Optimal stopping with discount and observation costs

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Abstract

For iid random variables in the domain of attraction of a max-stable distribution with discount and observation costs we determine asymptotic approximations of the optimal stopping values and asymptotically optimal stopping times. The results are based on Poisson approximation of the related embedded planar point processes. The optimal stopping problem for the limiting Poisson point processes can be reduced to differential equations for the boundaries. We obtain in several cases numerical solutions of the differential equations. In some cases the analysis allows to obtain explicit optimal stopping values. This approach leads to approximate solutions of the optimal stopping problem of these discrete time sequence.

Keywords: optimal stopping, max stable distributions, Poisson processes

AMS 1991 subject classification: 60 G40, 62 L15

1 Introduction

Kennedy and Kertz (1990, 1991, 1992) found approximations for the optimal stopping of iid sequences (X_i) in the domain of attraction of max-stable laws of type Λ, Φ_α and Ψ_α. They also determined approximations to the stopping problem with linear costs $(X_i - i\delta)$, respectively with geometrically discounted sequences $e^{-i\delta}X_i$ for infinite sequences as $\delta \to 0$ i.e. for the case of 'small' observation costs respectively discount factor close to 1.

A general approximation method for the optimal stopping of independent sequences was developed in a recent paper of Kühne and Rüschendorf (1998) (which we quote in the following by KR (98)). This method is based on Poisson approximation of the embedded planar point processes $N_n = \sum_{i=1}^n \varepsilon_{(\frac{i}{n}, X_{n,i})}$, where $X_{n,i} = \frac{X_i - b_n}{a_n}$ are normalized versions of X_i which ensure convergence of N_n to a Poisson process on $[0,1] \times \mathbb{R}$.

For the iid case and X_i with distribution function F in the domain of a max-stable law $G \in \{\Lambda, \Phi_\alpha, \Psi_\alpha\}$ convergence of N_n holds, $N_n \xrightarrow{\mathcal{D}} N$, where N is a Poisson process with intensity measure $\mu = \lambda_{[0,1]} \otimes \nu$ and

$$\begin{aligned} \nu([x,\infty)) &= e^{-x} && \text{if } F \in D(\Lambda) \\ \nu([x,\infty)) &= x^{-\alpha}, x > 0, && \text{if } F \in D(\Phi_\alpha) \\ \nu([x,\infty)) &= (-x)^{\alpha}, x < 0, && \text{if } F \in D(\Psi_\alpha),\ \omega_F = 0 \end{aligned} \tag{1.1}$$

(see Resnick (1987, p. 210)). a_n, b_n are the normalizations as in the limit theorems for maxima $M_n = \max\{X_1, \ldots, X_n\}$.

The optimal stopping problem for a Poisson process with intensity μ can be reduced to solving a differential equation of the form

$$\begin{cases} v'(t) = -\int\limits_{v(t)}^{\infty}\int\limits_{x}^{\infty} h(t,y)dy\,dx, \quad 0 \le t \le t_0 \\ v(t_0) = c \end{cases} \tag{1.2}$$

where h is a Lebesgue-density of μ, $(t,z) \to \int\limits_z^\infty \int\limits_x h(t,y)dy\,dx$ is assumed to be continuous (condition (D)) and t_o, c arise from the specific problem (see KR (98)).

The solution of (1.2) is the optimal stopping curve for the Poisson process, if v is assumed to satisfy the separation condition (S) that $(v - f)|[0,t] > c_t > 0$ for $t < t_0$, where f is decreasing and describes the lower boundary of the intensity μ. In the case of iid asymptotics in (1.1) $f = -\infty$ if $F \in D(\Lambda, \Psi_\alpha)$ and $f = 0$ if $F \in D(\Phi_\alpha)$. Note that the intensity of the limiting Poisson process N is infinite at the lower boundary f.

