Rü (1981b). By definition in (4.3) one can write $Z$ also directly as a function of $V$, $Z = \tau_F^*(V)$ where the functional dependence on $V$ is of the form

$$\tau_F^*(V) = (h_1(V_1), h_2(V_1, V_2), \ldots, h_n(V_1, \ldots, V_n)). \tag{4.5}$$

This is called *standard representation* of $F$. It is of the functional form in Theorem 4.1 with $\Theta = (V_2, \ldots, V_n)$.

If $F, G \in \mathcal{F}_n$, then pointwise comparison

$$\tau_F^{-1}(V) \leq \tau_G^{-1}(V) \text{ implies } F \leq_{st} G. \tag{4.6}$$

Here $\leq_{st}$ denotes the usual stochastic ordering w.r.t. monotonically nondecreasing functions. This criterion stated in Rü (1981b) implies many of the known stochastic ordering criteria.

As corollary of Theorem 4.1 we obtain the following result, allowing to increase one component convexly.

**Corollary 4.2** *Assume that $X, Y$ have the same copula and assume that the conditional distribution of $X_i|(X_1, \dots, X_{i-1}) = (x_1, \dots, x_{i-1})$, is stochastically increasing in $x_1$, $2 \leq i \leq n$.*

*a) If $X_1 \leq_{cx} Y_1$ and $F_i = G_i$, $2 \leq i \leq n$, then $X \leq_{dcx} Y$.*

*b) If $X_1 \leq_{icx} Y_1$ and $F_i = G_i$, $2 \leq i \leq n$, then $X \leq_{idcx} Y$.*

*Here $\leq_{icx}$, $\leq_{idcx}$ are the increasing convex resp. increasing directionally convex order.*

**Proof:**

a) Using the standard representation for the joint copula of $X$, $Y$ we see that w.l.g.

$$X_1 = F_1^{-1}(V_1),\ X_i = F_i^{-1} \circ f_i(V_1, \dots, V_i),\ i \geq 2,$$

and

$$Y_1 = G_1^{-1}(V_1),\ Y_i = X_i = F_i^{-1} \circ f_i(V_1, \dots, V_i),\ i \geq 2,$$

where $f_i(v_1, \dots, v_i)$ are nondecreasing in $v_1$ and where $U = (U_1, \dots, U_n)$ with $U_i = f_i(V_1, \dots, V_i)$, $i \geq 2$, $U_1 = V_i$ are the common copula vector of $X$, $Y$. Therefore, the assumption of Theorem 4.1 is fulfilled with

$$\begin{aligned}
&\Theta = (V_2, \dots, V_n),\\
&h_1(v_1, \vartheta) = G_1^{-1}(v_1) = h_1(v_1),\\
&g_1(v_1, \vartheta) = F_1^{-1}(v_1) = g_1(v_1) \text{ and}\\
&g_i(v_1, \vartheta) = h_i(v_1, \vartheta) = F_i^{-1} \circ f_i(v_1, (\vartheta_2, \dots, \vartheta_i)),\ i \geq 2.
\end{aligned}$$

b) follows similarly. □

**Remark 4.3** *a) One can iteratively apply Corollary 4.2 to obtain a $\leq_{dcx}$-comparison result in the case that $X_i \leq_{cx} Y_i$, $1 \leq i \leq n$. This comparison result under the assumption that the joint copula is conditionally increasing is due to Müller and Scarsini (2001).*

*b) A similar comparison result as in Corollary 4.2 can be formulated if $X$ and $Y$ have not necessarily the same copula. Let $X_i = F_i^{-1} \circ f_i(V_1, \dots, V_i)$, $Y_i = G_i^{-1} \circ g_i(V_1, \dots, V_i)$ be the standard representations of $X$, $Y$, let $\Theta = (V_2, \dots, V_n)$ and assume:*

