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On convex risk measures on L^p-spaces

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Abstract Much of the recent literature on risk measures is concerned with essentially bounded risks in L^∞. In this paper we investigate in detail continuity and representation properties of convex risk measures on L^p spaces. This frame for risks is natural from the point of view of applications since risks are typically modelled by unbounded random variables. The various continuity properties of risk measures can be interpreted as robustness properties and are useful tools for approximations. As particular examples of risk measures on L^p we discuss the expected shortfall and the shortfall risk. In the final part of the paper we consider the optimal risk allocation problem for L^p risks.

Key words convex risk measure – expected shortfall – optimal risk allocation

1 Introduction

Measuring the risk of a financial position is a complex process which is connected with many features in the financial market. One postulate is that measures of risk should include the aspect of *securization of risk* i.e. the possibility to transfer risks by hedging actions to the market. The price of the hedging action is one part of the risk of a position. The remaining part of the risk has to be evaluated based on the underlying probability model and on the preferences of the risk taker. Thus measuring of risk is connected with probalistic modelling, with pricing and with preferences – called the three P's in Lo (1999).

Let $\mathcal{C}$ be the set of available hedging actions including addition of capital, trading actions in basics X like $\int_0^T \xi_s dX_s$ or buying derivatives. Let π be a relevant pricing system and let finally $\mathcal{A}$ describe the set of acceptable positions (related to the preference system). Then a reasonable version of a risk measure is defined as

$$\varrho(X) = \inf\{\pi(Y); Y \in \mathcal{C}, \quad X + Y \in \mathcal{A}\}. \tag{1.1}$$

$\varrho(X)$ is the minimal price of a hedging action which added to the risk makes it acceptable. This point of view towards risk measures is proposed in recent papers by Scandolo (2004) and Frittelli and Scandolo (2006).

If all components of (1.1), the price π, the hedging set $\mathcal{C}$ as well as the acceptance set $\mathcal{A}$ are convex, then (1.1) defines a convex risk measure. Thus in order to study properties and applications of risk measures as in (1.1) a first step is to consider general convex risk measures and their properties when defined on L^p risks. Our paper is concerned in first line with continuity and representation properties of general convex risk measures on L^p. Results of this type are useful tools e.g. to describe basic risk components or to consider robustness properties of risk measures. It will be of future interest to obtain more specific results concerning relevant classes of examples as in (1.1). The results in this paper should be of use for investigating this type of risk measures in connection with typical applications of risk measures as to portfolio optimization (risk reward optimization), risk sharing and risk allocation or to describe the influence of risk components.

A risk measure is defined on a class $\mathfrak{X}$ of financial positions quantifying the risk of X by some number $\varrho(X)$. Any risk measure has a natural domain $\mathfrak{X} \subset L^0$ of definition. Here $L^0 = L^0(\Omega, \mathfrak{A}, P)$ is the class of all measurable elements on some nonatomic measure space $(\Omega, \mathfrak{A}, P)$. The σ-algebra $\mathfrak{A}$ is generally assumed to be countably generated. The value at risk $V_\alpha(X)$ for example is naturally defined on L^0, while the tail conditional expectation $TC_\alpha(X) = E(-X \mid X \leq q_\alpha(X))$ is naturally defined and finite on $\mathfrak{X} = \{X \in L^0; X^- \in L^1\}$. The structure and properties of convex risk measures has been investigated in detail in the literature in the case that $\mathfrak{X} = L^\infty$ is the class of essentially bounded risks (see the exhausting presentations in Delbaen (2002) and Föllmer and Schied (2004)).

It is however of interest to consider in more detail the case, where $\mathfrak{X} = L^p$ is an L^p-space, since the usual modelling of risks is typically by nonbounded random variables as e.g. normally or stable distributed random variables. This frame is naturally present in those literature which is more concentrated on applications of risk measures to realistic models like stable models or hyperbolic models. Specific risk measures are an essential ingredient of some complex optimization problems as e.g. of determining optimal portfolios according to optimization of risk-reward quotients. A very informative and well written presentation to this approach to risk management is given in Rachev, Menn, and Fabozzi (2005). The literature on the more theoretical representation and continuity properties of risk measures on L^p is not very numerous. For coherent finite risk measures Nakano (2004) considered the case $\mathfrak{X} = L^1$, while in a subsequent paper Inoue (2003) extented some of Nakano's results to L^p-spaces, $1 \leq p \leq \infty$. A very general approach to convex risk measures in the frame of Frechet lattices was developed in Frittelli and Gianin (2002) and in particular in the recent paper of Biagini and Frittelli (2006). The recent preprint of Cheridito and Li (2006), which we got to know only after finishing this work, contains results related to ours in the context of Orlicz spaces. In the present paper we supplement these papers by some more detailed discussion of various properties of risk measures, in particular continuity properties, which are useful for applications and are responsible for corresponding representation results. We also consider examples as the expected shortfall and the shortfall risk and discuss some applications to the risk allocation problem in this more general context.

We consider (possibly nonfinite) *convex risk measures* $\varrho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \le p \le \infty$, which are monetary risk measures i.e.

$$X \le Y \ [P] \text{ implies } \varrho(Y) \le \varrho(X) \qquad \textit{monotonicity} \tag{1.2}$$

$$\varrho(X+m) = \varrho(X) - m, \quad \forall m \in \mathbb{R} \qquad \textit{cash invariance} \tag{1.3}$$

and which are convex, i.e.

$$\varrho(\alpha X + (1-\alpha)Y) \le \alpha\varrho(X) + (1-\alpha)\varrho(Y), \quad \forall X, Y \in L^p. \tag{1.4}$$

A convex risk measure ϱ is called coherent if

$$\varrho(\lambda X) = \lambda\varrho(X) \text{ for all } \lambda \ge 0 \text{ and } X \in L^p. \tag{1.5}$$

Note that for $0 \le p < 1$ there do not exist non constant, finite convex risk measures (see Biagini and Frittelli (2006)). Cash invariance implies that

$$\varrho(X + \varrho(X)) = 0 \text{ for } X \in L^p \text{ and } \varrho(m) = \varrho(0) - m, \tag{1.6}$$

for all $m \in \mathbb{R}^1$. Let $\mathcal{A}_\varrho := \{X \in L^p; \varrho(X) \le 0\}$ denote the acceptance set of ϱ. Then for a monetary risk measure $\varrho \not\equiv \infty$ and with $\mathcal{A} = \mathcal{A}_\varrho$ it holds (like in the case $p = \infty$):

1. $\mathcal{A}$ is monotone and convex
2. ϱ is convex if and only if $\mathcal{A}$ is convex
3. $\varrho(X) = \begin{cases} \infty & \text{if } m + X \notin \mathcal{A} \quad \text{for all } m \in \mathbb{R} \\ \inf\{m \in \mathbb{R};\ m + X \in \mathcal{A}\} & \text{else} \end{cases}$
4. ϱ is coherent if and only if $\mathcal{A}$ is a convex cone.