Finally, convergence of the stopping problem of $\{X_1, \ldots, X_n\}$ to the stopping problem of N holds if additionally to the assumptions (S) and (D) also the $\{(M_n)_+\}$ are uniformly integrable (condition (G)) and the lower curve condition (L) holds, i.e. $\underline{\lim}\, u_n(1-\varepsilon) > -\infty$ for all $\varepsilon > 0$. Here $u_{n,1}, \ldots, u_{n,n}$ is the optimal stopping curve of $X_{n,1}, \ldots, X_{n,n}$ and $u_n(s) = u_{n,[ns]\vee 1}$. Convergence of the stopping problem means convergence of the optimal stopping times, stopping time distributions and stopping value.

This new approximation approach was applied to the optimal stopping of iid sequences with observation costs and df F in the max-domain of $\Lambda(x) = e^{-e^{-x}}$ in KR (98). It was also applied to iid sequences with discount costs for dfs F in the max-domain of Φ_α resp. Ψ_α.

Some related ideas can be found in the literature. An approximation of the optimal stopping of max-sequences by the optimal stopping of continuous time extremal processes has been observed in Flatau and Irle (1984). In this paper both stopping problems are of monotone kind and could be solved explicitely. The approximative optimal stopping of the max-sequence by that of the limiting process is not derived from some general approximation argument but is proved directly. It has also been observed in some papers that the optimal stopping problem has an easier solution in a related form with a Poisson number of points. Bruss and Rogers (1991) and Gnedin (1996) use this idea in the context of an optimal selection problem. An example of this kind is the house selling problem (see Chow, Robbins, and Siegmund (1971)) where a Poisson process with finite intensity of the form $\mu = \lambda \otimes \nu$ on $[0,1] \times \mathbb{R}^1$ is stopped. For this case the stopping problem could be reduced to the stopping of stationary Markov sequences and could be solved in some cases.

In this paper we apply the approximation approach in a more systematic way to further stopping problems of iid sequences with observation costs and/or discount costs where the iid random variables are in the domain of max-stable distributions. These kind of nonstationary sequences are a natural class of models for stopping problems. These models were also the subject of intensive study in extreme value theory after the iid case had been solved.

Some of the stopping problems considered in this paper do not admit closed solutions due to the complexity of the assumptions and in particular of the differential equation (1.2) but can be solved numerically. In these problems direct recursive solutions as in the papers of Kennedy and Kertz in the iid case seem to be not obtainable. For some details of the arguments and calculations we refer to KR (98) and to the dissertation of Kühne (1997).

2 Approximate optimal stopping rules

Let (Y_i) be an iid sequence of real random variables with df F in the domain of attraction of a max-stable law which is known to be of three possible types, $\Lambda, \Phi_\alpha, \Psi_\alpha$; $\Lambda(x) = e^{-e^{-x}}, x \in \mathbb{R}^1$, $\Phi_\alpha(x) = e^{-x^{-\alpha}}, x > 0$, and $\Psi_\alpha(x) = e^{-(x)^d}, x < 0$.

We consider at first random sequences (Y_i) with distribution function F in the domain of attraction of Φ_α or with $F \in D(\Psi_\alpha)$ and $\omega_F = 0$ (i.e. $Y_i \leq 0$). In both cases the normalizing sequences a_n, b_n can be choosen as $a_n = F^{-1}(1 - \frac{1}{n})$ (resp. $a_n = -F^{-1}(1 - \frac{1}{n})$) and $b_n = 0$. Consider a bounded discount sequence $c_i > 0$ and observation costs (d_i) such that (c_i) are either increasing or decreasing and

$$\lim \frac{c_{[nt]}}{c_n} = t^c, \quad t \in [0,1] \tag{2.1}$$

and

$$\lim \frac{d_{[nt]}}{a_n c_n} = d \quad \text{exist.} \tag{2.2}$$

Conditions of the type (2.1), (2.2) are common in extreme value theory (see de Haan and Verkade (1987)). They ensure convergence of the imbedded point processes N_n.

