Approximate optimal stopping of dependent sequences

Robert Kühne and Ludger Rüschendorf*
University of Freiburg

Abstract

We consider optimal stopping of sequences of random variables satisfying some asymptotic independence property. Assuming that the embedded planar point processes converge to a Poisson process we introduce some further conditions to obtain approximation of the optimal stopping problem of the discrete time sequence by the optimal stopping of the limiting Poisson process. This limiting problem can be solved in several cases. We apply this method to obtain approximations for the stopping of moving average sequences, of hidden Markov chains, and of max-autoregressive sequences. We also briefly discuss extensions to the case of Poisson cluster processes in the limit.

Keywords: Optimal stopping, Poisson processes, asymptotic independence, moving average processes, hidden Markov chains

1 Introduction

The aim of this paper is to extend the recent approach in Kühne and Rüschendorf (2000) (in the following abbreviated by KR (2000)) for approximatively optimal stopping of independent sequences $X_1, \dots, X_n$ to dependent sequences. The basic assumption in this approach is convergence of the embedded planar point process

$$N_n = \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, X_{n,i}\right)} \xrightarrow{\mathcal{D}} N \tag{1.1}$$

to some Poisson point process N with intensity measure μ with Lebesgue-density h. Here $X_{n,i} = \frac{X_i - b_n}{a_n}$ is some normalization of X_i induced from the central limit theorem for maxima. Then under some additional conditions it is proved that the optimal stopping problem of $X_1, \dots, X_n$ can be approximated by the optimal stopping of the Poisson process. Finally it is

* Corresponding author. *Address:* Department of Mathematical Stochastics, Eckerstr. 1, D-79104 Freiburg, Germany. *E-mail address:* ruschen@stochastik.uni-freiburg.de

shown that the optimal stopping of a Poisson process with intensity μ can be reduced to solving a differential equation of the form

$$\begin{aligned} v'(t) &= -\int_{v(t)}^{\infty}\int_{x}^{\infty} h(t,y)dy\,dx,\ 0\le t\le 1 \\ v(1) &= f(1). \end{aligned} \tag{1.2}$$

The solution of (1.2) is the optimal stopping curve u for the Poisson process. Here f is a decreasing function describing the lower boundary of the intensity and μ is assumed to have singularities only on the lower boundary; in technical terms μ is a Radon measure on $M_f = \{(t,y)\in[0,1]\times\overline{\mathbb{R}}; y > f(t)\}$ supplied with the relative topology of $[0,1]\times\mathbb{R}$. Additional to the continuity assumption (D) on μ as described above we need that v satisfies the separation condition

$$(S)\qquad (v-f)_{/[0,t]} > c_t > 0 \quad \text{for all} \quad t<1. \tag{1.3}$$

Finally for convergence of the stopping problems we need that the positive parts M_n^+ of the maxima $M_n = \max\{X_1,\dots,X_n\}$ are uniformly integrable (condition (G)) and the lower curve condition (L), i.e.

$$\underline{\lim}\, u_n(1-\varepsilon) > -\infty \quad \text{for all} \quad \varepsilon > 0, \tag{1.4}$$

where $u_{n,1}\dots,u_{n,n}$ is the optimal stopping curve of $X_{n,1},\dots,X_{n,n}$ and $u_n(s)$ $= u_{n,[n,s]\vee 1}$ is the functional form of $u_{n,i}$. Convergence of the stopping problem means convergence of the optimal stopping times, stopping time distributions, and stopping value.

To extend this program to dependent sequences we need some kind of asymptotic independence assumption (A) as well as a modification (L$'$) of the lower curve condition.

As application we derive approximative optimal stopping results for some finite moving average processes, for hidden Markov models (chain dependent sequences), and for max-autoregressive sequences. In a subsequent section we briefly consider the case where in the limit we obtain a Poisson-cluster process. This needs some alternative construction and allows to consider also some infinite MA-processes. For some details on the arguments we refer to the dissertation of Kühne (1997) on which this paper is based.

2 Approximation of optimal stopping

For general dependent sequences the point process approximation in (1.1) plus some conditions related to (D), (S), and (L) will not imply approximation of the stopping problems as the following simple example shows clearly.

Example 2.1 *Let (Y_i) be iid exponentially distributed and let T_n denote the optimal stopping time of $Y_1, \dots, Y_n$. Let $n_k = 10^{2^k}, k \geq 1$, and $m_k \in \{n_{k-1}+2, \dots, n_k\}$ be such that $Y_{m_k} = Y_{n_{k-1}+2} \vee \dots \vee Y_{n_k}$, and define*

$$X_i \ := \ \begin{cases} Y_i & i \neq m_k - 1 \\ -1 & i = m_k - 1. \end{cases}$$

Then (X_i) exhibit the same point process behavior as (Y_i) but have a completely different stopping behavior. Consider stopping of $(X_1, \dots, X_{n_k})$, then $T_k^0 = \inf\{i \in \{n_{k-1}+1, \dots, n_k\} : X_i = -1\} + 1$ is a stopping time with $X_{T_k^0} = X_{n_{k-1}+2} \vee \dots \vee X_{n_k}$ and so $\lim_{k\to\infty} EX_{T_k^0} - \log n_k = \lim_{k\to\infty}(EM_{n_k} - \log n_k) = \gamma = 0.5772 \ldots > 0 = \lim_{k\to\infty}(EY_{T_{n_k}})$ (see Kennedy and Kertz (1991)).

The essential local information allowing to stop at the maximum is lost by the asymptotic approximation of the point process.

Let $(X_{n,i})_{1 \leq i \leq n}$ be a double sequence of random variables with adapted filtration $\mathcal{F} = (\mathcal{F}_{n,1}, \dots, \mathcal{F}_{n,n})$. Consider the following assumptions:

Condition (A) (asymptotic independence) Assume that for $t \in [0,1)$

$$P^{N_n(\cdot \cap [t,1] \times \mathbb{R}) | \mathcal{F}_{n,[nt]-1}} \xrightarrow{P} P^{N(\cdot \cap [t,1] \times \mathbb{R})} \tag{2.1}$$

For a threshold stopping time τ with threshold v define $\tau^{\geq t}$ as the first time over threshold v after time t. Let u denote the optimal stopping curve of the Poisson process N.

Condition (L$'$) There exist non-increasing constants $u_{n,i}$ such that $u_{n,[nt]\vee 1} \to u(t), t \in [0,1)$, the optimal stopping curve of the limiting Poisson process, and for the threshold stopping time $T_n' = \inf\{i : X_{n,i} \geq u_{n,i}\}$ holds

$$\lim_{t\to 1} \limsup_{n\to\infty} E|X_{n,T_n'^{\geq [ns]}}| 1_{\{T_n'^{\geq [ns]} \geq [nt]\}} = 0 \ \text{ for all } \ s < 1.$$

Finally we consider the direct generalization of the lower curve condition. Define $\gamma_{n,i} = \operatorname{ess\,sup}\{E(X_{n,\tau}|\mathcal{F}_{n,i}); \tau \geq i\}$.

Condition (L)

$$\liminf_{n\to\infty} E\gamma_{n,[nt]} > -\infty, \ \text{ for all } \ t < 1.$$

We now can formulate our main approximation result. Let T denote the optimal stopping time of the limiting Poisson process $N = \sum \varepsilon_{(\tau_k, y_k)}$, let K^T denote the optimal stopping index and let T_n denote an optimal stopping time of $X_{n,i}$. (For more details on the notation see KR (2000).)