*1) $X_i | (X_1, \dots, X_{i-1}) = (x_1, \dots, x_{i-1})$ and*
*$Y_i | (Y_1, \dots, Y_{i-1}) = (x_1, \dots, x_{i-1})$ are stochastically increasing in $x_1$*

*2) $X_1 \leq_{cx} Y_1$ and*

*3) $X_i | \Theta = \vartheta \leq_{cx} Y_i | \Theta = \vartheta$, $2 \leq i \leq n$.*

$$\text{Then} \quad X \leq_{dcx} Y. \tag{4.7}$$

*c) Two general methods to increase convexly distributions are the following*

(C$_1$) *Let $Y_i = h_i \circ X_i$ with $h_i \uparrow$,*

$$h_i(y) - h_i(x) \geq y - x, \forall x < y \text{ and } EY_i = EX_i \tag{4.8}$$

(C$_2$) *Let $Y_i = h_i(X_i, \Theta)$, where $X_i$, $\Theta$ are independent and*

$$Eh_i(x_i, \Theta) = x_i \quad \forall x_i, \ 1 \leq i \leq n. \tag{4.9}$$

*Then under ($C_1$)*

$$X \leq_{ccx} Y \tag{4.10}$$

*where $\leq_{ccx}$ denotes the componentwise convex order. Under ($C_2$) we obtain convex ordering of the vectors*

$$X \leq_{cx} Y \tag{4.11}$$

*More generally ($C_2$) can be replaced by condition ($C_2'$):*

(C$_2'$) $Y_i = h_i(X, \Theta)$, *where $Eh_i(x, \Theta) = x_i$, $1 \leq i \leq n$, $X$, $\Theta$ independent.*

*Condition ($C_2'$) again implies convex ordering*

$$X \leq_{cx} Y \tag{4.12}$$

The argument in the proof of Corollary 4.2 allows to derive the following criterion for the $\leq_{wcs}$ ordering.

**Proposition 4.4** *Let $X, Y$ be random vectors with the same conditionally increasing copula and let $X_i \leq_{cx} Y_i, 1 \leq i \leq n$, then $X \leq_{wcs} Y$.*

**Proof:** W.l.g. let $X_1 \leq_{cx} Y_1$ and $X_i \stackrel{d}{=} Y_i, 2 \leq i \leq n$. We have to establish

$$\mathrm{Cov}(f_i(X_i), f(X_{(i+1)})) \leq \mathrm{Cov}(f_i(Y_i), f(Y_{(i+1)})), 1 \leq i \leq n,$$

for $f_i, f$ monotonically nondecreasing. Without loss of generality we consider the case $i = 1$. Since $X_{(2)} \stackrel{d}{=} Y_{(2)}$ we also may assume that $X_{(2)} = Y_{(2)}$ and $Ef(X_{(2)}) = 0$. Using the standard representation and notation from the proof of Corollary 4.2 we obtain conditionally given $\Theta = (V_2, \dots, V_n) = \vartheta$. $f_1(X_1) = h_1(V_1), f(X_{(2)}) = h_2(V_1)$ and $f_1(Y_1) = g_1(V_1)$ where $h_i, g_1$ are monotonically nondecreasing, $g_1 = g_1(\cdot, \vartheta)$. Thus conditionally given $\Theta = \vartheta$ by the Lorentz-Ky Fan Theorem (see( 4.1))

$$(f_1(X_1), f(X_{(2)})) \leq_{dcx} (f_1(Y_1), f(Y_{(2)}))$$

and therefore

$$\mathrm{Cov}(f_1(X_1), f(X_{(2)})) = Ef_1(X_1)f(X_{(2)}) \leq \mathrm{Cov}(f_1(Y_1), f(Y_{(2)})). \qquad \square$$

Proposition 4.4 implies that the directionally convex ordering result of Müller and Scarsini (2001) stating that under the conditions of Proposition 4.4 $X \leq_{dcx} Y$ is also obtained as a consequence of our ordering result in Theorem 3.1 for the wcs-ordering.