Define

$$X_i = c_i Y_i + d_i, \quad i \in \mathbb{N}. \tag{2.3}$$

We consider the optimal stopping problem for $X_1, \ldots, X_n$, i.e. the problem to determine

$$V^n = \sup\{EX_\tau;\ \tau \text{ a stopping time} \leq n\}. \tag{2.4}$$

Let T_n denote an optimal stopping time for problem (2.4).

Theorem 2.1 *Consider the optimal stopping problem* (2.4) *and assume* (2.1), (2.2) *with* $c > -\frac{1}{\alpha}$.

a) If $F \in D(\Phi_\alpha), \alpha > 1, d \geq 0$, *then the differential equation*

$$\begin{cases} u'(t) &= -\dfrac{t^{c\alpha}}{\alpha - 1}(u(t) - dt^{1/\alpha + c})^{1-\alpha} \\ u(1) &= d \end{cases} \tag{2.5}$$

has a unique continuous, monotonically nonincreasing solution u.

If (T_n) *denotes the sequence of optimal stopping times, then*

$$E\frac{X_{T_n}}{\widehat{a}_n} \to u(0), \quad \text{where } \widehat{a}_n = c_n a_n. \tag{2.6}$$

The threshold stopping times $T_n^u := \inf\left\{i \leq n; X_i \geq u\left(\frac{i}{n}\right)\right\}$ *are asymptotically optimal.*

b) If $F \in D(\Psi_\alpha), \alpha > 0, d \geq 0$, *then the differential equation*

$$\begin{cases} u'(t) &= -\dfrac{t^{c\alpha}}{1+\alpha}\left(-u(t) + dt^{-c-1/\alpha}\right)^{1+\alpha} \\ u(1) &= -\infty \end{cases} \tag{2.7}$$

has a unique continuous, monotonically nonincreasing solution on $[0, 1]$ *and*

$$E\frac{X_{T_n}}{\widehat{a}_n} \to u(0). \tag{2.8}$$

The threshold stopping times T_n^u *are asymptotically optimal.*

Proof:

a) In the first step we prove the point process convergence

$$N_n = \sum_{i=1}^n \varepsilon_{\left(\frac{i}{n}, \frac{X_i}{\widehat{a}_n}\right)} \xrightarrow{\mathcal{D}} N, \tag{2.9}$$

where N is a Poisson process with intensity given by

$$\frac{d\mu(\cdot \times [x, \infty))}{d\lambda_{[0,1]}}(t) = t^{c\alpha}(x - dt^{1/\alpha + c})^{-\alpha}, \quad x \geq 0.$$

For the proof we note that $\frac{X_i}{\widehat{a}_n} = \frac{c_i Y_i + d_i}{c_n a_n} = \frac{c_i}{c_n}\frac{Y_i}{a_n} + \frac{d_i}{c_n a_n}$. Define $\gamma_{n,\frac{i}{n}} = \frac{c_i}{c_n}$, $\tau_{n,\frac{i}{n}} = \frac{d_i}{c_n a_n}$; then for $t \neq 0$, $(t_n, y_n) \to (t, y)$ implies

$$R_n(t_n, y_n) := (t_n, \gamma_{n,t_n} y_n + \tau_{n,t_n}) \to (t, \gamma_t y + \tau_t) =: R_0(t, y), \tag{2.10}$$

where $\gamma_t = \lim_n \frac{c_{[nt]}}{c_n} = t^c$ and

$$\begin{aligned}\tau_t &= \lim \frac{d_{[nt]}}{a_n c_n} = \lim \frac{d_{[nt]}}{a_{[nt]}c_{[nt]}} \frac{a_{[nt]}c_{[nt]}}{a_n c_n} \\ &= dt^c t^{1/\alpha} = dt^{c+1/\alpha}.\end{aligned}$$