Theorem 2.2 (Approximation of stopping problems) *Assume that $N_n = \sum_{i=1}^n \varepsilon_{\left(\frac{i}{n}, X_{n,i}\right)}$ converges on M_f to a Poisson process N with intensity μ satisfying (D). Let the optimal stopping curve u be the unique solution* *of* (1.2) *and satisfy the separation condition (S). Finally assume conditions (A), (G), and (L′). Then*

$$\begin{array}{ll} a) & E(\gamma_{n,[nt]}|\mathcal{F}_{n,[nt]-1}) \xrightarrow{P} u(t), t \in [0,1) \qquad (2.2)\\ b) & (T_n, X_{n,T_n}) \xrightarrow{\mathcal{D}} (T, y_{K^T}) \text{ and } EX_{n,T_n} \to Ey_{K^T} = u(0)\\ c) & (T'_n) \text{ is an asymptotically optimal sequence of stopping times,}\\ & \text{i.e. } EX_{n,T'_n} \to u(0). \end{array}$$

Proof Theorem 2.2 a): Define for $1 \le \ell < m \le n$

$$M_{n,\ell,m} = X_{n,\ell} \vee \cdots \vee X_{n,m}, \quad M_n = M_{n,1,n}.$$

For $t \in [0,1)$ we define $a^i_{n,k} = [n(t+(1-t)\frac{i}{k})], 0 \le i \le k, X'^{\,i}_{n,k} = M_{n,a^{i-1}_{n,k},a^i_{n,k}}$ and the filtration $\mathcal{F}' = (\mathcal{F}'_{n,1}, \dots, \mathcal{F}'_{n,k}) = (\mathcal{F}_{n,a^1_{n,k}}, \dots, \mathcal{F}_{n,a^k_{n,k}})$. This induces a finite stopping problem for $\overline{X}_{n,i} = X'^{\,j}_{n,k}$ if $a^{j-1}_{n,k} < i \le a^j_{n,k}$ with corresponding filtration $\overline{\mathcal{F}}_{n,i}$ which majorizes the corresponding stopping problem of $X_{n,i}$ since $\overline{X}_{n,i} \ge X_{n,i}$ for all i. From point process convergence and the continuous mapping theorem we conclude that

$$\begin{aligned} \left(X'^{\,1}_{n,k}, \dots, X'^{\,k}_{n,k}\right) \;\to\; & \left(M_{t,t+(1-t)\frac{1}{k}}, \dots, M_{t+(1-t)\frac{k-1}{k},n}\right)\\ & =: \left(X'^{\,1}_k, \dots, X'^{\,k}_k\right), \end{aligned} \tag{2.3}$$

where $M_{s,t} = \sup\{y_k;\ \tau_k \in (s,t]\}$ is the corresponding max in the Poisson process. Here and in the following we assume w.l.g. almost sure convergence of the points (see the corresponding argument in KR (2000)).

We prove by backward induction from k–1 to 1 convergence of the optimal stopping behavior:

$$E\left(\gamma'^{\,i+1}_{n,k}|\mathcal{F}'_{n,i}\right) \xrightarrow{P} u'_{k,i+1}, \qquad i = 0, \dots, k-1, \tag{2.4}$$

where $(u'_{k,i})$ is the optimal stopping curve of the independent sequence $(X'^{\,i}_k)$.

We start with the induction step $i+1 \to i$.

Consider the Bellmann equation

$$E\left(\gamma'^{\,i+1}_{n,k}|\mathcal{F}'_{n,i}\right) = E\left(X'^{\,i+1}_{n,k} \vee E\left(\gamma'^{\,i+2}_{n,k}|\mathcal{F}'_{n,i+1}\right)|\mathcal{F}'_{n,i}\right) \tag{2.5}$$

where by the induction assumption $E\left(\gamma'^{\,i+2}_{n,k}|\mathcal{F}'_{n,i+1}\right) \xrightarrow{\mathcal{D}} u'_{k,i+1}$. This implies conditional convergence $P^{E\left(\gamma'^{\,i+2}_{n,k}|\mathcal{F}'_{n,i+1}\right)|\mathcal{F}'_{n,i}} \xrightarrow{P} \varepsilon_{u'_{k,i+1}}$. Define the mapping $M^0\left(\sum \varepsilon_{s_i,z_i)}\right) = \sup_i z_i$ and $N^i_n = \sum_{\substack{t+(1-t)\frac{i-1}{k}\\ <\frac{j}{n} \le t+(1-t)\frac{i}{k}}} \varepsilon_{\left(\frac{j}{n},X_{n,j}\right)}$. Then

$$P^{X'^{i+1}_{n,k}|\mathcal{F}'_{n,i}} = P^{M_{n,a^{i-1}_{n,k}+1,a^i_{n,k}}|\mathcal{F}'_{n,i}} = P^{M^0(N^i_n)|\mathcal{F}'_{n,i}}. \tag{2.6}$$

By the independence assumption (A) holds

$$P^{N_n^i|\mathcal{F}'_{n,i}} \xrightarrow{P} P^{N^i}, \text{ where } N^i = \sum_{\frac{i-1}{k}<\tau_j\le t+(1-t)\frac{i}{k}} \varepsilon_{(\tau_j,y_j)}.$$

Therefore, we conclude as in Resnick (1987, section 4)

$$P^{X'^{i+1}_{n,k}|\mathcal{F}'_{n,i}} \xrightarrow{P} P^{M^0 N^i} = P^{X'^i_k}. \tag{2.7}$$

For sequences of random variables X_n, X in a separable metric space holds: $X_n \overset{P}{\to} X$ if there exist measurable sets m_n with $P(m_n) \to 0$ such that pointwise

$$X_n 1_{m_n^c} + X 1_{m_n} \to X. \tag{2.8}$$

Therefore, we obtain measurable sets $m_n, P(m_n) \to 0$ such that pointwise for all ω

$$P^{N'_n|\mathcal{F}'_{n,i}} 1_{m_n^c} + P^{N^i} 1_{m_n} \xrightarrow{\mathcal{D}} P^{N^i}.$$

This implies by the continuous mapping theorem

$$\begin{array}{rcl} P^{M^0 N'_n|\mathcal{F}'_{n,i}} 1_{m_n^c} + P^{M^0 N^i} 1_{m_n} & \xrightarrow{\mathcal{D}} & P^{M^0 N^i} \\ \text{i.e.} \qquad P^{X'^{i+1}_{n,k}|\mathcal{F}'_{n,i}} & \xrightarrow{P} & P^{X'^{i+1}_k}. \end{array} \tag{2.9}$$

Together, we obtain from (2.9), (2.7)

$$P^{X'^{\,i+1}_{n,k} \vee E(\gamma'^{\,i+2}_{n,k}|\mathcal{F}'_{n,i+1})|\mathcal{F}'_{n,i}} \xrightarrow{P} P^{X'^{\,i+1}_k \vee u'_{k,i+1}}. \tag{2.10}$$

To conclude from (2.10) convergence of conditional expectations as in (2.4) we next establish uniform integrability. By assumption (G) (M_n^+) is uniformly integrable. Therefore, also $(E(M_n^+|\mathcal{F}'_{n,i+1}))$ is uniformly integrable and so with $X_n = E(M_n^+|\mathcal{F}'_{n,i+1}) 1_{\{E(M_n^+|\mathcal{F}'_{n,i+1})>L\}}$ holds

$$\lim_{L\to\infty} \limsup_{n\to\infty} EX_n = 0. \tag{2.11}$$

From the integrability condition (L$'$) we conclude as in the proof of Theorem 3.2 in KR (2000):

$$\lim_{n\to\infty} EX_{n,T'_n \ge [nt]} = u(t) = E_{y_{K^T} \ge t}, \quad t \in [0,1). \tag{2.12}$$

The positive parts of these random variables are by (2.11) uniformly integrable. Therefore, with $t = \frac{k-1}{k}$ we obtain $\left\{X_{n,T'_n \ge \left[n\frac{k-1}{k}\right]}\right\}_{n\in\mathbb{N}}$ is u.i. This implies that $\left\{X'^{\,k}_{n,k}\right\}_{n\in\mathbb{N}}$ is u.i., again using (2.11), and, therefore,

$$\lim_{L\to-\infty} \limsup_{n\to\infty} EX'^{\,k}_{n,k} 1_{\{X'^{\,k}_{n,k}<L\}} = 0 \text{ for any } k.$$