Conversely, any $\mathcal{A} \subset L^p$, $\mathcal{A} \neq \emptyset$ convex and monotone such that

$$\inf\{m \in \mathbb{R};\ m + Y \in \mathcal{A}\} > -\infty \quad \text{for all } Y \in L^p \tag{1.7}$$

induces a convex risk measure $\varrho_\mathcal{A}$ by

$$\varrho_\mathcal{A}(X) := \inf\{m \in \mathbb{R};\ m + X \in \mathcal{A}\}, \quad X \in L^p, \text{ and } \mathcal{A} \subset \mathcal{A}_{\varrho_\mathcal{A}}. \tag{1.8}$$

Condition (1.7) prevents, that $\varrho_\mathcal{A}$ attains the value $-\infty$. For $\mathcal{A} \subset L^\infty$ it is sufficient to replace (1.8) by

$$\inf\{m \in \mathbb{R};\ m \in \mathcal{A}\} > -\infty \tag{1.9}$$

2 Representation of convex risk measures on L^p

The basic tool for the representation of convex functionals is the Fenchel–Moreau Theorem which for reference purpuses we restate here (see Ekeland and Teman (1974)).

Theorem 2.1 (Fenchel–Moreau Theorem) *Let (E,τ) be a locally convex topological vectorspace with topological dual E^*. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be proper (i.e. $f \not\equiv \infty$) convex and lower semicontinuous. Then f is identical to the doubly conjugate f^{**} i.e.*

$$f(x) = \sup_{x^* \in E^*} (< x^*, x > - f^*(x^*)) \qquad \text{for all } x \in E, \tag{2.1}$$

where $f^*(x^*) = \sup_{x \in E} (< x^*, x > - f(x))$, $x^* \in E^*$, is the conjugate of f.

To apply this representation theorem to convex risk functionals the following extension of a classical theorem of Namioka stating that positive linear functionals on a Fréchet-lattice are continuous is most useful.

Theorem 2.2 (Extended Namioka Theorem, Biagini and Frittelli (2006)) *Let (E,τ) be a Fréchet-lattice and $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper, convex, increasing function. Then f is continuous on $I_f := \operatorname{int}(dom\, f)$, where $dom\, f = \{x \in E : f(x) < \infty\}$ denotes the domain of f.*

As consequence this result implies the following continuity properties of convex risk measures.

Corollary 2.3 *Let $\varrho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \le p \le \infty$, be a proper convex risk measure. Then*

1. *ϱ is continuous on I_ϱ w.r.t. relative norm topology.*
2. *Any finite convex risk meaure on L^p, $1 \le p \le \infty$ is continuous an L^p.*
3. *If $\varrho : L^\infty \to \mathbb{R} \cup \{\infty\}$ is a proper convex risk measure, then ϱ is finite and continuous on L^∞.*

In general nonfinite convex risk measures on L^p are not lower semicontinuous except for $p = \infty$. For $p = \infty$ convex risk measures are even Lipschitz continuous (see Föllmer and Schied (2004, Lemma 4.3)). Let $\mathcal{M} = \mathcal{M}_1$ denote the class of all P-continuous probability measures on $(\Omega, \mathfrak{A})$. We identify, the normed positive part of the dual space of L^p with

$$\mathcal{Q}_p := \mathcal{M}_1^q = \left\{ Q \in M_1;\ \frac{dQ}{dP} \in L^q \right\} \quad \text{for } 1 \le p < \infty, \tag{2.2}$$

where q is the conjugate index and

$$\begin{aligned} \mathcal{Q}_\infty &:= \mathcal{M}_{1,f} \\ &= \{Q \in ba(P); Q \text{ is a normed finite additive } P\text{-continuous measure}\} \end{aligned} \tag{2.3}$$

in case $p = \infty$. We call a risk measure ϱ on L^p *representable* if ϱ has a representation of the form

$$\left\{ \begin{array}{l} \varrho(X) = \sup_{Q \in \mathcal{Q}} (E_Q(-X) - \alpha(Q)) \text{ for some } \mathcal{Q} \subset \mathcal{Q}_p \\ \text{and } \alpha : \mathcal{Q} \to \mathbb{R} \cup \{\infty\}, \text{ such that } \inf_{Q \in \mathcal{Q}} \alpha(Q) \in \mathbb{R} \end{array} \right\} \tag{2.4}$$

As consequence of the Fenchel–Moreau theorem one obtains the following representation result of convex risk measures (see Inoue (2003), Nakano (2004), Dana (2005), Biagini and Frittelli (2006)).

Theorem 2.4 (Representation of convex risk measures on $\boldsymbol{L^p}$)

a) Let $\varrho : L^p \to \mathbb{R} \cup \{\infty\}$ be a proper, convex, lower semicontinuous (w.r.t. $\|\cdot\|_p$) risk measure on L^p, then

1) $\varrho(X) = \sup_{Q \in \mathcal{Q}_p} (E_Q(-X) - \varrho^*(Q)), X \in L^p$ *for* $1 \le p \le \infty$. (2.5)

2) $\varrho^*(Q) = \sup_{X \in \mathcal{A}_\varrho} E_Q(-X)$. (2.6)

b) A monetary risk measure ϱ on L^p, $1 \le p \le \infty$, is representable if and only if ϱ is convex and lower semicontinuous w.r.t $\|\cdot\|_p$.

Remark 2.1 a) In comparison to the case $p = \infty$ where the representation is based on finitely additive measures the representation in the case $p < \infty$ is restricted to probability measures. From this point of view the case $1 \le p < \infty$ is more pleasant than the case $p = \infty$. All finite convex risk measures on L^p, $p < \infty$, are norm continuous by Corollary 2.3 and thus have a representation as in (2.5).

b) In the case $p = \infty$, $\mathcal{Q}_\infty = \mathcal{M}_{1,f}$ is weak-$*$-compact and any convex risk measure is upper semi-continuous w.r.t. weak-$*$-topology and thus the sup in (2.5) is attained and

$$\varrho(X) = \max_{Q \in \mathcal{Q}_\infty} (E_Q(-X) - \varrho^*(Q)), \quad X \in L^\infty, \tag{2.7}$$

see Föllmer and Schied (2004, Theorem 4.15).

Coherent risk measures on L^p have a simplified representation and allow to show that the sup is attained as in (2.7) for the L^∞-case. The proof of these properties is based on the following lemmas.

Lemma 2.5 *Let $f : E \to \mathbb{R}$ be a convex, positively homogeneous function on a normed Fréchet-lattice $(E, ||\cdot||)$. Then f is Lipschitz-continuous, i.e. there exists a $C < \infty$ such that*

$$|f(x) - f(y)| \le C||x - y||, \quad \forall x, y \in E. \tag{2.8}$$

Proof By the extended Namioka Theorem (Theorem 2.2) f is $||\cdot||$- continuous. This implies similarly to the proof of boundedness of continuous linear functionals that f is bounded. Therefore, for some constant $C < \infty$

$$f(x) \le |f(x)| \le C||x||, \quad x \in E.$$

This implies using convexity and positive homogeneity

$$f(x) - f(y) \leq 2f\Big(\frac{1}{2}x - \frac{1}{2}y\Big) = f(x-y) \leq C||x-y||$$

and

$$f(y) - f(x) \leq f(y-x) \leq C||x-y||,$$

i.e. (2.7) follows. □

For the case $E = L^p$ and for coherent risk measures the equivalence of continuity and the Lipschitz property was already stated in Inoue (2003, Lemma 2.1).

Corollary 2.6 *Any finite coherent risk measure $\varrho : L^p \to \mathbb{R}$ is Lipschitz continuous.*

The following two results generalize the corresponding results in Föllmer and Schied (2004, Corollary 4.8) for the case $E = L^\infty$.