Remind that $\widetilde{N}_n = \sum_{i=1}^n \varepsilon_{(\frac{i}{n}, \frac{Y_i}{a_n})} \to \widetilde{N}$ a Poisson process with intensity $\widetilde{\mu} = \lambda\!\!\!\lambda_{[0,1]} \otimes \nu, \quad \nu([x,\infty)) = x^{-\alpha}, x > 0$ (see Resnick (1987, p. 210)).
From uniform convergence of $R_n \to R_0$ on $[0,1]$ for $t > 0$ we conclude that $N_n = R_n \widetilde{N}_n \xrightarrow{\mathcal{D}} N := R_0 \widetilde{N}$, where R_n, R_0 operate on the points of the point process. N is a Poisson process with intensity $\mu = (\widetilde{\mu})^{R_0}$. Then

$$\begin{aligned}\mu([0,t] \times [x,\infty)) &= \widetilde{\mu}\left(\left\{(s,y); \quad 0 \le s \le t, s^c y + ds^{c+1/\alpha} \ge x\right\}\right) \\ &= \int_0^t ds (s^{-c}x - ds^{1/\alpha})^{-\alpha}.\end{aligned}$$

Therefore, $\frac{d\mu(\cdot \times [x,\infty))}{d\lambda\!\!\!\lambda_{[0,1]}}(t) = t^{c\alpha}(x - dt^{1/\alpha + c})^{-\alpha}$. Since

$$\int_{u(x)}^{\infty} \frac{1}{(xt^{-c} - dt^{1/\alpha})^{\alpha}} \, dx = -\frac{t^{c\alpha}}{\alpha - 1} \left(u(t) - dt^{1/\alpha + c}\right)^{1-\alpha},$$

u solves the differential equation in (1.2) characterizing optimal stopping curves for the limiting Poisson process. The regularity conditions stated in the introduction are easily checked in this case and imply convergence of the stopping problem.

b) is proved analogously to a). To prove uniqueness of the solution of (2.7) (which is needed as $u(1) = -\infty$, see KR (98)) assume that u_1, u_2 are different solutions. Then $u_1(0) \neq u_2(0)$ since a solution v is uniquely determined by its initial value $v(0) = c > -\infty$. Assume $u_1(0) > u_2(0)$; then by (2.5), $u_1'(t) \ge u_2'(t)$, $t \in [0,1)$ and so $u_1(t) \ge u_2(t) + u_1(0) - u_2(0)$, $t \in [0,1)$. For any solutions v holds: $u_0(t) + b \ge v(t) \ge u_0(t) + b(1-t)^{1/\alpha + c}$. For t close to one this leads to a contradiction with $u_1(0) - u_2(0) > 0$. □

Remark 2.2 For $d < 0$ in part a) of Theorem 2.1 the separation condition is difficult to analyse. It can be checked numerically in each case considered and then leads to the same approximation result as in the case $d \ge 0$.

Theorem 2.1 reduces the optimal stopping problem of $X_1, \ldots, X_n$ to solving the differential equations (2.5), (2.7). This can be done in general only numerically.

Example 2.3 In the following example we consider the case Φ_α with $\alpha = 2, c = 1, d = 1$. The optimal stopping value $u(0)$ for the limiting Poisson process together with (2.6) and (2.8) determines the asymptotic behaviour of the stopping value V^n. It is calculated with Maple (note the singularity of the derivative in $t = 1$).

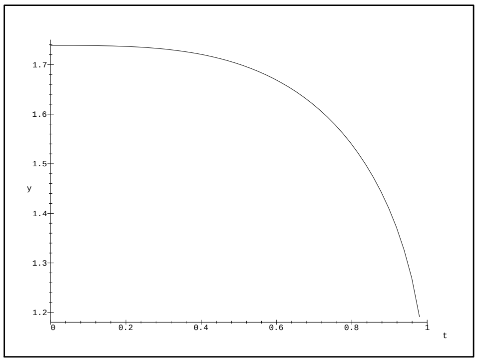

Figure 1:
Optimal stopping boundary $u(t)$,
$\alpha = 2, c = 1, d = 1$

Next we consider in more detail the case $F \in D(\Phi_2)$ and $c_i = 1, \forall i$, i.e. the Φ_2-case with observation costs only. Let $(d_i) \subset \mathbb{R}$ with $d_i \geq 0, d_{i+1} \geq d_i$ or $d_i \leq 0, d_{i+1} \leq d_i$ and $d = \lim_n \frac{d_n}{a_n}$ exists and define

$$X_i = Y_i + d_i.$$

In this case we obtain a more detailed analysis of the differential equation and derive an implicit representation of the solution. This yields an explicit expression for the optimal stopping value $u(0)$.