We finally consider some functional models based on random sequences $(U_i)$, $(V_i)$, $V$, where $V$ and $(V_i)$ are independent of $(U_i)$. Let $(U_i)$, $(V_i)$ be random sequences and let $V$ and $(V_i)$ be one-dimensional and independent of $(U_i)$. From these we build up functional models $X = (X_1, \dots, X_n)$, $Y = (Y_i)$, $Z = (Z_i)$ and $W = (W_i)$ by

$$\begin{aligned} X_i &= g_i(U_i, V_i), & Y_i &= g_i(U_i, V) \\ Z_i &= \widetilde{g}_i(U_i, V_i), & W_i &= \widetilde{g}_i(U_i, V). \end{aligned} \tag{4.13}$$

Here $(U_i)$ is some basic stochastic variable while $V_i$ is some external random source whose influence is given by the functionals $g_i$, $\widetilde{g}_i$.

We make the following assumptions:

(A1) $(U_i)$ are independent.

(A2) $V_i \sim V$, $1 \leq i \leq n$.

(A3) $g_i(u, \cdot), \widetilde{g}_i(u, \cdot)$ are monotonically nondecreasing.

Let $\leq_{ccx}$ denote the component-wise convex order. The first part of the following result is stated in Bäuerle (1997, Theorem 3.1) for independent $U_i$.

**Proposition 4.5** *Under (A2), (A3) holds*

*a)* $$X \leq_{sm} Y, \qquad Z \leq_{sm} W\,. \tag{4.14}$$

*b) If additionally $g_i(u_i,\,\cdot\,) \leq_{cx} \widetilde{g}_i(u_i,\,\cdot\,)$, for all $u_i$, then*

$$Y \leq_{dcx} W \tag{4.15}$$

**Proof:**

a) For $\varphi \in \mathcal{F}^{sm}$ holds by the Lorentz theorem in (1.7)

$$\begin{aligned} E\varphi(X) &= E_U E(\varphi(X) \mid U_1 = u_1, \ldots, U_n = u_n) \\ &= E_U E\varphi(g_1(u_1, V_1), \ldots, g_n(u_n, V_n)) \\ &\leq E_U E\varphi(g_1(u_1, V), \ldots, g_n(u_n, V)) \\ &= E\varphi(g_1(U_1, V), \ldots, g_n(U_n, V)) \;=\; E\varphi(Y). \end{aligned}$$

Here $E_U$ denote the marginal expectation w.r.t. the random vector $U$ The proof of $Z \leq_{sm} W$ is similar.

b) The proof of $X \leq_{dcx} W$ follows from a) and the Ky Fan–Lorentz theorem in 4.1 by a conditioning argument on the $U_i$.

□

We next compare vectors $X, Y, Z, W$ under the assumption that the marginals increase convexly.

**Proposition 4.6** *Under conditions (A1), (A2), (A3) the following holds: If for all $v$, $\widetilde{g}_i(U_i, v) \leq_{cx} g_i(U_i, v)$ then*

$$Z \leq_{ccx} X, W \leq_{ccx} Y \text{ and } Z \leq_{dcx} Y. \tag{4.16}$$

**Proof:** By Proposition 4.5, $X \leq_{sm} Y$. Further for any component-wise convex function $\varphi$ holds when conditioning under $V_i = v_i$ and using the assumption on $g_i, \widetilde{g}_i$:

$$\begin{aligned} E\varphi(X) &= E_V E\varphi(g_1(U_1, v_1), \ldots, g_n(U_n, v_n)) \\ &\geq E_V E\varphi(\widetilde{g}_1(U_1, v_1), \ldots, g_n(U_n, v_n)) \\ &= E\varphi(Z), \text{ i.e. } Z \leq_{ccx} X. \end{aligned}$$

$E_V$ is the marginal expectation w.r.t. $V$. Here at the inequality we used that by (A1) $\quad (g_1(U_1, v_1), \ldots, g_n(U_n, v_n)) \leq_{ccx} (\widetilde{g}_1(U_1, v_1), \ldots, \widetilde{g}_n(U_n, v_n))$. So we get $Z \leq_{ccx} X \leq_{sm} Y$, implying $Z \leq_{dcx} Y$. The inequality $W \leq_{ccx} Y$ is similar.