Proposition 2.7 *Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper convex, lower semicontinuous (lsc) positive homogeneous function on a locally convex topological vector space (E, τ). Then*

$$f^*(x^*) \in \{0, \infty\}, \quad \forall x^* \in E^*.$$

Proof For $x^* \in E^*$ holds

$$\begin{aligned} f^*(x^*) &= \sup_{x \in E}(x^*(x) - f(x)) \\ &= \sup_{\lambda x \in E}(x^*(\lambda x) - f(\lambda x)) \\ &= \lambda f^*(x^*), \quad \forall \lambda \geq 0. \end{aligned}$$

Thus $f^*(x^*) \in \{0, \infty\}$. □

Proposition 2.8 *Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper convex, lsc, positively homogenous function on a normed vectorspace $(E, ||\cdot||)$. Then*

$$f(x) = \max_{x^* \in \mathcal{Q}} x^*(x) \tag{2.9}$$

with $\mathcal{Q} := \{x^* \in E^*;\ f^*(x^*) = 0\}$.

Proof By the Fenchel–Moreau Theorem (Theorem 2.1) and Proposition 2.7

$$f(x) = \sup\{x^*(x);\ x^* \in E^*,\ f^*(x^*) = 0\}. \tag{2.10}$$

We have to prove that the sup is attained. By Brezis (1999, Proposition I.9) f is $\sigma(E^*, E)$ lower semicontinuous. Thus

$$\{x^* \in E^*; f^*(x^*) = 0\} = \{x^* \in E^*; f^*(x^*) \leq 0\} \tag{2.11}$$

ist $\sigma(E^*, E)$-closed. Further,

$$\{x^* \in E^*;\ f^*(x^*) = 0\} \quad \text{is } ||\cdot||_{E^*}\text{-bounded.} \tag{2.12}$$

To argue for (2.12) note that

$$f^*(x^*) = 0 \Leftrightarrow x^*(x) \le f(x), \quad \forall x \in E$$

and

$$\begin{aligned} f^*(x^*) \le 0 &\Leftrightarrow \sup_{x\in E}(x^*(x) - f(x)) = 0 \\ &\Leftrightarrow \inf_{x\in E}(-x^*(x) + f(x)) = 0 \\ &\Leftrightarrow x^*(-x) \ge -f(x), \quad \forall x \in E \\ &\Leftrightarrow x^*(x) \ge -f(-x), \quad \forall x \in E. \end{aligned}$$

Thus for $x \in E$, $\{x^*(x); x^* \in E^*, f^*(x^*) = 0\}$ is bounded below by $-f(-x)$ and above by $f(x)$. But pointwise boundedness of $\mathcal{Q}$ implies norm-boundedness of w.r.t. $||\cdot||_{E^*}$.

By Alaoglu's Theorem $\mathcal{Q}$ is $\sigma(E^*, E)$-compact. Thus for all $x \in E$ the continuous functional $x^* \to x^*(x)$ attains its supremum on $\mathcal{Q}$. $\square$

As consequence we now obtain a more specific version of the representation result in (2.4) to L^p in the case of coherent risk measures.

Theorem 2.9 (Representation of coherent risk measures on L^p)
Let $\varrho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \le p \le \infty$; then

a) ϱ is a proper, $\|\cdot\|_p$-lsc, coherent risk measure
$\Leftrightarrow \exists \mathcal{Q} \subset \mathcal{Q}_p$ such that

$$\varrho(X) = \max_{Q\in\mathcal{Q}} E_Q(-X), \quad X \in L^p. \tag{2.13}$$

$\Leftrightarrow$ ϱ is a finite, $||\cdot||_p$-continuous, coherent risk measure.
b) In case $p = \infty$ holds:
ϱ is a finite coherent risk measure (2.14)
$\Leftrightarrow \varrho(X) = \max_{Q\in\mathcal{Q}} E_Q(-X)$ for some $Q \subset \mathcal{Q}_\infty$.

Proof a) Denote the equivalences by 1), 2), 3). Then
1) $\Rightarrow$ 2) follows from Propositions 2.8 and 2.7.
2) $\Rightarrow$ 3) The properties of a coherent risk measure are easy to establish. Finiteness of ϱ follows by definition and thus $I_\varrho = \{X \in L^p; \varrho(X) < \infty\}$ $= L^p$. By the extended Namioka Theorem (Theorem 2.2) we obtain that ϱ is $||\cdot||_p$-continuous.
3) $\Rightarrow$ 1) is obvious.

b) Since coherent risk measures on L^∞ are $||\cdot||_\infty$-continuous (see Corollary 2.3). b) follows from a). $\square$

In the next step we extend the attainment result in (2.13) to the representation of finite convex risk measures on L^p, $1 \leq p < \infty$. Let ϱ have a representation of the form (see (2.4))

$$\varrho(X) = \sup_{Q \in \mathcal{Q}} \left(E_Q(-X) - \varrho^*(Q) \right), \tag{2.15}$$

with $\mathcal{Q} \subset \{Q \in \mathcal{M}_p;\ \varrho^*(Q) < \infty\}$. Assuming

$$\varrho(0) = - \inf_{Q \in \mathcal{Q}} \varrho^*(Q) < \infty \tag{2.16}$$

we obtain $\varrho^*(Q) \geq a := -\varrho(0)$, $\forall\, Q \in \mathcal{Q}$. In fact finiteness of ϱ is then equivalent to norm-boundedness of $\mathcal{D} := \{\frac{dQ}{dP};\ Q \in \mathcal{Q}\} \subset L^q$.

Proposition 2.10 *Let ϱ be a proper, convex lsc risk measure on L^p with representation* (2.15) *and satisfying* (2.16)*. Then it holds:*

ϱ is finite if and only if $\mathcal{D} \subset L^q$ is norm-bounded.

Proof If $\mathcal{D}$ is norm-bounded, then by (2.15) and (2.16)

$$E_Q(-X) - \varrho^*(Q) \leq \|X\|_p \left\| \frac{dQ}{dP} \right\|_q - \varrho^*(Q) \leq \|X\|_p \sup_{Q \in \mathcal{Q}} \left\| \frac{dQ}{dP} \right\|_q - a$$

and thus $\varrho(X) < \infty$.

Converseley, if $\mathcal{D}$ is not norm-bounded in L^q, then $\forall\, n \in \mathbb{N}$ there exist $X_n \in L^p$, $Q_n \in \mathcal{Q}$ such that $X_n \leq 0$, $\|X_n\|_p = 1$, and $E_{Q_n}(-X_n) \geq n^3$. Defining $X := \sum \frac{1}{n^2} X_n \in L^p$, then $X \leq \frac{X_n}{n^2}$ for all n and thus

$$\varrho(X) \geq \varrho\left(\frac{X_n}{n^2}\right) \geq E_{Q_n}\left(\frac{X_n}{n^2}\right) - a \geq n - a$$

for all $n \in \mathbb{N}$. This implies $\varrho(X) = \infty$. $\square$

As consequence we obtain for finite convex risk measures the analogon of the representation in Theorem 2.9 for coherent risk measures.