Theorem 2.4 *a) If $|d| < 2\sqrt{2}$, then*

$$\frac{EX_{T_n}}{a_n} \to \sqrt{2}\, e^{d \frac{\frac{\pi}{2} - \tan^{-1}\left(\frac{d}{\sqrt{8-d^2}}\right)}{\sqrt{8-d^2}}},$$

where T_n are the optimal stopping rules.

Furthermore, with

$$F(t,y) := \left(y - \frac{\sqrt{t}}{2} d\right)^2 + t\left(2 - \frac{d^2}{4}\right) - 2e^{2d \frac{\tan^{-1}\left(2 \frac{y - \frac{\sqrt{t}}{2} d}{\sqrt{8-d^2}\,\sqrt{t}} - \tan^{-1} \frac{d}{\sqrt{8-d^2}}\right)}{\sqrt{8-d^2}}}$$

for $t \in (0,1], y \in \mathbb{R}$ the following conditions

$$\begin{cases} 1.) \quad F(t, u(t)) = 0 \quad \text{for } t \in (0,1] \\ 2.) \quad u(0) = \sqrt{2}\, e^{d \frac{\frac{\pi}{2} - \tan^{-1} \frac{d}{\sqrt{8-d^2}}}{\sqrt{8-d^2}}} \\ 3.) \quad u(t) \geq d \quad \text{for } t \in [0,1] \end{cases} \tag{2.11}$$

determine uniquely a monotonically nonincreasing function on $[0,1]$. *The corresponding threshold stopping time* T_n^u *is asymptotically optimal.*

b) *For* $d > 2\sqrt{2}$ *holds*

$$\frac{EX_{T_n}}{a_n} \to \sqrt{2}\left(\frac{d^2 + d\sqrt{d^2-8} - 4}{4}\right)^{\frac{d}{2\sqrt{d^2-8}}} \tag{2.12}$$

and

$$F(t,y) = \left(y - \frac{\sqrt{t}}{2}d\right) + t\left(2 - \frac{d^2}{4}\right) - 2e^{2c\,\frac{\tan h^{-1}\left(\frac{c}{d^2-8}\right) - \tan h^{-1}\left(2\,\frac{y-\frac{\sqrt{t}}{2}d}{\sqrt{d^2-8}}\right)}{\sqrt{d^2-8}}}$$

determines an asymptotical optimal threshold stopping time as in a).

c) *If* $d = 2\sqrt{2}$ *then* $\frac{EX_{T_n}}{a_n} \to \sqrt{2}e$ *and* $F(t,y) = y - \sqrt{2t} - \sqrt{2}\,e^{\frac{y-2\sqrt{2t}}{y-\sqrt{2t}}}$ *determines an asymptotic optimal threshold stopping time as in a).*

Proof: As $\frac{d_n}{a_n} \to d$ we have for $t \in [0,1]$

$$\gamma_t := \lim \frac{d_{[nt]}}{a_n} = \lim \frac{d_{[nt]}a_{[nt]}}{a_{[nt]}a_n} = d\sqrt{t}.$$