Denoting $P_{n,\omega} = P^{X'^{\,i+1}_{n,k} \vee E\left(\gamma'^{\,i+2}_{n,k} | \mathcal{F}'_{n,i+1}\right)} | \mathcal{F}'_{n,i}(\omega) 1_{m_n^c}(\omega) + P^{X'^{\,i+1}_k \vee u'_{k,i+1}} 1_{m_n}(\omega)$ and $P_{0,\omega} = P^{X'^{\,i+1}_k \vee u'_{k,i+1}}$ and let $X^\omega_{n,i}, M^\omega_n$ be random variables on a probability space $(\Omega', \mathcal{A}', Q)$ with $X^\omega_{ni} \le M^\omega_n$ and

$$\begin{aligned} Q^{M^\omega_n} &= P^{E(M_n^+ | \mathcal{F}'_{n,i+1}) | \mathcal{F}'_{n,i}}(\omega) \\ Q^{X^\omega_{n,i}} &= P^{E(X'^{\,k}_{n,k} | \mathcal{F}'_{n,i+1}) | \mathcal{F}'_{n,i}}(\omega). \end{aligned}$$

Then for $\omega \in m_n^c$ holds

$$\begin{aligned} &\int_L^\infty x dP_{n,\omega} \\ &= E(X'^{\,i+1}_{n,k} \vee E(\gamma'^{\,i+2}_{n,k} | \mathcal{F}'_{n,i+1}) 1_{\{X'^{\,i+1}_{n,k} \vee E(\gamma'^{\,i+2}_{n,k} | \mathcal{F}'_{n,i+1}) \ge L\}} | \mathcal{F}'_{n,i})(\omega) \end{aligned}$$

and we obtain

$$\begin{aligned} &\lim_{L\to\infty} \limsup_{n\to\infty} \int_L^\infty x dP_{n,\omega} \\ &\le \lim_{L\to\infty} \limsup_{n\to\infty} E_Q X^\omega_{n,i} 1_{\{X^\omega_{n,i} \ge L\}} + \lim_{L\to\infty} \limsup_{n\to\infty} E_Q M^\omega_n 1_{\{M^\omega_n \ge L\}} = 0. \end{aligned}$$

A similar estimate holds for the lower tail. From the uniform integrability of $P_{n,\omega}$ we conclude $\int x dP_{n,\omega} \to \int x dP_{0,\omega}$, i.e.

$$E(X'^{\,i+1}_{n,k} \vee E(\gamma'^{\,i+2}_{n,k} | \mathcal{F}'_{n,i+1}) | \mathcal{F}'_{n,i}) \xrightarrow{P} E(X'^{\,i+1}_k \vee u'_{k,i+2}).$$

This implies

$$E(\gamma'^{\,i+1}_{n,k} | \mathcal{F}'_{n,i}) \xrightarrow{\mathcal{D}} u'_{k,i+1}. \tag{2.13}$$

This proves the induction step. The beginning of the induction with $i = k-1$ is similar but simpler since the second conditional term is not present in this case.

In particular (2.4) implies

$$E(\gamma'^{\,1}_{n,k} | \mathcal{F}'_{n,0}) \xrightarrow{P} u'_{k,1}.$$

For the limit as $k \to \infty$ one obtains as in the proof of Theorem 2.5 in KR (2000) convergence of the point process of maxima to N:

$$N^k = \sum_{i=1}^k \varepsilon_{\left(t+(1-t)\frac{i}{k}, X'^{\,i}_k\right)} \xrightarrow{\mathcal{D}} N(\cdot \cap [t,1] \times \mathbb{R}).$$

Define $u^k(t) = u'_{k,[kt]\vee 1}, t < 1$. N^k fulfills the assumptions of the approximation theorem (KR (2000, Theorem 3.2)). Therefore, we obtain $u^k \to u$. Since $u^k \ge u, \forall k$, this implies that

$$\lim_{n\to\infty} P(E(\gamma_{n,[nt]} | \mathcal{F}_{[nt]-1}) \ge u(t) + \varepsilon) = 0, \text{ for all } \varepsilon > 0.$$

(2.12) and condition (G) imply $E\left(\gamma_{n,[nt]}|\mathcal{F}_{[nt]-1}\right) \xrightarrow{P} u(t)$ for $t \in [0,1)$ and so a) holds. $\square$

Proof of b, c: In the next step of the proof we establish convergence as in (2.4) also for the random time point T. Let $(\widetilde{T}_n)$ be a squence of stopping time such that $\left(\frac{\widetilde{T}_n}{n}, X_{n,\widetilde{T}_n}\right) \xrightarrow{\mathcal{D}} (T, Y)$. By a Skorohod type argument as in KR (2000) we assume that convergence is a.s. Our aim is to prove

$$E\left(\gamma_{n,\widetilde{T}_n+1}|\mathcal{F}_{\widetilde{T}_n}\right) \xrightarrow{P} u(T) \text{ on } \{T < 1\} \tag{2.14}$$

For the proof we use a discretization argument and define for $k \in \mathbb{N}, k \leq n, x \in [0,1)$

$$\begin{aligned} g_k(x) &= \tfrac{1}{k}\inf\{i \in \mathbb{N}; i \geq kx\} = \tfrac{\lceil kx\rceil}{k} \\ \text{and } g_k^n(x) &= \lceil n g_k\left(\tfrac{x}{n}\right) + 1\rceil \wedge n, \text{ for } 1 \leq x \leq n-1. \end{aligned} \tag{2.15}$$

g_k^n maps a number in $\{1, \ldots, n-1\}$ to the nearest number of the form $\lceil \frac{ni}{k} + 1\rceil, i = 1, \ldots, k-1, n$. With $e_k^n = \left(\max_{\widetilde{T}_n < i < g_k^n(\widetilde{T}_n)} X_{n,i} - \gamma_{n,g_k^n(\widetilde{T}_n)}\right)_+$ we obtain

$$\begin{aligned} E\left(\gamma_{n,g_k^n(\widetilde{T}_n)}|\mathcal{F}_{n,\widetilde{T}_n}\right) &\leq E\left(\gamma_{n,\widetilde{T}_{n+1}}|\mathcal{F}_{n,\widetilde{T}_n}\right) \\ &\leq E\left(\gamma_{n,g_k^n(\widetilde{T}_n)} \vee \max_{T_n < i < g_k^n(\widetilde{T}_n)} X_{n,i}|\mathcal{F}_{n,\widetilde{T}_n}\right) \\ &= E\left(\gamma_{n,g_k^n(\widetilde{T}_n)}|\mathcal{F}_{n,\widetilde{T}_n}\right) + E\left(e_k^n|\mathcal{F}_{n,\widetilde{T}_n}\right). \end{aligned} \tag{2.16}$$

Since $\lim_{k\to\infty} g_k(x) = x, \forall x \in [0,1)$ and g_k attains only k values we obtain from (2.4)

$$E\left(\gamma_{n,g_k^n(\widetilde{T}_n)}|\mathcal{F}_{n,\widetilde{T}_n}\right) \to u(g_k(T)) \text{ on } \left\{T \leq \tfrac{k-1}{k}\right\}$$

and, therefore,

$$\lim_{k\to\infty}\lim_{n\to\infty} E\left(\gamma_{n,g_k^n(\widetilde{T}_n)}|\mathcal{F}_{n,\widetilde{T}_n}\right) = u(T) \text{ on } \{T < 1\}. \tag{2.17}$$

Consider next the behavior of e_k^n. We state

$$\lim_{k\to\infty}\limsup_{n\to\infty} P\left(E\left(e_k^n|\mathcal{F}_{n,\widetilde{T}_n}\right) 1_{\{\widetilde{T}_n < n\frac{k-1}{k}\}} \geq \varepsilon\right) = 0 \text{ for all } \varepsilon > 0. \tag{2.18}$$

Note that $e_k^n \underset{n\to\infty}{\longrightarrow} e_k = \left(\max_{T<\tau_i<g_k(T)} y_i - u(g_k(T))\right)_+$. For almost all $\omega \in \Omega$ there exists $k_\omega \in \mathbb{N}$ such that

$$\#\{(\tau_i(\omega), y_i(\omega));\ T(\omega) < \tau_i(\omega) \leq g_k(T(\omega)), y_i(\omega) > y_{K^{T(\omega)}}\} = 0 \text{ for } k \geq k_\omega.$$

Otherwise there would exist infinitely many points $\tau_i(\omega)$ close to $T(\omega)$ such that $y_i^\omega > u(\tau_i(\omega))$ in contradiction to the separation assumption on the Poisson process N. This implies $e_k \underset{k\to\infty}{\longrightarrow} 0$ a.s.