Theorem 2.11 (Representation of finite convex risk measures) *Let $\varrho : L^p \to \mathbb{R} \cup \{\infty\}$ be a risk measure with $\varrho(0) < \infty$. Then ϱ is a finite convex risk measure if and only if ϱ has a representation of the form*

$$\varrho(X) = \max_{Q \in \mathcal{Q}} \left(E_Q(-X) - \varrho^*(Q) \right) \tag{2.17}$$

for some representaion set Q such that $\mathcal{D} = \{\frac{dQ}{dP};\ Q \in \mathcal{Q}\} \subset L^q$ is weakly compact.

Proof If ϱ is finite, then by Theorem 2.4 and Proposition 2.10 the density set $\mathcal{D} = \{\frac{dQ}{dP};\ Q \in \mathcal{Q}\}$ of a representation set $\mathcal{Q}$ is norm-bounded. W.l.g. we can assume that $\mathcal{D}$ is weakly closed and thus by the Banach Alaoglu theorem $\mathcal{D}$ is weakly compact. For any $X \in L^p$ the mapping $Q \to E_Q(-X) - \varrho^*(Q)$ is usc w.r.t. $\sigma(L^q, L^p)$ and thus the sup in the representation of ϱ is attained (see (2.6), (2.4)).

If conversely a representation as in (2.17) holds, then $\mathcal{D}$ can be chosen convex and thus $\mathcal{D}$ is norm-bounded. By Proposition 2.10, therefore, ϱ is a finite convex risk measure. □

Remark 2.2 Nakano (Theorem 1.2, 2004) characterizes lsc finite coherent risk measures on L^1 by a representation of the form

$$\varrho(X) = \sup_{Q \in \mathcal{Q}} E_Q(-X), \quad X \in L^1 \tag{2.18}$$

with some subset $\mathcal{Q} \subset \mathcal{Q}_1$, such that $\mathcal{D} := \{\frac{dQ}{dP};\ Q \in \mathcal{Q}\} \subset L^\infty$ is weak $*$-closed. The attainment of the sup as is Theorem 2.9 is not considered in that paper. Inoue (2003) characterizes finite continuous coherent risk measures on L^p by a representation as in (2.15) and the norm-boundedness condition

$$\sup_{Q \in \mathcal{Q}} \left|\left| \frac{dQ}{dP} \right|\right|_q < \infty, \tag{2.19}$$

corresponding to Proposition 2.10 After having finished this paper we got to know a recent preprint of Cheridito and Li (2006) who establish a similar attainment result as in Theorem 2.11 in the more general context of Orlicz spaces. The method of proof of Cheridito and Li (2006) however is different and more involved compared to our proof.

In the final part of this section we consider convex risk measures restricted to some cash-invariant subsets $M \subset L^p$, i.e. $X \in M$ and $m \in \mathbb{R}$ implies $X + m \in M$. M describes some restrictions on the class of risks, as for example induced by restrictions on the market or by postulates of some regulation authorities. Trading and exchanging of risks is only allowed within M. This is relevant in particular for the restricted optimal risk allocation problem (see e.g. Heath and Ku (2004) and Burgert and Rüschendorf (2006a)).

The following representation result extends Filipovic and Kupper (2006) who consider the case $p = \infty$. For $\varrho : L^p \to \mathbb{R} \cup \{\infty\}$ define the restriction ϱ^M of ϱ to M by

$$\varrho^M(X) := \begin{cases} \varrho(X), & X \in M, \\ \infty, & else. \end{cases} \tag{2.20}$$

Theorem 2.12 (Representation of restricted convex risk measures) *Let $\varrho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \le p \le \infty$, be a proper, convex lsc risk measure. Let*

$M \subset L^p$ be convex, closed and cash invariant, $M \not\equiv \emptyset$. Then the restricted risk measure ϱ^M is proper, convex, cash invariant, $\sigma(L^p, L^q)$-continuous and

$$\varrho^M(X) = \sup_{Q\in\mathcal{Q}_p} \Big(E_Q(-X) - \big(\varrho^M\big)^*(Q)\Big), \quad \forall X \in M. \tag{2.21}$$

In case $p = \infty$ of if $p < \infty$ and ϱ is finite the sup *in (2.21) is attained.*

Proof By definition and using the properties of M the restriction ϱ^M of ϱ is proper, convex and cash invariant. Further for $c \in \mathbb{R}$, $\{\varrho^M \leq c\} = \{\varrho \leq c\} \cap M$ is convex and norm closed and thus by a wellknown criterion $\sigma(L^p, L^q)$-closed. Therefore, ϱ^M is $\sigma(L^p, L^q)$-lsc.

The Fenchel-Moreau Theorem can be applied to the monotone cash invariant risk measure ϱ and yields the representation

$$\varrho(X) = \sup_{Q\in\mathcal{Q}_p} (E_Q(-X) - \varrho^*(Q)), \quad X \in L^p. \tag{2.22}$$

It can also be applied to ϱ^M and yields

$$\varrho^M(X) = \sup_{Q\in L^q} (E_Q(-X) - (\varrho^M)^*(Q)), \quad X \in L^p, \tag{2.23}$$

where $Q \in L^q$ is identified with the signed measure $\frac{dQ}{dP}P$. Note that ϱ^M is not monotone in general and therefore we can not restrict in (2.23) to normed positive measures but have to take the sup over $\mathcal{Q} \in L^q$. Generally, for $Q \in \mathcal{Q}_p$ we have

$$\begin{aligned}\varrho^*(Q) &= \sup_{X\in L^p} (E_Q(-X) - \varrho(X)) \\ &\geq \sup_{X\in L^p} (E_Q(-X) - \varrho^M(X)) = (\varrho^M)^*(Q).\end{aligned}$$

By (2.22) and (2.23) we obtain, therefore, for $X \in M$

$$\begin{aligned}\varrho^M(X) = \varrho(X) &= \sup_{Q\in\mathcal{Q}_p} (E_Q(-X) - \varrho^*(Q)) \\ &\leq \sup_{Q\in\mathcal{Q}_p} (E_Q(-X) - (\varrho^M)^*(Q)) \\ &\leq \sup_{Q\in L^p} (E_Q(-X) - (\varrho^M)^*(Q)) = \varrho^M(X).\end{aligned}$$

This implies that the representation in (2.23) can be restricted to $Q \in \mathcal{Q}_p$ and thus yields (2.21).

In the case $p = \infty$, $\mathcal{Q}_\infty$ is $\sigma((L^\infty)', L^\infty)$-compact (see Filipovic and Kupper (2006, Lemma 3.6)). Further the function $\mathcal{Q}_\infty \to \mathbb{R}$, $Q \to E_Q(-X) - (\varrho^M)^*(Q)$ is $\sigma((L^\infty)', L^\infty)$-usc and thus also ϱ^M is usc and attains the supremum. For the case $1 \leq p < \infty$ the attainment follows from Theorem 2.11.
□

3 Pointwise continuity of convex risk measures on L^p

By the extended Namioka Theorem all finite convex risk measures on L^p are $\| \|_p$-continuous. In this section we investigate pointwise continuity properties of convex risk measures on L^p and consequences for their representation. Pointwise continuity properties are an important tool for approximation arguments and in particular imply qualitative robustness properties of risk measures which are important for their applications. Continuity properties of convex risk measures on L^∞ and consequences for their representation have been investigated in detail in Föllmer and Schied (2004). Jouini, Schachermayer, and Touzi (2006) have etablished the interesting fact that all finite convex, law invariant risk measures on L^∞ are $\sigma(L^\infty, L^1)$-usc. Thus by the characterization of continuity properties in Föllmer and Schied (2004, Theorem 4.31) they are Fatou continuous and allow a representation based on probability measures. In a recent paper Biagini and Frittelli (2006) have established some continuity properties of risk functionals in a general framework of Riesz spaces and Fréchet spaces.