This implies as in (2.9) point process convergence

$$N_n = \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, \frac{X_i}{a_n}\right)} \quad \xrightarrow{\mathcal{D}} \quad N = \sum_r \varepsilon_{\left(\tau_i, Y_i + d\sqrt{\tau_i}\right)} \tag{2.13}$$

to a Poisson process with intensity given by $\frac{d\mu(\cdot\times[x,\infty))}{d\lambda\!\!\!\lambda_{[0,1]}}(t) = (x - d\sqrt{t})^{-2}$. Consider the optimal stopping curve differential equation for N

$$\begin{cases} u'(t) &= -\displaystyle\int_{u(t)}^{\infty} \frac{1}{(x-d\sqrt{t})^2}\,dx = -\frac{1}{(u(t)-d\sqrt{t})} \\ u(1) &= d \end{cases} \tag{2.14}$$

a) Some detailed analysis (see Kühne (1997)) yields that the implicit equation

$$F(t, u(t)) = 0, \quad t \in (0,1]$$

has a unique solution $u(t)$ whose continuity and differentiability follows from an implicit function theorem and u satisfies equation (2.11). Since $u(t) > d\sqrt{t}$ we have $u'(t) < 0$ i.e u is strictly monotonically nonincreasing on $(0,1)$ and $u(t) > d$, $t \in (0,1)$. Assume that $d' = \lim_{t\uparrow 1} u(t) > d$. Then $d_n = u(t_n) \to d' > d$ for any $t_n \uparrow 1$ and $F(t_n, d_n) = 0, \forall n$. Continuity of F implies $F(1, d') = 0$; a contradiction to the fact that $F(1,x) \neq 0$ for $x \in (d,\infty)$. So u is continuous in $t=1$ and it is the unique solution of (2.11). We may extend F continuously to $[0,1]\times[d,\infty)$ by

$$F^1(t,y) = \begin{cases} F(t,y) & t \in (0,1] \\ y^2 - 2e^{2d\frac{\frac{\pi}{2}-\tan^{-1}\frac{d}{\sqrt{8-d^2}}}{\sqrt{8-d^2}}} & t=0. \end{cases}$$

Then $F^1(0,y)=0$ if and only if $y=\sqrt{2}\,e^{d\frac{\frac{\pi}{2}-\tan^{-1}\frac{d}{\sqrt{8-d^2}}}{\sqrt{8-d^2}}}$ and one can argue that u can be extended continuously to 0 by this value.

From the general approximation theorem in KR(98) as described in the introduction we obtain a). For more details of this argument we refer to Kühne (1997).

The proof of b), c) is similar. The involved differential equations are solved numerically using Maple. $\square$

Example 2.5 The following example shows the optimal stopping curve u for values of $u(1)=d$ from -2.8 up to 5.

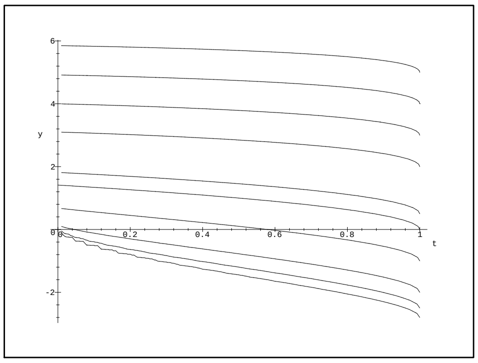

Figure 2:
Optimal stopping curve $u(t)$
for $d=-2.8,\dots,d=5$

Remark 2.6 For an infinite iid sequence (X_i) the optimal stopping problem for $X_1-\delta$, $X_2-2\delta,\dots,$ $\delta>0$ is treated in Chow, Robbins, and Siegmund (1971). The optimal stopping time is given by

$$T_1^\delta = \inf\{i \in \mathbb{N};\ X_i \geq V_1(\delta)\}, \tag{2.15}$$

where $E(X_1-V_1(\delta))_+ = \delta$ and $V_1(\delta)=V(X_1-\delta, X_2-2\delta,\dots)$. The infinite problem is easier than the finite problem. It can be reduced to the stopping of a max sequence and can be solved by a monotone case argument. Kennedy and Kertz (1992) prove point process convergence results: If $F \in D(G)$ is in the domain of an extreme value distribution G, then there are constants $a_1(\delta), b_1(\delta), c_1(\delta)$ such that for $\delta \to 0$

$$\sum_{i=1}^{\infty} \varepsilon_{\left(\frac{i}{c_1(\delta)}, a_1(\delta)(X_i - i\delta - b_1(\delta))\right)} \xrightarrow{\mathcal{D}} N,$$

where N is a Poisson process on $[0,\infty] \times (-\infty,\infty]$. Based on this result the asymptotic distribution of the stopped random variables is approximated for $\delta \to 0$. Similar results are also given in the case of geometrically discounted variables $e^{-i\delta}X_i$. We could use our general approach to prove approximation of the optimal stopping problem to that of the limiting Poisson process in this case too. Instead we will give an extension of these results to the case with geometrically discounted and with linear observation costs of type $b \cdot i$ simultaneously.