□

**Remark 4.7** *The random vectors $Y, Z$ and $X, W$ which are compared w.r.t. $\leq_{dcx}$ in Proposition 4.6 do not have the same dependence structure (copula). The vectors $X, Y, Z, W$ considered in Proposition 4.6 are not necessarily positive dependent. Since we do not assume independence of the $(V_i)$, any $F \in \mathcal{F}_n$ can be represented in the form $(g_i(U_i, V_i))$, with $g_i$ satisfying (A3). Thus the comparison results of Propositions 4.5 and 4.6 concern a large class of models.*

# 5 Multivariate marginals

We consider random vectors $X = (X_1, \dots, X_n)$, where $X_i$ are $k_i$-dimensional random vectors with df's $F_i$ and corresponding probability measures $P_1, \dots, P_n$. The df $F = F_X \in \mathcal{F}(F_1, \dots, F_n)$ has (known) multivariate marginals. In comparison to the case of one-dimensional marginals the multivariate marginals case has not been considered a lot in the literature even if it seems to be natural that for several applications one can control (determine) the joint distribution of some subgroups and would like to control the influence of dependence between the subgroups. Some general results and principles for these kind of problems have been discussed in Rü (1991a, 1991b). In the following we consider the analog of the classical Fréchet-bounds and obtain as consequence some bounds on the integrals on $\Delta$-monotone functions. Let $k = \sum_{i=1}^n k_i$ be the dimension of $X$.

**Theorem 5.1 (Fréchet-bounds for multivariate marginals)**

*a) If $n = 2$, $A \in \mathcal{B}^k$ is closed, and $\pi_1(x_1, x_2) = x_1$ is the first projection then*

$$\begin{aligned} M(A) &:= \sup\{P(A); P \in M(P_1, P_2)\} && (5.1) \\ &= 1 - \sup\left\{P_2(O) - P_1\left(\pi_1\left(A \cap (\mathbb{R}^{k_1} \times O)\right)\right); O \subset \mathbb{R}^{k_2} \text{ open}\right\}, \end{aligned}$$

*b)*

$$\begin{aligned} F^-(x) := \left(\sum_{i=1}^n F_i(x_i) - (n-1)\right)_+ &\leq F(x_1, \dots, x_n) && (5.2) \\ &\leq \min_{i \leq n} F_i(x_i) =: F^+(x) \end{aligned}$$

$$\begin{aligned} \bar{F}^-(x) := \left(\sum_{i=1}^n \bar{F}_i(x_i) - (n-1)\right)_+ &\leq \bar{F}(x_1, \dots, x_n) && (5.3) \\ &\leq \min_{i \leq n} \bar{F}_i(x_i) =: \bar{F}^+(x) \end{aligned}$$

*where $\bar{F}_i(x_i) = P(X_i \geq x_i)$, $\bar{F}(x_1, \dots, x_n) = P(X_i \geq x_i, 1 \leq i \leq n)$ are the multivariate survival functions. The bounds in (5.2), (5.3) are sharp.*

**Proof:**

a) is a consequence of Strassen (1965, Theorem 11) (see Rü (1982, 1986)).