The following development of continuity properties for convex risk measures extends results from Föllmer and Schied (2004) to the case of L^p-spaces. It supplements and details some of the results in Biagini and Frittelli (2006) in the case of L^p-spaces. In the following definition we resume some relevant continuity properties of risk measures (see also Föllmer and Schied (2004, Chapter 4.2)).

Definition 3.1 *Let ϱ be a risk functional $\varrho : L^p \to \mathbb{R} \cup \{\infty\}, 1 \le p \le \infty$, on L^p.*

1. *ϱ is called* continuous from above*, if $(X_n) \subset L^p$, $X_n \downarrow X \quad P$ a.s. to some $X \in L^p$ implies that $\lim \varrho(X_n) = \varrho(X)$.*
2. *ϱ is called* continuous from below*, if $(X_n) \subset L^p$, $X_n \uparrow X \quad P$ a.s. to some $X \in L^p$ implies that $\lim \varrho(X_n) = \varrho(X)$.*
3. *ϱ is called* Fatou-continuous *(has the Fatou property), if $(X_n) \subset L^p$, $|X_n| \le Y \; P$ a.s. for some $Y \in L^p$ and $X_n \to X \quad P$ a.s. for some $X \in L^p$ implies $\varrho(X) \le \liminf \varrho(X_n)$.*
4. *ϱ is called Lebesgue-continuous if $(X_n) \subset L^p$, $X_n \to X \; P$ a.s., $X \in L^p$ and $|X_n| \le Y \; P$ a.s. for some $Y \in L^p$ implies $\lim \varrho(X_n) = \varrho(X)$.*

The following theorem shows that for the class of finite convex risk measures on L^p, $p < \infty$ all pointwise continuity properties are fulfilled.

Theorem 3.1 (Continuity of finite convex risk measures) *Let $\varrho : L^p \to \mathbb{R}$, $1 \le p < \infty$, be a finite convex risk measure. Then it holds:*

1. *ϱ is $\sigma(L^p, L^q)$-lsc.*
2. *$\mathcal{A}_\varrho$ is $\sigma(L^p, L^q)$-closed.*
3. *ϱ has the Fatou property.*
4. *ϱ is continuous from above.*
5. *ϱ is continuous from below.*
6. *ϱ is Lebesgue-continuous.*

Proof 1. By the extended Namioka Theorem (2.2), ϱ is $\| \|_p$-continuous. From Aliprantis and Border (1994, Corollary 4.73 and Example 4.67) τ-lower semicontinuity of a function $f : E \to \mathbb{R} \cup \{\infty\}$ on a topological vectorspace (E, τ) is equivalent to $\sigma(E, E^*)$-lsc of f. Thus 1. follows
2. $\mathcal{A}_\varrho = \{X \in L^p; \varrho(X) \leq 0\}$ is by 1. closed.
3. and 6. If $X_n \to X$ P a.s. and $|X_n| \leq Y$ P a.s. for some $Y \in L^p$ then by the majorized convergence theorem of Lebesgue $X_n \to X$ in L^p. As ϱ is $\| \|_p$-continuous, $\lim \varrho(X_n) = \varrho(X)$, ϱ has the Fatou property and ϱ is even Lebesgue-continuous.
4. follows from 3. observing that for $X_n \downarrow X$ P a.s. $|X_n| \leq \max\{|X_1|\ |X|\} \in L^p$.
5. similarly follows from 3. $\square$

For not necessarily finite risk measures the following Lemma extends Föllmer and Schied (2004, Lemma 4.20), who consider the case $p = \infty$.

Lemma 3.2 *Let $\varrho : L^p \to \mathbb{R} \cup \{\infty\}$ be a proper, convex risk measure on L^p, $1 \leq p \leq \infty$. Then it holds:*
ϱ is continuous from above $\Leftrightarrow$ ϱ has the Fatou property.

Proof "$\Rightarrow$" Is similar to the proof of the majorized convergence theorem as based on the monotone convergence theorem.

"$\Leftarrow$" For $(X_n) \subset L^p$, $X_n \downarrow X$ P a.s. for some $X \in L^p$ holds by monotonicity of ϱ, $\varrho(X_n) \leq \varrho(X)$, for all $n \in \mathbb{N}$. On the other hand by the Fatou property $\liminf \varrho(X_n) \geq \varrho(X)$ since $|X_n| \leq \max(|X_1|, |X|)$, $\quad \forall n \in \mathbb{N}$. Together we obtain $\lim \varrho(X_n) = \varrho(X)$. $\square$

The following result shows that $\| \|_p$-lsc convex risk measures ϱ are continuous from above and thus have the Fatou property also for non-finite ϱ.

Theorem 3.3 *Let $\varrho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \leq p \leq \infty$ be a proper, convex risk measure on L^p. Then the following are equivalent:*

1. *ϱ is $\sigma(L^p, L^q)$-lsc*
2. *ϱ is $\| \|_p$-lsc*
3. *$\varrho(X) = \sup_{Q \in \mathcal{M}_1^q}(E_Q(-X) - \varrho^*(Q)), \quad \forall X \in L^p$*
4. *ϱ is continuous from above*
5. *ϱ has the Fatou property*

Proof 1. $\Leftrightarrow$ 2. This holds true using the same argument as in the proof of 1. of Theorem 3.1.
2. $\Leftrightarrow$ 3. holds by Theorem 2.4.
4. $\Leftrightarrow$ 5. holds by Lemma 3.2.
2. $\Leftrightarrow$ 4. follows as in the proof of Lemma 3.2. $\square$

For completeness reasons we state the essential continuity results for the case of $p = \infty$ from the literature.

Theorem 3.4 (Continuity and representation, Föllmer and Schied (2004, Theorem 4.31)) *Let $\varrho : L^\infty \to \mathbb{R}$ be a finite convex risk measure. Then the following are equivalent:*

1. *ϱ has a representation on $\mathcal{M}_1$, the set of all P-continuous probability measures, i.e.*
$$\varrho(X) = \sup_{Q \in \mathcal{M}_1} (E_Q(-X) - \varrho^*(Q)), X \in L^\infty. \tag{3.1}$$
2. *ϱ is continuous from above.*
3. *ϱ is Fatou-continuous.*
4. *ϱ is $\sigma(L^\infty, L^1)$-lsc.*
5. *$\mathcal{A}_\varrho$ is $\sigma(L^\infty, L^1)$-closed.*

For the following interesting result on law invariant risk measures on L^∞ it is assumed that the underlying probability space $(\Omega, \mathfrak{A}, P)$ is an atomless, separable complete metric space with Borel σ-Algebra $\mathfrak{A}$.

Theorem 3.5 (Law invariant risk measures, Jouini et al. (2006)) *Let $\varrho : L^\infty \to \mathbb{R}$ be a finite convex, law invariant, risk measure. Then*
$$\varrho(X) = \sup_{Q \in L^1} (E_Q(-X) - \varrho^*(Q)), X \in L^\infty \tag{3.2}$$
and ϱ is $\sigma(L^1, L^\infty)$-lsc.