By the first part of this proof $(\gamma_{n,[nt]})_{n\in\mathbb{N}}$ is uniformly integrable. Since $(M_n^+)_{n\in\mathbb{N}}$ is uniformly integrable we obtain $\left(E\left(e_k^n|\mathcal{F}_{n,\widetilde{T}_n}\right)1_{\{\widetilde{T}_n\le n\frac{k-1}{k}\}}\right)_{n\in\mathbb{N}}$ is u.i. and (2.18) follows as in the first part of proof. From (2.16) we obtain

$$\begin{aligned} 0 &\le E\left(\gamma_{n,\widetilde{T}_{n+1}}|\mathcal{F}_{n,\widetilde{T}_n}\right) - E\left(\gamma_{n,g_k^n(\widetilde{T}_n)}|\mathcal{F}_{n,\widetilde{T}_n}\right) \\ &\le E\left(e_k^n|\mathcal{F}_{n,\widetilde{T}_n}\right). \end{aligned}$$

Thus

$$\begin{aligned} \lim_{k\to\infty}\limsup_{n\to\infty} P\bigg(\Big(E\big(\gamma_{n,\widetilde{T}_{n+1}}|\mathcal{F}_{n,\widetilde{T}_n}\big) - E\big(\gamma_{n,g_k^n(\widetilde{T}_n)}|\mathcal{F}_{n,\widetilde{T}_n}\big)\Big) \\ \cdot 1_{\left\{\widetilde{T}_n\le n\frac{k-1}{k}\right\}} \ge \varepsilon\bigg) = 0. \end{aligned}$$

for all $\varepsilon > 0$. So with (2.17) this implies (2.14).

In the final step we prove convergence of the optimal stopping time T_n. Define $\widehat{T}_n = \inf\{i : X_{n,i} \ge u\left(\frac{i}{n}\right)\}$. We state that

$$P\left(\widehat{T}_n = T_n\right) \to 1. \tag{2.19}$$

By the approximation result for threshold stopping times KR (2000, Proposition 2.4) holds

$$\left(\frac{\widehat{T}_n}{n}, X_{n,\widehat{T}_n}\right) \xrightarrow{D} (T, y_{K^T}). \tag{2.20}$$

Assuming a.s. convergence and $T < 1$, we obtain by (2.14)

$$E(\gamma_{n,\widehat{T}_n+1}|\mathcal{F}_{\widehat{T}_n}) \xrightarrow{P} u(T). \tag{2.21}$$

By the continuity assumption (D) $y_{K^T} > u(T)$. Then for $\omega \in \Omega$ and $n \ge n_\omega$, $X_{n,\widehat{T}_n}(\omega) \ge \left(2\frac{y_{K^T}-u(T)}{3} + u(T)\right)(\omega)$. From (2.21) for $n \ge n'_\omega$

$$E(\gamma_{n,\widehat{T}_n+1}|\mathcal{F}_{\widehat{T}_n})(\omega) \le \left(\frac{y_{K^T}-u(T)}{3} + u(T)\right)(\omega).$$

Together this implies for $n \ge n_\omega \vee n'_\omega : X_{n,\widehat{T}_n}(\omega) > E(\gamma_{n,\widehat{T}_n+1}|\mathcal{F}_{\widehat{T}_n})(\omega)$, i.e.

$$P(\widehat{T}_n \ge T_n) \to 1. \tag{2.22}$$

Conversely, suppose for some subsequence (n'') holds

$$P(\widehat{T}_{n''} > T_{n''}) \to d > 0. \tag{2.23}$$

Then with some subsequence $n' \subset (n'')$ such that $\left(\frac{T_{n'}}{n'}, X_{n',T_{n'}}\right)$ converges and $\frac{T_{n'}}{n'} \to T'$ we obtain as in the proof of (2.22), $P(T_{n'} \geq \widehat{T}_{n'}) \to 1$ in contradiction to (2.23). This implies (2.19).

Asymptotic optimality of T'_n follows similar to the proof of the approximation theorem in KR (2000, Theorem 4.5). Thus the proof of a), b), c) is complete. $\square$

A useful result to establish the point process convergence in Theorem 2.2 is the following theorem. A simple proof of this result was given in Kühne (1997) based on the conditioning argument as in Jakubowski (1986), Beska et al. (1982)

Theorem 2.3 (Point process convergence. Durrett and Resnick (1978, Th. 3.1)) *Let $(X_{n,i})$ be a sequence of random variables with filtration $(\mathcal{F}_{n,i})$, μ a measure on $[0,1] \times (a,\infty)$, and for all $x > a$ with $\mu([0,1] \times \{x\}) = 0$ holds:*

$$\sum_{i=1}^{[nt]} P(X_{n,i} > x | \mathcal{F}_{n,i-1}) \xrightarrow{P} \mu([0,t] \times [x,\infty))$$

and

$$\sup_{1 \leq i \leq n} P(X_{n,i} > x | \mathcal{F}_{n,i-1}) \xrightarrow{P} 0.$$

Then $N_n = \sum \varepsilon_{\left(\frac{i}{n}, X_{n,i}\right)} \xrightarrow{D} N$ a Poisson process on $[0,1] \times (a,\infty)$ with intensity $\mu/[0,1] \times (a,\infty)$.

3 Optimal stopping of finite moving average processes

Let

$$X_i = \sum_{j=1}^{k} c_j Y_{i-j}, \quad i \in \mathbb{N}, \tag{3.1}$$

denote a finite moving average (MA) process where (Y_i) are iid r.v.s with distribution function F in the domain of an extreme value distribution Λ, Ψ_α, Φ_α. Point process convergence of MA-processes has been intensively studied. We will investigate approximatively optimal stopping of MA-sequences in some of these cases.

3.1 $F \in D(\Psi_\alpha)$.

Theorem 3.1 *Let $k \in \mathbb{N}, F \in D(\Psi_\alpha), \alpha > 0$ be integrable with right endpoint of the support $\omega_F = 0$ and assume $c_j > 0$. Then we obtain with $a_n = F^{-1}(n^{\frac{-1}{k}})\frac{\Gamma(\alpha k+1)^{\frac{1}{\alpha k}}}{\Gamma(\alpha+1)^{\frac{1}{\alpha}}}\prod_{i=1}^{k} c_i^{\frac{1}{k}}$ convergence of the optimal stopping problem:*

a) $$E\frac{X_{T_n}}{a_n} \to -\left(\frac{\alpha k}{\alpha k+1}\right)^{-\frac{1}{\alpha k}}$$

b) $$P\left(\frac{X_{T_n}}{a_n} \le x\right) \to \begin{cases} 1, & x \ge 0 \\ 1-(-x)^{\alpha k}\frac{1}{2+\frac{1}{\alpha k}}, & 0 > x \ge -(\frac{\alpha k}{\alpha k+1})^{-\frac{1}{\alpha k}} \\ (\frac{\alpha k+1}{\alpha k})^{\frac{\alpha k+1}{\alpha k}}(-x)^{-\alpha k-1}, & x < -(\frac{\alpha k}{\alpha k+1})^{-\frac{1}{\alpha k}} \end{cases}$$

c) *$T_n' = \inf\{i \le n : X_i \ge w_{n-i}\}$ with $w_n = a_n\left(\frac{\alpha k}{\alpha k+1}^{-\frac{1}{\alpha k}}\right)$ defines an asymptotically optimal sequence of stopping times.*

Proof: For the proof we apply Theorem 2.2

1) *Point process convergence:* By Davis and Resnick (1991) holds

$$\sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n},\frac{X_i}{\tilde{a}_n}\right)} \xrightarrow{\mathcal{D}} \widetilde{N} \tag{3.2}$$

where $\tilde{a}_n = F^{-1}(n^{-\frac{1}{k}})$. $\widetilde{N}$ is a Poisson process with intensity $\lambda_{[0,1]} \otimes \nu_k, \nu_k([x,0]) = c(\alpha,k)(-x)^{\alpha k}, x \le 0$, where $c(\alpha,k) = \frac{\Gamma(\alpha+1)^k}{\Gamma(k\alpha+1)\prod_{i=1}^k c_i^\alpha}$. Replacing $\tilde{a}_n$, by a_n, the point process converges to a Poisson point process with intensity $\lambda_{[0,1]} \otimes \nu', \nu'([x,0]) = (-x)^{\alpha k}, x \le 0$.