b) In the case $n=2$ b) follows from a). The case $n\geq 2$ follows from a general result in Rü (1981a) which states that for $P\in M(P_1,\ldots,P_n)$ where the marginals $P_i$ are defined on general polish spaces and for any measurable sets $A_1,\ldots,A_n$ holds

$$\Big(\sum_{i=1}^{n} P_i(A_i)-(n-1)\Big)_+ \leq P(A_1\times\ldots\times A_n)\leq \min_{i\leq n} P_i(A_i) \tag{5.4}$$

and the bounds in (5.4) are sharp. □

**Remark 5.2** *a) In the case of one-dimensional marginals* (5.2), (5.3) *are the classical Fréchet-bounds.* $F^+$ *the upper Fréchet-bound is a df while the lower Fréchet-bound* $F^-$ *is a df only in exceptional cases with large jumps (for details see Dall'Aglio (1972)).*

*b)* Bounds for tails of $\Psi(X_1,X_2)$: *Part a) of Theorem 5.1 allows to obtain sharp upper and lower bounds for tail probabilities* $P(\Psi(X_1,X_2)\geq t)$ *of general functionals* $\Psi(X_1,X_2)$*. One obtains much simpler forms for monotonically nondecreasing function* $\Psi$ *however:*

*If* $\Psi$ *is monotonically nondecreasing upper-semicontinuous and* $A_\Psi^- := \{(u,v)$ *minimal in* $\mathbb{R}^2 : \Psi(u,v)\geq t\}$*, then*

$$P(\Psi(X_1,X_2)\geq t)\leq \inf_{(u,v)\in A_\Psi^-(t)} (F_1(u)+F_2(v))\,. \tag{5.5}$$

*In analogy to the case of combined risks where* $\Psi(x,y)=x+y$ *(see Rü (1982)) one could call the bounds in* (5.5) *the infimal (resp. supremal)* $\Psi$*-convolution of* $P_1,P_2$*.*

In the one-dimensional case the comonotone random vectors $(F_1^{-1}(U),\ldots,$ $F_n^{-1}(U))$ attain the upper Fréchet-bound (i.e. have df $F^+$) and by (1.7) are the riskiest random vectors. In the multivariate case there typically will not exist comonotone vectors $(X_1,\ldots,X_n)$ with $X_i\sim F_i$ in the sense that $(X_1,\ldots,X_n)=(f_1(U),\ldots,f_n(U))$ with nondecreasing, $f_i:[0,1]\to\mathbb{R}^{k_i}$. Even in the case that $k_i=k_1$, $1\leq i\leq n$, and $F_1=F_2=\ldots=F_n$ the *natural* comonotone vector with identical components will not attain the sharp upper Fréchet-bound $F^+(x)$ and thus does not yield the riskiest portfolio distribution.

**Proposition 5.3 (Comonotone random vector and Fréchet-bounds)**

*a) In general the upper and lower Fréchet-bounds for $k_i \geq 2$ do not define distribution functions.*

*b) If $F_1 = F_2 = \ldots = F_n$ is a $k_1$-dimensional df, and $X_1 \sim F_1$, then the comonotone random vector $X = (X_1, \ldots, X_n)$ has df*

$$F(x) = F_1(x_1 \wedge \ldots \wedge x_n) \leq F^+(x) = \min_{1 \leq i \leq n} F_1(x_i), \quad x_i \in \mathbb{R}^k. \tag{5.6}$$

*In general there is strict inequality in* (5.6).

**Proof:**

b) follows directly from the definition. Only for $k_1 = 1$ equality holds in (5.6) in general.

a) The reason for a) is the following. Let w.l.o.g. $n = 2$, $k_1 = k_2 = 2$. Assume that for $G, H \in \mathcal{F}_2$ with one-dimensional marginals $G_1, G_2, H_1, H_2$ the lower Fréchet-bound $F^- = F^-(G, H)$ were a four dimensional df. Then for $X \sim F^-$ we conclude from (5.2) that $(X_1, X_3) \sim F_2^-(G_1, H_1)$, $(X_1, X_4) \sim F_2^-(G_1, H_2)$, $(X_2, X_3) \sim F_2^-(G_2, H_1)$ and $(X_2, X_4) \sim F_2^-(G_2, H_2)$. This however would imply strong positive correlation of $(X_3, X_4)$ and of $(X_1, X_2)$ which is not according to our assumption. Except for some exceptional cases with big jumps (as in Dall'Aglio's classical 1972 paper) we would obtain that $(X_1, X_2) \sim F_2^+(G_1, G_2)$ and $(X_3, X_4) \sim F_2^+(H_1, H_2)$. Similarly, the argument for the upper Fréchet-bound yields the same conclusion. □