Remark 3.1 a) Theorem 3.5 is the basis to show that any law invariant, convex risk measure on L^∞ has a Kusuoka type representation via mixtures of average value at risk measures (see Jouini et al. (2006, Theorem 1.1)).
b) The proof of the $\sigma(L^1, L^\infty)$-lsc in Theorem 3.5 is similar to that of Theorem 3.1. For details see Kaina (2007, Korollar 4.2.19).

Finally, the following result of Föllmer and Schied (2004, Theorem 4.31) and (Jouini et al., 2006, Theorem 5.2), combines continuity properties of convex risk measures on L^∞ with representability on $\mathcal{M}_1$ and attainment of the supremum.

Theorem 3.6 (Convex risk measures on L^∞) *Let $\varrho : L^\infty \to \mathbb{R}$ be a convex risk measure on L^∞ with $\sigma(L^\infty, (L^\infty)')$-conjugate $\varrho^* : (L^\infty)' \to \mathbb{R} \cup \{\infty\}$. Then it holds:*

a) *The following conditions are equivalent:*
   1. *ϱ is Lebesgue-continuous.*
   2. $\varrho(X) = \sup_{Q \in \mathcal{M}_1} (E_Q(-X) - \varrho^*(Q), X \in L^\infty,$ (3.3)

      $\{\varrho^* \leq c\} \subset L^1$ for all $c \in \mathbb{R}$ and $\{\varrho^* \leq c\}$ is uniformly integrable for all $c > \max\{0, -\varrho(0)\}$.
   3. *ϱ is continuous from below.*
   4. *ϱ is Fatou continuous and* $\operatorname{dom} \varrho^* = \{\varrho^* < \infty\} \subset L^1$.

5. ϱ is Fatou continuous and $\{\varrho^* \le c\}$ is a $\sigma(L^1, L^\infty)$-compact subset of L^1 for all $c > -\varrho(0)$.

b) 1. - 5. imply that the sup *in* (3.3) *is attained*

c) If L^1 is separable, then 1. - 5. are equivalent with the attainment of the sup *in* (3.3).

4 The expected shortfall and the shortfall risk

A classical example of a coherent risk measure is the expected shortfall risk introduced in the early papers on risk measures in mathematical finance (see Delbaen (2002), Acerbi and Tasche (2002), Rockafellar and Uryasev (2002), Föllmer and Schied (2004), and Tasche (2002) and many further references therein).

The expected shortfall risk measure ES_α, $\alpha \in (0,1)$ is naturally defined on L^1 (or even more generally on $\{X \in L^0;\ EX^- < \infty\}$). Let

$$q_\alpha(X) = \inf\{x \in \mathbb{R} : P(X \le x) \ge \alpha\} = q_\alpha^-(X) \tag{4.1}$$

denote the lower α-quantile of X. Then define for $X \in L^1$ the *expected shortfall risk* by

$$ES_\alpha(X) = \frac{1}{\alpha}(E(X1_{\{X \le q_\alpha(X)\}} + q_\alpha(X)(\alpha - P(X \le q_\alpha(X))). \tag{4.2}$$

In the literature one can find further notions for the expected shortfall risk measure like "average value at risk" or "conditional value at risk", which are explained by the following representations. The expected shortfall can be represented as

$$ES_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha q_\lambda(X)d\lambda \tag{4.3}$$

(see Acerbi and Tasche (2002, Proposition 3.2)) as well as

$$ES_\alpha(X) = \frac{1}{\alpha}E((q-X)^+] - q, \quad q = q_\alpha(X) \tag{4.4}$$

(see Föllmer and Schied (2004, Lemma A.19)). For a further representation we use that the distributional transform of X defined as $U = F(X,V)$ with $F(x,\lambda) = P(X < x) + \lambda P(X = x)$, is uniformly distributed on $(0,1)$ and it holds that $X = F^{-1}(U)$ P a.s.. From this property one sees that ES_α can also be defined as conditional value at risk

$$ES_\alpha(X) = -E(X \mid U \le \alpha) = CV_\alpha(X) \tag{4.5}$$

(see Burgert and Rüschendorf (2006a)).

$ES_\alpha(X)$ is a coherent risk measure on L^1 (see Tasche (2002)). $ES_\alpha(X)$ is obviously law invariant. The following result extends the representation of $ES_\alpha(X)$ by scenario measures from the case of L^∞-risks (see Föllmer and Schied (2004, Theorem 4.47)) to the case of L^1-risks.

Theorem 4.1 (Representation of the expected shortfall on L^1) *The expected shortfall risk measure $ES_\alpha : L^1 \to \mathbb{R}$ is a coherent risk measure on L^1 with representation*

$$ES_\alpha(X) = \max_{Q \in \mathcal{Z}_\alpha} E_Q(-X), \tag{4.6}$$

where $\mathcal{Z}_\alpha := \{Q \in \mathcal{M}_1;\ \frac{dQ}{dP} \le \frac{1}{\alpha}\ P\ a.s.\}$. In particular ES_α has all the continuity properties stated in Theorem 3.1.

Proof It remains to establish (4.6). Since $ES_\alpha(X) \in \mathbb{R}^1$ for all $X \in L^1$, it is by the extended Namioka Theorem continuous on L^1. Thus ES_α has by Proposition 2.8 and Theorem 2.12 a representation of the form

$$ES_\alpha(X) = \max_{Q \in \mathcal{Q}} E_Q(-X), \quad X \in L^1, \tag{4.7}$$

where $\mathcal{Q} = \{Q \in \mathcal{M}_1^\infty;\ (ES_\alpha)^*(Q) = 0\}$.

In the first step we show that for $Q \notin \mathcal{Z}_\alpha$

$$(ES_\alpha)^*(Q) = \sup_{X \in L^1} (E_Q(-X) - ES_\alpha(X)) = \infty. \tag{4.8}$$

The argument for (4.8) can be given similarly to the proof of the corresponding statement in the L^∞-case in Föllmer and Schied (2004, pg. 181).

By (4.8) we conclude that for $Q \in \mathcal{M}_1^\infty$, $ES_\alpha^*(Q) = 0$ implies that $Q \in \mathcal{Z}_\alpha$. Thus by (4.7) we obtain

$$(ES_\alpha)^*(X) = \max_{Q \in \mathcal{Z}_\alpha} E_Q(-X), \quad X \in L^1. \quad \square$$

The shortfall risk is an extension of the expected shortfall. For a convex, increasing loss function $\ell : \mathbb{R} \to \mathbb{R}$, ℓ not identically constant, assume that for some $1 \le p \le \infty$, $X \in L^p$ implies

$$E\ell(-X) < \infty. \tag{4.9}$$

Define for some $x_0 \in \mathbb{R}$ the acceptance set

$$\mathcal{A}_p := \{X \in L^p;\ E\ell(-X) \le x_0\} \tag{4.10}$$

and denote by

$$SR_p(X) := \inf\{m \in \mathbb{R}; X + m \in \mathcal{A}_p\}, \quad X \in L^p \tag{4.11}$$

the generated risk measure on L^p. $SR_p(X)$ is called *shortfall risk*. For $p = \infty$ this risk measure has been investigated in detail in Föllmer and Schied (2004, section 4.9). Thus we restrict to the case $p < \infty$.