2) *Condition (G)* is trivially satisfied, as $\frac{M_n^+}{a_n} = 0$ for all n.

3) (D) and uniqueness of the optimal stopping curve of N by the differential equation in (1.2) has been proved in KR (2000, Theorem 4.5). Note that $f \equiv -\infty$ in this case.

4) *Asymptotic independence (A):* Note that

$$\begin{aligned} &P^{N_n(\cdot\cap[t,1]\times\mathbb{R})|X_1,\dots,X_{[nt]-1}} \\ &= P^{\sum_{i=[nt]}^{[nt]+k}\varepsilon_{\left(\frac{i}{n},\frac{X_i}{a_n}\right)}+\sum_{i=[nt]+k+1}^{n}\varepsilon_{\left(\frac{i}{n},\frac{X_i}{a_n}\right)}|X_1,\dots,X_{[nt]-1}} \end{aligned}$$

By the normalization we have that $\sum_{i=[nt]}^{[nt]+k}\varepsilon_{\left(\frac{i}{n},\frac{X_i}{a_n}\right)} \xrightarrow{\mathcal{D}} 0$, where 0 denotes the zero-measure. Also

$$P^{\sum_{i=[nt]+k+1}^{n}\varepsilon_{\left(\frac{i}{n},\frac{X_i}{a_n}\right)}|X_1,\dots,X_{[nt]-1}} = P^{\sum_{i=[nt]+k+1}^{n}\varepsilon_{\left(\frac{i}{n},\frac{X_i}{a_n}\right)}} \xrightarrow{\mathcal{D}} P^{N_{[t,1]}}, \tag{3.3}$$

which implies $P^{N_n|[t,1]|X_1,\dots,X_{[nt]-1}} \xrightarrow{P} P^{N_{[t,1]}}$.

5) *Condition (L′):* For $\ell = 1, \dots, k$ define

$$T_n^\ell = \inf\{i \le n : i+\ell = 0 \operatorname{mod} k, X_i \ge w_{n-i}\}.$$

Then by the independence of X_i involved we obtain from Kennedy and Kertz (1991)

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} E\frac{X_{T_n^\ell}}{a_n} 1_{\{T_n^\ell > n-[n\varepsilon]\}} = 0.$$

This implies using $T_n' = T_n^1 \wedge \dots \wedge T_n^k$ and $0 \ge X_{T_n'} \ge X_{T_n^1} \wedge \dots \wedge X_{T_n^k}$
$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} E\tfrac{X_{T_n'}}{a_n} 1_{\{T_n' > n-[n\varepsilon]\}} \ \ge\ \lim_{\varepsilon\to 0}\lim_{n\to\infty} \sum_{\ell=1}^k E\left(\tfrac{X_{T_n^\ell}}{a_n} 1_{\{T_n^\ell > n-[n\varepsilon]\}}\right) = 0.$$

This implies (L′).

From Kennedy and Kertz (1990, p. 398) we use that for $F \in D(G)$

$$n(1-F(w_n)) \to x \iff \frac{w_n - b_n}{a_n} \to -\log G(x). \tag{3.4}$$

This implies that

$$\frac{w_{[nt]}}{a_n} \to -(1-t)^{\frac{-1}{\alpha}}\left(\tfrac{\alpha}{1+\alpha}\right)^{\frac{-1}{\alpha}} = u(t), \tag{3.5}$$

which is the optimal stopping curve in the limiting Poisson process. Therefore, (T_n') is an asymptotically optimal stopping sequence and convergence in a), b) follows from Theorem 2.2 □

3.2 $F \in D(\Lambda)$.

Consider $F = F_{\alpha,p} \in D(\Lambda)$, where for $p > 0, \alpha \in \mathbb{R}$

$$1 - F_{\alpha,p}(x) \sim Kx^\alpha e^{-x^p} \text{ as } x \to \infty. \tag{3.6}$$

Consider a finite MA-process $X_i = \sum_{j=1}^k c_j Y_{i-j}$, where (Y_i) are iid, $Y_i \sim F$.

Proposition 3.2 (Point process convergence. Rootzen (1986, Th. 6.3)) *Let $F = F_{\alpha,p}$ have a density $f \in C^1$ with $f(x) \sim K'x^{\alpha'}e^{-x^p}$, $\alpha' = \alpha + p - 1$, $K' = Kp$. Assume that*

$$\limsup_{x\to\infty}\left|\frac{xD'(x)}{D(x)}\right| < \infty, \tag{3.7}$$

where $D(x) = \begin{cases} f(x)e^{x^p}, & x > 0 \\ f(x), & x < 0 \end{cases}$ *, and let* a_n, b_n *be the normalizing constants of the associated iid sequence* Z_i *where* $Z_i \stackrel{d}{=} X_i$ *(see Rootzen (1986, (5.3) and (5.5))). Then*

$$\sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, \frac{X_i - b_n}{a_n}\right)} \xrightarrow{\mathcal{D}} N \tag{3.8}$$

a Poisson process with intensity $\mu = \lambda_{[0,1]} \otimes \nu$, $\nu([x,\infty)) = e^{-x}$, $x \in \mathbb{R}$, *if any of the following conditions holds:*
1) $c_i > 0$, $\forall i$ *and* $e^{cx} f'(x)$ *is bounded for* $x \in (-\infty, 0]$ *for some* $c \geq 0$.
2) $f(-z)$ *fulfills the assumptions on* $f(z)$ *with some* $p' > p$ *and some* α', K'.

The normal distribution satisfies the assumptions above. Under additional assumptions also convergence for infinite MA-processes are proved in Rootzen (1986). As consequence we obtain from Theorem 2.2

Corollary 3.3 *Suppose* $F = F_{\alpha,p}$ *fulfills the conditions of Proposition 3.2, then (formally with* $c = 0$*, see the following Remark 3.4 a))*

1) $\dfrac{EX_{T_n} - b_n}{a_n} \to -\log(1+c)$

2) $P\left(\dfrac{X_{T_n} - b_n}{a_n} \leq x\right) \to \begin{cases} 1 - \dfrac{1}{2}\dfrac{e^{-x}}{1+c}, & x \geq -\log(1+c) \\ \dfrac{1}{2}e^{x}(1+c), & x < -\log(1+c) \end{cases}$

3) *Let* w_n *be constants with* $n(1 - F(w_n)) \to 1$, *then for any* $\varepsilon \in [0,1)$

$$\begin{aligned} T_n' \;=\; \inf\Bigg\{ & i \leq n : (i \geq n - [n\varepsilon] \text{ and } X_i \geq w_{n-i}) \\ & \text{or } \left(i < n - [n\varepsilon] \text{ and } \frac{X_i - b_n}{a_n} \geq u^I\left(\tfrac{i}{n}\right) := \log \frac{1 - \left(\frac{i}{n}\right)^{1+c}}{1+c} \right) \Bigg\} \end{aligned}$$

is an asymptotically optimal sequence of stopping times.

Proof: The proof is analogous to the proof of Theorem 3.1. (G) is implied by the uniform integrability of the normalized maxima of the iid sequence and the k-independence of X_i. □

Remark 3.4 *a) Note that the general formulation with* $c \neq 0$ *corresponds to the optimal stopping of* $X_i + d_i$ *with additional observation costs* $d_i > 0$, $d_i \uparrow$ *where* $\frac{d_n - d_{[nt]}}{a_n} \to -c\log t$ *and with normalizations* $\widehat{b}_i = b_i + c_i$. *The optimal stopping curve in the limit then is* $u_c^I(t) = \log \frac{1-t^{1+c}}{1+c}$ *(see KR (2000, Theorem 4.3)).*

b) *A similar result as in Corollary 3.3 (with the same stopping limits) also holds for* $p = 1$ *if there exists exactly one index* i *with* $c_i = \max\{c_j, 1 \leq j \leq m\}$ *and if any of the following assumptions from Rootzen (1986, Theorem 7.5) holds:*

1) $c_i \geq 0$

2) $F(z) = O\left(e^{\frac{-|z|^p}{\gamma}}\right)$ *as* $z \to -\infty$ *for* γ *such that* $\min c_j \gamma^{\frac{1}{p}} = \max_j c_j$.