**Remark 5.4** *The discussion in the proof of a) in Proposition 5.3 shows that up to some exceptional cases (with big jumps) only for $G = F_2^+(G_1, \ldots, G_n)$ and $H = F_2^+(H_1, \ldots, H_n)$ the Fréchet-bounds are df's. This holds in particular true if $G_i$, $H_i$ are continuous df's.*

For $\Delta$-monotone functions $f$ one can conclude from (5.2), (5.3) upper and lower bounds for the integrals.

**Theorem 5.5** *Let $f : \mathbb{R}^k \to \mathbb{R}^1$ be $\Delta$-monotone and assume that for $1 \leq i \leq k$, $\lim_{x_i \to -\infty} f(x_1, \ldots, x_i, \ldots, x_k) = 0$ for all $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k$. Then for any $F \in \mathcal{F}(F_1, \ldots, F_n)$*

$$\int \bar{F}^-(x)\ df(x) \leq \int f dF \leq \int \bar{F}^+(x)\ df(x) \tag{5.7}$$

*if the integrals exist.*

**Proof:** The proof follows from the Fréchet-bounds in (5.2), (5.3) applied to the partial integration formula in Rü (1980, proof of Theorem 3). By this formula one obtains for any $a \in \mathbb{R}^k$

$$\int_{[a,\infty)} \Delta_a^x f dF = \int_{[a,\infty)} \bar{F}(x) df(x). \tag{5.8}$$

For $a \to -\infty$ the integrals converge by our assumption on $f$ to yield

$$\int f dF = \int \bar{F}(x) df(x) \tag{5.9}$$

and so (5.4) follows from (5.3). □

**Remark 5.6** *The bounds in* (5.4) *for the integrals are not sharp in general, since $F^+$, $F^-$ are not df's.*

**Example 5.7 (Antithetic and comonotone variates)** *In the following examples we investigate some* natural *multivariate extensions of antithetic and comonotone variates for various examples of functions $\varphi$. In particular we also give an example where the comonotone random vector yields the lowest risk for a directionally convex function $\varphi$.*

*Consider as example the case $k_i = 2$, $1 \leq i \leq n$ and $F_i = F_2^-(G_i, H_i)$, $1 \leq i \leq n$, i.e. for a uniform random variable $U$, $(G_i^{-1}(U), H_i^{-1}(1-U)) \sim F_i$, $1 \leq i \leq n$. Some alternative random vectors with df's in $\mathcal{F}(F_1, \ldots, F_n)$ are*

$$\begin{aligned} W^+ &= ((G_1^{-1}(U), H_1^{-1}(1-U), \ldots, (G_n^{-1}(U), H_n^{-1}(1-U))), \\ Z &= ((G_1^{-1}(U_1), H_1^{-1}(1-U_1)), \ldots, (G_n^{-1}(U_n), H_n^{-1}(1-U_n))), \\ \textit{and } W^- &= ((G_1^{-1}(U), H_1^{-1}(1-U)), (G_2^{-1}(1-U), H_2^{-1}(U)), \ldots) \end{aligned} \tag{5.10}$$