Proposition 4.2 (Shortfall risk) *Under assumption* (4.9) *the shortfall risk SR_p, $1 \le p < \infty$, is a finite, convex, $\|\ \|_p$-continuous risk measure on L^p. Thus SR_p has all the pointwise continuity properties in Theorem 3.1. Further SR_p has a representation of the form*

$$SR_p(X) = \max_{Q \in \mathcal{Q}_p} (E_{\mathcal{Q}}(-X) - SR_p^*(Q)), \quad X \in L^p. \tag{4.12}$$

Proof Since $\mathcal{A}_p$ is convex and monotone it follows from (4.9) that SR_p defines a finite convex risk measure on L^p. Thus by Theorem 3.1 SR_p is continuous from above, from below and Lebesgue continuous and the acceptance set $\mathcal{A}_p$ is $\sigma(L^p, L^q)$-closed. The representation property in (4.12) is then a consequence of Theorems 2.4 and 2.11. □

Remark 4.1 Föllmer and Schied (2004, Theorem 4.16) establish in the case $p = \infty$ that the minimal penalty function (the conjugate) $(SR_p)^* = \alpha^p_{\min}$ is given by

$$(SR_p)^*(Q) = \inf_{\lambda>0} \frac{1}{\lambda}\left(x_0 + E\ell^*\left(\lambda\frac{dQ}{dP}\right)\right), \quad Q \in \mathcal{M}_1. \tag{4.13}$$

Formula (4.13) also extends in the same form to $1 \le p < \infty$. For the proof note that for any $\lambda > 0$ and X in the acceptance set $\mathcal{A}^p_\varrho$ with $\phi = \frac{dQ}{dP}$ it holds

$$-X\phi = \frac{1}{\lambda}(-X)(\lambda\phi) \le \frac{1}{\lambda}(\ell(-X) + \ell^*(\lambda\phi)).$$

In consequence

$$\alpha^p_{\min}(Q) = (SR_p)^*(Q) = \sup_{X\in\mathcal{A}^p_\varrho} E_Q(-X) \le \frac{1}{\lambda}(x_0 + E\ell^*(\lambda\phi)). \tag{4.14}$$

On the other hand since $\mathcal{A}^p_\varrho = \{X \in L^p; SR_p(X) \le 0\} \supset \mathcal{A}^\infty_\varrho$ if follows that

$$\alpha^p_{\min}(Q) = \sup_{X\in\mathcal{A}^\checkmark_\varrho} E_Q(-X) \ge \sup_{X\in\mathcal{A}^\infty_\varrho} E_Q(-X) = \alpha^\infty_{\min}(Q).$$

This implies that (4.13) also holds for $1 \le p < \infty$.

Corollary 4.3 *The minimal penalty function of the shortfall risk SR_p on L^p is given by*

$$(SR_p)^*(Q) = \inf_{\lambda>0} \frac{1}{\lambda}\left(x_0 + E\ell\left(\lambda\frac{dQ}{dP}\right)\right), \quad Q \in \mathcal{M}_p. \tag{4.15}$$

Example 4.4 *For $\ell(x) = \max(\frac{1}{p}x^p, 0)$, $p > 1$ holds*

$$\ell^*(z) = \begin{cases} \frac{1}{q}z^q, & z \ge 0, \\ \infty & \textit{else}. \end{cases}$$

For $Q \in \mathcal{M}_p$ holds $\phi = \frac{dQ}{dP} \in L^q$ and the infimum in (4.3) is attained for $\lambda_Q = (\frac{px_0}{E\phi^q})^{\frac{1}{q}}$. Thus one obtains the explicit result:

$$(SR_p)^*(Q) = (px_0)^{\frac{1}{p}} E\left[\left(\frac{dQ}{dP}\right)^q\right]^{\frac{1}{q}} < \infty, \quad Q \in \mathcal{M}_p \tag{4.16}$$

(see also Föllmer and Schied (2004, Example 4.109)).

5 Optimal risk allocation (risk sharing) in L^p

Consider in a market model $(\Omega, \mathfrak{A}, P)$ with n economic agents endowed with finite convex risk measures $\varrho_i : L^p \to \mathbb{R}$, $1 \le i \le n$ and risky positions $\xi_1, \dots, \xi_n \in L^p$. Let $X = \sum_{i=1}^n \xi_i$ be the total risk. The problem is whether there are "better" allocations in the class of all allocations of X to the agents

$$\mathcal{A}(X) = \mathcal{A}_p(X) = \left\{ (X_1, \dots, X_n)\,; \quad X_i \in L^p, \quad \sum_{i=1}^n X_i = X \right\}. \tag{5.1}$$

An allocation $(X_i) \in \mathcal{A}(X)$ is called *Pareto optimal allocation (POA)* if for any other allocation $(Y_i) \in \mathcal{A}(X)$ such that

$$\varrho_i(Y_i) \le \varrho_i(X_i),\ 1 \le i \le n,\ \text{it holds that}\ \varrho_i(Y_i) = \varrho_i(X_i),\ 1 \le i \le n. \tag{5.2}$$

Besides Pareto optimality as postulate on an allocation it is natural to postulate that any risk exchange form ξ_i to X_i for the i-th agent leads to an individual improvement. This postulate is called individual rationality postulate.

(IR) X satisfies the individual rationality constraint if

$$\varrho_i(X_i) \le \varrho_i(\zeta_i), 1 \le i \le n. \tag{5.3}$$

It is however easy, based on the cash invariance, to construct from Pareto optimal allocations those which also satisfy the IR condition (see Jouini et al. (2006)) and thus we restrict to the consideration of the PO-property.

The optimal allocation problem is a classical problem in insurance and finance. It is usually considered in the frame of risks on L^∞ (see for example Barrieu and Karoui (2005), Burgert and Rüschendorf (2006a), Rüschendorf (2006), Jouini et al. (2006)). Based on the continuity and representation properties in the preceeding sections it is possible to extend the relevant results to risks in L^p. We shall concentrate mainly on the case $1 \le p < \infty$. We define the *infimal convolution risk measure* on L^p

$$\widehat{\varrho}(X) = \varrho_1 \wedge \ldots \wedge \varrho_n(X) = \inf \left\{ \sum_{i=1}^n \varrho_i(X_i); \quad (X_i) \in \mathcal{A}(X) \right\} \tag{5.4}$$

By definition $\widehat{\varrho}$ is convex, monotone and cash-invariant and $\widehat{\varrho}(X) < \infty$ on L^p. To ensure finiteness of $\widehat{\varrho}$ we assume

(E) The intersection of the *scenario* sets $\operatorname{dom} \varrho_i^*$ of the risk measures ϱ_i is nonempty,

$$\bigcap_{i=1}^n \operatorname{dom} \varrho_i^* \ne \phi. \tag{5.5}$$

Condition (E) implies that $\widehat{\varrho}$ is a finite convex risk measure and

$$\widehat{\varrho}^* = \sum_{i=1}^n \varrho_i^* \quad \text{with } \operatorname{dom} \widehat{\varrho}^* = \bigcap_{i=1}^n \operatorname{dom} \varrho_i^*. \tag{5.6}$$

When all ϱ_i are law invariant risk measures, then assumption (E) is satisfied if we assume that $\varrho_i(0) = 0$, $1 \leq i \leq n$. Condition (E) is related to an equilibrium condition for the market introduced in Heath and Ku (2004) (see also Burgert and Rüschendorf (2006a,b), Rüschendorf (2006)). It describes that the view towards risk of the n agents is not too different and does not allow *risk-arbitrage*.