3) $F(z) \sim K|z|^\alpha e^{\frac{-|z|^p}{\gamma}}$ *for* γ *such that* $\min_j c_j \gamma^{\frac{1}{p}} = \max_j c_j$.

This class contains for p = 1 the exponential distribution.

c) *Related stopping results also hold true for the class* $S_r(\gamma)$. *Let* $\omega_F = \infty$ *for some distribution function* F *on* $\mathbb{R}$. *Then* $F \in S_r(\gamma)$ *for some* $\gamma \geq 0$ *if* $\lim_{x\to\infty} \frac{1-F*F(x)}{1-F(x)} = d \in (0,\infty)$ *and* $\lim_{x\to\infty} \frac{1-F(x-y)}{1-F(x)} = e^{\gamma y}$, $\forall y \in \mathbb{R}$.

$S_r(0)$ *contains the log-normal distribution as well as the distribution function* F *defined by* $1 - F(x) = e^{-\frac{x}{(\log x)^\alpha}}$, $x > 1$, $\alpha > 0$.

If $F = F_{\alpha,p}, p \in (0,1)$, *then* $F \in S_r(0)$. *If* $F = F_{\alpha,p}$, $p = 1$, $\alpha < -1$, *then* $F \in S_r(1)$ *(see Davis and Resnick (1988)).*

Also, if $F \in D(\Lambda) \cap S_r(\gamma)$ *is integrable* $c_j > 0$ *for all* j, *then point process convergence as in* (3.8) *holds if* $\gamma = 0$ *or* $c_i = \max\{c_j\}$ *for some unique index* i. *(In Davis and Resnick (1988) in fact a convergence result for infinite MA-processes is proved.) As in Corollary 3.3 we obtain the asymptotics of the stopping problem. It seems to be difficult to establish condition (L′) for the infinite case in this example.*

d) *For MA-processes with polynomial tails and* $F \in D(\Phi_\alpha)$ *the limiting point processes typically are cluster Poisson processes which need a different technique and will be dealt in short in chapter 6.*

4 Hidden Markov chains (chain dependent sequences)

Let $(J_n)_{n\in\mathbb{N}}$ be an aperiodic irreducible Markov chain with m states, transition matrix $(p_{ij})_{1\leq i,j\leq m}$ and stationary distribution $\pi_1, \ldots, \pi_m$. $(X_i)_{i\in\mathbb{N}}$ is called hidden Markov chain (or chain-dependent) if

$$\begin{aligned} & P(J_n = j, X_n \leq x \mid J_0, \ldots, J_{n-1}, X_1, \ldots, X_{n-1}) \\ = \;& P(J_n = j, X_n \leq x \mid J_{n-1}) = p_{J_{n-1},j} F_{J_{n-1}}(x), \end{aligned}$$

for some distribution functions $F_1, \ldots, F_m$. The *Markov chain* J_n chooses the distribution function at the nth state.

Proposition 4.1 (Point process convergence. Durrett and Resnick (1978, Example 3.1)) *Let (X_i) be chain-dependent and assume that*

$$\lim_{n\to\infty} n \sum_{i=1}^{m} \pi_i \overline{F}_i(a_n x + b_n) = \nu(x, \infty) \tag{4.1}$$

for some constants a_n, b_n and some non-degenerate measure ν on $\mathbb{R}$. Then

$$N_n = \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, \frac{X_i - b_n}{a_n}\right)} \xrightarrow{\mathcal{D}} N \tag{4.2}$$

a Poisson process with intensity $\lambda_{[0,1]} \otimes \nu$ where $x \to e^{-\nu(x,\infty)}$ is a df of an extreme value type.

Proof: For the proof observe that

$$\begin{aligned}
&\sum_{i=1}^{[nt]} P(X_{n,i} > x | \mathcal{F}_{n,i-1}) \\
&= \sum_{i=1}^{[nt]} P(X_i > x a_n + b_n | J_{i-1}) = \sum_{i=1}^{[nt]} (1 - F_{J_{i-1}}(x a_n + b_n)) \\
&= \sum_{j=1}^{m} \sum_{i=1}^{[nt]} (1 - F_j(x a_n + b_n)) 1_{\{J_{i-1} = j\}} \\
&= \sum_{j=1}^{m} \#\{j : J_i = j, i = 0, \ldots, n-1\} (1 - F_j(x a_n + b_n)) \\
&\sim \sum_{j=1}^{m} n t \pi_j (1 - F_j(x a_n + b_n)) \sim t \nu(x, \infty).
\end{aligned}$$

This implies the result by applying Theorem 2.3. □

We now establish approximation of the optimal stopping problem of this class of hidden Markov chains if the limit of type Φ_α. Then the optimal limiting stopping curve is given by $u(t) = \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}} (1-t)^{\frac{1}{\alpha}}$.

Theorem 4.2 (Optimal stopping of hidden Markov chain) *Assume condition (4.1) where $\nu([x, \infty)) = e^{-x^\alpha}$ for some $\alpha > 1$ then with T_n the optimal stopping time of the hidden Markov chain $X_1, \ldots, X_n$ holds:*

1) $\dfrac{E X_{T_n}}{a_n} \to \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}}$

2) $$P\left(\frac{X_{T_n}}{a_n} \le x\right) \to \begin{cases} 1 - x^{-\alpha} \frac{1}{2 - \frac{1}{\alpha}}, & x \ge \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}} \\ \frac{\alpha}{2\alpha-1} \left(\frac{\alpha-1}{\alpha}\right)^{\frac{\alpha-1}{\alpha}} x^{\alpha-1}, & 0 < x < \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}} \\ 0 & x \le 0. \end{cases}$$

3) $T_n' := \inf\left\{1 \le i \le n:\; X_i \ge \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}}\left(1-\frac{i}{n}\right)^{\frac{1}{\alpha}}\right\}$ *defines an asymptotically optimal sequence of stopping times.*

Proof: For the proof we establish the conditions of Theorem 2.2.

1) Point process convergence of $N_n \to N$ is stated in Proposition 4.1. Condition (D) for the intensity measure has been established in KR (2000, Theorem 4.4, case $c = 0$)).

2) Condition (A): $P^{N_n|[t,1]|\mathcal{F}_n^i} \xrightarrow{P} P^{N|[t,1]}$, with $i = [nt-1]$. Observe that

$$\begin{aligned} P^{N_n(\cdot\cap[0,1]\times\mathbb{R})|\mathcal{F}_n^i} &= P^{N_n(\cdot\cap[0,1]\times\mathbb{R})|J_{[nt-1]}} \\ &= \sum_{j=1}^m P^{N_n(\cdot\cap[0,1]\times\mathbb{R})|J_{[nt-1]}=j} 1_{\{J_{[nt-1]}=j\}} \\ &\to P^{N(\cdot\cap[0,1]\times\mathbb{R})} \quad P \text{ a.s.} \end{aligned}$$

For the proof we just have to repeat the arguments in the proof of Proposition 4.1 to establish point process convergence.

3) (G) : $\left(\frac{M_n^+}{a_n}\right)_{n\in\mathbb{N}}$ is uniformly integrable. For the proof let for $k \in \{1,\dots,m\}$, $(Y_i^k)_{i\in\mathbb{N}}$ be iid with d.f. F_k independent of $(Y_i^j)_{i\in\mathbb{N}}$ for $j \ne k$ and of $(J_n)_{n\in\mathbb{N}_0}$. Then at stage n X_i can be defined as Y_i^k with probability $p_{J_{n-1},k}$ and, therefore, $X_i \le Y_i^1 \vee \dots \vee Y_i^m$. With $\overline{G}_j = 1 \wedge \frac{1}{\pi_j}\sum_{i=1}^m \pi_i \overline{F}_i$ holds $\overline{F}_j \le \overline{G}_j$ and $G_j \in D(\Phi_\alpha)$. Therefore, there exist iid r.v.s $(Z_i^j)_{j\in\mathbb{N}}$, $Z_i^j \sim G_j$, $Z_i^j \ge Y_i^j$, and $P^{Y_i^j} \in D(\Phi_\alpha)$. This implies uniform integrability of $\left\{\frac{1}{a_n}Y_1^k \vee \dots \vee Y_n^k\right\}, 1 \le k \le m$, and, therefore, of $\left(\frac{1}{a_n}\bigvee_{k=1}^m\bigvee_{i=1}^n Y_i^k\right)$. Finally this implies uniform integrability of $\left(\frac{1}{a_n}M_n^+\right)$.