*where $(U_i)$ are independent uniform. $W^+$ is a generalized comonotone vector, $Z$ the independent vector and $W^-$ a generalized antithetic vector. Then for $x = (x_1, \ldots, x_n), x_i = (y_i, z_i)$ we obtain*

$$\begin{aligned} F_{W^+}(x) &= (\min G_i(y_i) + \min H_i(z_i) - 1)_+, \\ F_Z(x) &= \prod_{i=1}^n (G_i(y_i) + H_i(z_i) - 1)_+, \\ F^+(x) &= \min(G_i(y_i) + H_i(z_i) - 1)_+, \textit{and} \\ F^-(x) &= \Big(\sum_{i=1}^n (G_i(y_i) + H_i(z_i) - 1)_+ - (n-1)\Big)_+ \end{aligned} \tag{5.11}$$

*$F_{W^+}, F_Z$ are not uniformly comparable with each other.*

*Consider $\varphi_1(x) = (\sum y_i + \sum z_i)^2$ and the special case $G_i = G, H_i = H$, both with expectations zero. Then*

$$E\varphi_1(W^+) = \operatorname{Var}\Big(\sum_{i=1}^n G^{-1}(U) + H^{-1}(1-U)\Big)$$

$$= n^2 \operatorname{Var}(G^{-1}(U) + H^{-1}(1-U)). \tag{5.12}$$

$$\begin{aligned} E\varphi_1(Z) &= \operatorname{Var}\Big(\sum_{i=1}^n (G^{-1}(U_i) + H^{-1}(1-U_i))\Big) \\ &= n \operatorname{Var}(G^{-1}(U) + H^{-1}(1-U)). \end{aligned} \tag{5.13}$$

*So* $E\varphi_1(W^+) = nE\varphi_1(Z)$. *The comonotone vector* $W^+ = (X_1, X_1, \dots, X_1)$, $X_1 = (G^{-1}(U), H^{-1}(1-U))$ *induces a much higher variance of the sum than the independent vector* $Z = (X_1, \dots, X_n)$ *where* $X_i$ *are independent,* $X_i \sim X_1$. *In fact in this case we obtain from* (1.7) *that*

$$E\varphi_1(W^+) \geq E\varphi_1(X) \tag{5.14}$$

*for all* $X$ *with marginals* $F_i = F_2^-(G, H)$, *i.e. the risk measured by* $\varphi_1$ *is maximal for the comonotone vector (this is true for all marginals* $F_i$*).*

*For* $\varphi_2(x) = \max y_i + \max z_i$, *the sum of the maximal risks in the first and second components the situation is different* $E\varphi(W^+) = EG^{-1}(U) + EH^{-1}(1-U)$ *is the smallest possible value. From extreme value theory it is known that the value of the independent vector* $Z$, $E\varphi_2(Z)$, *is of the order* $a_n = G^{-1}(1 - \frac{1}{n}) + H^{-1}(1 - \frac{1}{n})$ *under the corresponding domain of attraction conditions. It was shown in the classical 1976 paper of Lai and Robbins that this is close to the maximal possible value attained by* maximally dependent *rv's. The fact that* $E\varphi_2(W^+) \leq E\varphi_2(Z)$ *is to be expected in this case since* $\varphi_2$ *is* $\Delta$*-antitone.*

*Finally consider* $\varphi_3(x) = \sum_{i=1}^{n-1} (y_i z_{i+1} + y_{i+1} z_i)^2$ *and let* $G$, $H$ *have support in* $\mathbb{R}_+$. *Then* $\varphi_3$ *is directionally convex (so a proper measure of risk). For the comonotone vector* $W^+$ *we obtain*

$$E\varphi_3(W^+) = (n-1)E(G^{-1}(U)H^{-1}(1-U) + H^{-1}(U)G^{-1}(1-U))^2. \tag{5.15}$$

*Thus in this case the comonotone vector* $W^+$ *yields the smallest possible risk. The reason is the strong negative dependence between the components of the marginal distributions. The largest possible value is attained in this case by the antithetic vector*

$$W^- = ((G^{-1}(U), H^{-1}(1-U)), (G^{-1}(1-U), H^{-1}(U)), \dots). \tag{5.16}$$

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