Proposition 5.1 *Consider the risk allocation problem with finite convex risk measures ϱ_i on L^p and assume condition (E). Then for an allocation $(\xi_i) \in \mathcal{A}_p(X)$ the following statements are equivalent:*

1. *(ξ_i) is a Pareto optimal allocation*
2. *$\widehat{\varrho}(X) = \sum_{i=1}^n \varrho_i(\xi_i)$*
3. *$\exists Q \in \mathcal{Q}_p$ such that $\varrho_i(\xi_i) = E_Q(-\xi_i) - \varrho_i^*(Q)$, $1 \leq i \leq n$.*

Proof The proof of Proposition 5.1 is similar to the proof in the case $p = \infty$ (see Jouini et al. (2006)).
2. $\Rightarrow$1. is obvious.
1. $\Rightarrow$ 2. can be seen by a simple reallocation argument using cash-invariance (see Burgert and Rüschendorf (2006a)).
2. $\Rightarrow$ 3. Let $Q \in \mathcal{M}_p, Q \in \partial\widehat{\varrho}(X)$ i.e. $\widehat{\varrho}(X) = E_Q(-X) - \widehat{\varrho}^*(Q)$. Then condition 2. and (5.6) imply

$$\widehat{\varrho}(X) = E_Q(-X) - \widehat{\varrho}^*(Q) = \sum_{i=1}^n (E_Q(-\xi_i) - \varrho_i^*(Q)) = \sum_{i=1}^n \varrho_i(\xi_i)$$

Since $\varrho_i(\xi_i) \leq E_Q(-\xi_i) - \varrho_i^*(Q)$, $1 \leq i \leq n$, this implies 3.
3. $\Rightarrow$ 2. Using 3. and the duality relations, we obtain

$$\begin{aligned}\widehat{\varrho}(X) \geq E_Q(-X) - \widehat{\varrho}^*(Q) &= E_Q\Big\{ -\sum_{i=1}^n \xi_i \Big\} - \sum_{i=1}^n \varrho_i^*(Q) \\ &= \sum_{i=1}^n \varrho_i(\xi_i) \geq \widehat{\varrho}(X).\end{aligned}$$

Thus the equality follows. □

From the practical point of view it is of importance to give a suitable version of the allocation problem also in the case that the equilibrium condition (E) does not hold. This has been given and analyzed in Burgert and Rüschendorf (2006a,b) and Rüschendorf (2006) for risks in L^∞. The idea is to restrict the class of decompositions in a suitable way to avoid *risk arbitrage*. We define for $X \in L^p$ a decomposition $X = \sum_{i=1}^n X_i$, $X_i \in L^p$ to be *admissible* if $X(\omega) \geq 0$ implies that $X_i(\omega) \geq 0$ and $X(\omega) \leq 0$ implies that $X_i(\omega) \leq 0$ for $1 \leq i \leq n$.

Thus the postulate of admissibility of an allocation prevents unrestricted borrowing in the market. We define the class of all admissible allocations of X by

$$\mathrm{Ad}(X) := \{(X_i) \in \mathcal{A}(X); (X_i) \text{ is admissible}\}. \tag{5.7}$$

Further we define the admissible infimal convolution

$$\varrho_*(X) := \inf\left\{\sum_{i=1}^n \varrho_i(X_i); (X_i) \in \mathrm{Ad}(X)\right\}. \tag{5.8}$$

The admissible infimal convolution risk measure ϱ_* has a useful dual representation.

Proposition 5.2 *Let ϱ_i be finite convex risk measures on L^p, $1 \leq i \leq n$. Then the admissible infimal convolution ϱ_* has the dual representation*

$$\begin{aligned}\varrho_*(X) = \sup\bigg\{ \int X_- d \wedge Q_j - \int X_+ d \vee Q_j - \sum \varrho_j^*(Q_j); \\ Q_j \in \operatorname{dom} \varrho_j^*,\ 1 \leq j \leq n\bigg\},\end{aligned} \tag{5.9}$$

where $\wedge Q_j$ resp. $\vee Q_j$ are the measures whose densities are the infima resp. maxima $\wedge \frac{dQ_j}{dP}$ resp. $\vee \frac{dQ_j}{dP}$ of the densities of Q_j with respect to P.

The proof of Proposition 5.2 is based on a nonconvex minimax theorem similar to that of Theorem 3.5 resp. Theorem 3.4 in Burgert and Rüschendorf (2006a,b).

ϱ_* is a monotone, convex risk functional. To obtain a convex risk measure, we have to ensure cash invariance. To that aim we define

$$\widehat{\varrho}_*(X) := \inf\{m \in \mathbb{R}; X + m \in \mathcal{A}\} = \inf\{m \in \mathbb{R};\ \varrho_*(X+m) \leq 0\}, \tag{5.10}$$

where $\mathcal{A} = \mathcal{A}_{\varrho^*} = \{X; \varrho_*(X) \leq 0\}$ is the acceptance set of ϱ_*. $\widehat{\varrho}_*$ is called *convex admissible infimal convolution.* Note that Proposition 5.2 gives a means to check the condition $X + m \in \mathcal{A}$. The following result confirms our notion of admissable allocations. Essentially weaker restrictions on the allocation do not prevent risk arbitrage.

Theorem 5.3 (Convex admissable infimal convolution risk measure) *Let $\varrho_1, \dots, \varrho_n$ be finite convex risk measures on L^p, $1 \leq p \leq \infty$. Then the convex admissible infimal convolution $\widehat{\varrho}_*$ is the largest convex risk measure ϱ with $\varrho \leq \varrho_*$.*
Under the equilibrium condition (E) holds $\widehat{\varrho}_* = \widehat{\varrho}$.

Remark 5.1 a) The argument for Theorem 5.3 is similar to that in Burgert and Rüschendorf (2006a). Theorem 5.3 justifies the introduction of the restriction on the class of decompositions in (5.7). Essentially weaker restrictions do not lead to a convex risk measure. Also based on the dual characterization in Proposition 5.2 it is possible to determine $\widehat{\varrho_*}$ for several concrete examples. In the case of coherent risk measures the representation simplifies essentially since $\varrho_j^*(Q_j) = 0$ for $Q_j \in \operatorname{dom} \varrho_j^*$. This implies that $X \in \mathcal{A}_{\varrho^*}$ if and only if for all $X \in L^p$ and $Q_j \in \mathcal{Q}_j = \operatorname{dom} \varrho_j^*$

$$\int X_- d \wedge Q_j \leq \int X_+ d \vee Q_j. \tag{5.11}$$

b) There are further natural systems of restrictions which lead to the same convex risk measure. One restriction of this type is to postulate that $|X_i| \leq |X|$ for an admissible decomposition (X_i) of X (see Burgert and Rüschendorf (2006a,b)).

c) A practical conclusion of Theorem 5.3 is to insist for the risk allocation problem on admissibility if the risk traders have an essential different view towards risk.

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