4) Condition (L) follows from $\lim_{n\to\infty} E\frac{X_n}{a_n} = 0$, point process convergence $N_n \xrightarrow{\mathcal{D}} N$ and KR (2000, Theorem 4.4).

Now the statement of Theorem 4.2 follows from Theorem 2.2 and the calculations in KR (2000, Theorem 4.4) for the limiting stopping problem. $\square$

5 Max-AR(1) sequences

Let (Y_i) be an iid sequence with d.f. $F \in D(\Psi_\alpha)$, $\alpha > 1$. Further let (Z_i) be a sequence of independent r.v.s, $0 \le Z_i \le 1$, independent of (Y_i) and let

X_0 be an r.v. with $EX_0^+ < \infty$. A max-autoregressive process of order 1 (max-AR(1)) is defined by

$$X_i = Z_i \max\{X_{i-1}, Y_i\}, \quad i \geq 1. \tag{5.1}$$

The extremes of this process were investigated in Alpuim, Catkan, and Hüsler (1995). Also the clustering properties of the exceedance point process in case $F \in D(\Phi_\alpha)$ has been investigated in that paper (see also the references there). The cases $F \in D(\Lambda)$, $D(\Psi_\alpha)$ do not exhibit this clustering property and so are simpler to deal with. To derive the optimal stopping approximations we can not apply Theorem 2.2 directly since we do not have point process convergence of the inbedded point process to a Poisson process. Instead we argue by comparison with a related point process.

Theorem 5.1 *Assume that* $F \in D(\Phi_\alpha), \alpha > 1, 0 \leq Z_i \leq 1, \frac{1}{n}\sum_{i=1}^{[nt]} EZ_i^\alpha \to td, 0 \leq t \leq 1$, *and* $EZ_i^\alpha \leq \beta$ *for some* $\beta < 1$, *then with* (a_n) *the max-normalization of* F *and* $\widehat{a}_n = a_n d^\alpha$ *holds:*

1) $$E\frac{X_{T_n}}{a_n} \to \left(\tfrac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}}$$

2) $$T_n' := \inf\left\{1 \leq i \leq n: \; X_i \geq \left(\tfrac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}} \left(1 - \tfrac{i}{n}\right)^{\frac{1}{\alpha}}\right\}$$

defines an asymptotically optimal sequence of stopping times.

Proof: Define $\widetilde{X_i} := Z_i Y_i, \; i \geq 1$, then

$$\widetilde{N}_n := \sum_{i=1}^n \varepsilon_{\left(\frac{i}{n}, \frac{\widetilde{X}_i}{a_n}\right)} \xrightarrow{\mathcal{D}} N \tag{5.2}$$

a Poisson process with intensity defined by $\mu([0,t] \times [x, \infty)) = tx^{-\alpha}$.

For the proof of (5.2) we establish the condition of Theorem 2.3. Let $1 - F(x) = L(x)x^{-\alpha}$ with some $L \in RV_0$, then

$$\begin{aligned}
P(Y_i Z_i > x\widehat{a}_n) &= \int P\left(Y_i > \widehat{a}_n \frac{x}{z_i}\right) dP^{Z_i}(z_i) \\
&= \int L\left(x\frac{\widehat{a}_n x}{z_i}\right)\left(\frac{x a_n}{z_i}\right)^{-\alpha} \frac{1}{d} dP^{Z_i}(z_i) \\
&= \left(\int L\left(x\frac{\widehat{a}_n}{z_i}\right)\left(\frac{1}{z_i}\right)^{-\alpha} dP^{Z_i}(z_i)\right) x^{-\alpha} a_n^{-\alpha}\frac{1}{d}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{[nt]} P(Y_i Z_i > x\widehat{a}_n) &= \sum_{i=1}^{[nt]} \int L\left(x\frac{\widehat{a}_n}{z_i}\right)\left(\frac{1}{z_i}\right)^{-\alpha} dP^{Z_i}(z_i) x^{-\alpha} a_n^{-\alpha}\frac{1}{d} \\
&\sim \frac{x^{-\alpha}}{d}\frac{1}{n}\sum_{i=1}^{[nt]} \int Z_i^\alpha dP \;\longrightarrow\; tx^{-\alpha}.
\end{aligned}$$

Finally,

$$\sup_{1\le i\le n}\{P(\widetilde{X}_i > x\widehat{a}_n)\} \le \sup_{1\le i\le n}\{P(Y_i > x\widehat{a}_n)\} \to 0.$$

This implies (5.2) by Theorem 2.3 on point process convergence.

The point process N fulfills (D) and $\left(\frac{\widetilde{M}_n^+}{a_n}\right)_{n\in\mathbb{N}}$, is uniformly integrable. The optimal stopping curve of N is $u(t) = u_\alpha^{II}(t) = \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}}(1-t)^{\frac{1}{\alpha}}$ (cf. KR (2000, Theorem 4.4)). Further, by the approximation result for optimal stopping of independent sequences, KR (2000, Theorem 4.4),

$$\widetilde{T}_n' := \inf\left\{i : \widetilde{X}_i \ge \widehat{a}_n u(\frac{i}{n})\right\} \tag{5.3}$$

is asymptotically optimal and

$$E\frac{X_{\widetilde{T}_n'}}{a_n} \to u(0). \tag{5.4}$$

This implies that $u(0)$ is a lower bound for the asymptotically optimal stopping value of X_i.

In the next step we prove that

$$\limsup_{n\to\infty}\frac{1}{\widehat{a}_n}E\gamma_{n,1} \le u(0). \tag{5.5}$$

i.e. $u(0)$ is in fact identical with the asymptotically optimal stopping value. For the proof note that

$$X_{[nt]} \le M_{n,0,[(n-\sqrt{n})t]}\prod_{i=[(n-\sqrt{n})t]}^{[nt]} Z_i \vee M_{n,[(n-\sqrt{n})t]+1,[nt]}$$

and

$$\frac{M_{n,[(n-\sqrt{n})t]+1,[nt]}}{a_n} \xrightarrow{P} 0.$$

From $M_n = X_1 \vee Y_2Z_2 \vee \ldots \vee Y_nZ_n$ and KR (2000, Proposition 4.1) we obtain that $\left(\frac{M_n^+}{a_n}\right)_{n\in\mathbb{N}}$ is uniformly integrable. Then $\left(\frac{M_{n,0,[(n-\sqrt{n})t]}}{a_n}\right)$ converges in distribution and in L^1 and $E\prod_{i=[(n-\sqrt{n})t]}^{[n,t]} Z_i \le \beta^{\sqrt{n}-1} \to 0$, which implies $\prod_{i=[(n-\sqrt{n})t]}^{[nt]} Z_i \xrightarrow{P} 0$ since $0 \le Z_i \le 1$. Together we obtain that

$$E\frac{X_{[nt]}}{a_n} \xrightarrow{P} 0. \tag{5.6}$$

Since $M_{n,[ns],[nt]} = X_{[ns]} \vee Y_{[ns]+1}Z_{[ns]+1} \vee \ldots \vee Y_{[nt]}Z_{[nt]}$ this implies

$$P^{\frac{M_{n,[ns],[nt]}-b_n}{a_n}|\mathcal{F}_{[ns]-1}} \xrightarrow{\mathcal{D}} P^{M_{s,t}} \tag{5.7}$$

Convergence as in (5.7) was the basic starting step in the proof of Theorem 2.2. Following the further steps of this proof we obtain (5.5). Obviously stopping of X_i majorizes stopping of $\widetilde{X}_i$ and equality holds in (5.5); i.e. part 1) is proved.

To prove asymptotic optimality of T_n' we have to establish that

$$E\frac{X_{T_n'}}{\widehat{a}_n} \to u(0). \tag{5.8}$$

To that purpose we prove that T_n' only stops asymptotically at time points where $X_i = \widetilde{X}_i$. Define $T_n'' = \sup\{i \le T_n' : X_i = \widetilde{X}_i\}$. Then

$$P(T_n'' = T_n') \to 1 \tag{5.9}$$

We prove (5.9) by contradiction. By definition $X_{T_n''} \ge X_{T_n''+1} \ge \ldots \ge X_{T_n'}$. W.l.g. assume that $\left(\frac{T_n'}{n}, \frac{X_{T_n'}}{\hat{a}_n}, \frac{T_n''}{n}, \frac{X_{T_n''}}{\hat{a}_n}\right) \xrightarrow{\mathcal{D}} (T', y, T'', y'')$, assume pointwise convergence for all $\omega \in \Omega$ and $y_k \neq u(\tau_k), \forall k \in \mathbb{N}, \tau_i \neq \tau_j, i \neq j$. Let T be the optimal stopping time of N and let $\frac{T_n'(\omega) - T_n''(\omega)}{n} \to 0$, then

$$T_n'(\omega) = T_n''(\omega) \text{ for all } n > n_\omega. \tag{5.10}$$

For the proof note that $\frac{\widetilde{T}_n'}{n} \to T$ and $T < 1$ a.s. Since $X_i \ge \widetilde{X}_i$, $T_n' \le \widetilde{T}'$ and so $T' < 1$ a.s. This yields $T'(\omega) = T''(\omega)$ and by continuity $y'(\omega) \le u(T'(\omega))$ and, therefore, $y''(\omega) \ge u(T'(\omega))$. By the first part of the proof $(T''(\omega), y''(\omega))$ can be assumed to be a point of N. Therefore, we obtain from (D) that $y''(\omega) > u(T''(\omega))$ and so $\frac{X_{T_n''(\omega)}}{\hat{a}_n} > u\left(\frac{T_n''(\omega}{n}\right)$ for $n > n_\omega$.

This implies that at time $T_n''(\omega)$ the condition in the definition of $T_n'(\omega)$ is fulfilled i.e. $T_n'(\omega) = T_n''$ for $n \ge n_\omega$. Assume now that (5.8) does not hold. Then by the previous argument $P(T'' < T') > 0$ i.e., there exists $q \in (0,1) \cap \mathbb{Q}$ with $P(T'' < q < T' < 1) > 0$. Then from definition of T_n'' and the above ordering relation $\liminf_{n\to\infty} P\left(\frac{X_{[qn]}}{\hat{a}_n} > \varepsilon\right) > 0$ for some $\varepsilon > 0$ in contradiction to (5.6).

Thus we obtain $T_n'(\omega) = T_n''(\omega)$ for $n > n_\omega$ and, therefore, $T_n'(\omega) = \widetilde{T}_n'(\omega)$ for $n > n_\omega$ which implies asymptotic optimality of T_n'. $\square$

6 Some extensions to Poisson cluster processes

The results in this paper can also be extended partially to the case where one has point process convergence to some Poisson cluster process, as is typical for more general MA-processes. Consider e.g. iid r.v.s (Y_i) with distribution function $F \in D(\Phi_\alpha)$. Let $\alpha > 1$, $c_j \in \mathbb{R}$, $j \in \mathbb{N}$ with $c_j \neq 0$ for at least one index j, say $c_1 \neq 0$, and $\sum_{j=1}^{\infty} |c_j|^\delta < \infty$ for some $0 < \delta < 1$. Consider the infinite MA-process

$$X_i = \sum_{j=1}^{\infty} c_j Y_{i+1-j}. \tag{6.1}$$

Assume any of the following conditions:

1) $c_j \geq 0$ for all j

2) $c_j \leq 0$, for all j and also $F(0-\cdot) \in D(\Phi_\alpha)$

3) $P(|Y_1| > x) \in RV_{-\alpha}$, and $\lim_{x\to\infty} \frac{P(Y_1>x)}{P(|Y_1|>x)} = p$ exists. If $p=1$ then $c_j > 0$ for some j; if $p=0$ then $c_j < 0$ for some j.

Under any of these conditions point process convergence holds on $[0,1] \times (0,\infty]$

$$N_n = \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, \frac{X_i}{a_n}\right)} \xrightarrow{\mathcal{D}} N = \sum_{k=1}^{\infty} \sum_{\substack{i=1 \\ c_i \neq 0}}^{\infty} \varepsilon_{(\tau_k, c_i y_k)} \tag{6.2}$$

where $N' = \sum \varepsilon_{\tau_k, y_k}$ is a Poisson process with intensity $\mu = \lambda_{[0,1]} \otimes \nu$ with $\nu([x,\infty]) = x^{-\alpha}$ (see Resnick (1987, Chapter 4.5)).

So the limiting point process N is a Poisson cluster process with deterministic cluster based on the underlying Poisson process N'. For the optimal stopping we will observe at stage n approximatively these limiting clusters and so it is intuitively clear that essentially the underlying Poisson process determines asymptotically the optimal stopping problem. At any observed large point one should wait until the point of the cluster arrives with the biggest coefficient. This program can be made precise and the method of the proof of Theorem 2.2 can be extended to yield the following theorem which we only formulate under conditions 1), 2) (details of the argument will be given elsewhere).

Theorem 6.1 *Consider the infinite moving average process in* (6.1) *with* $F \in D(\Phi_\alpha), \alpha > 1$ *and either condition 1) or 2). Also assume w.l.g. that* $\sup_i\{|c_i|\} = 1$. *Then we obtain*

1) $$\frac{EX_{T_n}}{a_n} \to \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}},$$

2) $$P\left(\frac{X_{T_n}}{a_n} \leq x\right) \to \begin{cases} 1 - x^{-\alpha}\frac{1}{2-\frac{1}{\alpha}}, & x \geq \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}} \\ \frac{\alpha}{2\alpha-1}\left(\frac{\alpha-1}{\alpha}\right)^{\frac{\alpha-1}{\alpha}} x^{\alpha-1}, & 0 < x < \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}} \\ 0, & x \leq 0. \end{cases}$$

Define $m = \inf\{i: \ c_i = \sup_j\{|c_j|\}\}$, $w = \sup\{i; \ |c_i| \geq |c_1|\}$, *then*

$$\begin{aligned} T_n' \quad := \quad & \inf\left\{ i \geq w+1 : X_i \geq a_n \frac{c_1}{c_m} u_\alpha^{II}\left(\frac{i}{n}\right), \right. \\ & \left. X_{i-1} \vee \ldots \vee X_{i-w} < \frac{1}{2} a_n \frac{c_1}{c_m} u_\alpha^{II}\left(\frac{i}{n}\right) \right\} + m - 1 \end{aligned} \tag{6.3}$$

defines an asymptotically optimal sequence of stopping times. Here $u_\alpha^{II}(t) = \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}} (1-t)^{\frac{1}{\alpha}}$

u^{II} is the optimal stopping curve of the underlying Poisson process N' and the stopping sequence T_n' , is defined according to the idea sketched above.

The case of random clusters in the limiting process is technically more involved even if the basic idea again is simple. When reaching a *large* random cluster one compares the first observed point with the max expected value of the points in the cluster to come. This max however is not fixed but has to be estimated from an initial small part of the data like the first $\sqrt{n}$ of the data. The details are somewhat technically involved and have been done in Kühne (1997) for the case of processes of the form $X_i = Y_i + Y_{i-1}$ with a random cluster of two points in the limit and $X_i = Y_i + bY_{i-1} + Y_{i-2}$ with a three point random cluster for $F \in D(\Lambda) \cap S_r(1)$. The method extends to more general cases. The formulation of the asymptotically optimal stopping time is somewhat involved but the leading idea again is to reduce this problem essentially to the underlying Poisson process and modify the optimal stopping curve by consideration of the cluster structure.

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