Theorem 2.7 *Let (Y_i) be iid integrable with distribution function $F \in D(\Lambda)$ and let $b \geq 0$ and $X_i = X_i^\delta = e^{-i\delta}Y_i - \frac{bi}{a_1(\delta)c_1(\delta)}, i \in \mathbb{N}$. Let T^δ denote the optimal stopping time of $X_1^\delta, X_2^\delta, \ldots$. Then*

$$\lim_{\delta \to 0} a_1(\delta)(EX_{T^\delta} - b_1(\delta)) = -\log(1+b).$$

Proof: We use the idea of the proof of the general approximation theorem in KR (98) which does not apply directly as we consider an unbounded time interval here. With $X_i' := e^{-i\delta}Y_i, i \in \mathbb{N}$, by Theorem 2.2 of Kennedy and Kertz (1992) on $[0,\infty) \times (-\infty,\infty]$ holds for $\delta \to 0$:

$$\sum_{i=1}^{\infty} \varepsilon_{\left(\frac{i}{c_i(\delta)}, a_1(\delta)(X_i' - b_1(\delta)\right)} \xrightarrow{\mathcal{D}} N' = \sum \varepsilon_{(\tau_i', Y_i')} \tag{2.16}$$

where N' is a Poisson process with intensity given by $\frac{d\mu_1}{ds\,dy} = e^{-s-y}$. Since $X_i = X_i' - \frac{b_i}{a_1(\delta)c_1(\delta)}$ we obtain from the continuous mapping theorem

$$\sum_{i=1}^{\infty} \varepsilon_{\left(\frac{i}{c_1(\delta)}, a_1(\delta)(X_i - b_1(\delta))\right)} \xrightarrow{\mathcal{D}} N = \sum \varepsilon_{(\tau_i', Y_i' - b\tau_i')} \tag{2.17}$$

N is a Poisson process with intensity given by $\frac{d\mu_2}{ds\,dy} = e^{-s-y-bs}$. N satisfies the differentiability condition (D) in KR (98) (see the introduction). Next we prove an analogue of the lower bound condition (L) (see the introduction):

$$\liminf_{n \to \infty} u_n(t) > -\infty, \quad t > 0. \tag{2.18}$$

To prove (2.18) choose for $t > 0$ fixed $t' > t$ and consider stopping times $\leq t'$. For the case without discount and observation costs (2.18) holds true by the results of Kennedy and Kertz (1992). Since on $[0, t']$ observation costs and the discounts are bounded below this holds true in general.

The optimal stopping curve of the Poisson process is given by the differential equation (1.2) which becomes

$$u'(t) = -e^{-u(t)}e^{-t-bt}, \quad u(\infty) = -\infty. \tag{2.19}$$

Equation (2.19) has the unique solution

$$u(t) = -t(1+b) - \log(1+b).$$

For any $t > 0$ $(u_n(t))$ is bounded above and by (2.18) there exists a converging subsequence with limit c (see Kennedy and Kertz (1992, p. 262)). As in the proof of the approximation theorem in KR (98), this implies that on $[0, t]$ $u_n(s) \to u(s)$ where the limit u solves (2.18) with $u(t) = c$. So the limit of each converging subsequence of $(u_n(t))$ solves this differential equation which implies $u_n(t) \to u(t)$. For $t = 0$ we obtain $u_n(0) \to u(0) = -\log(1+b)$. □

Remark 2.8 Similar extensions can be given in the cases Φ_α, Ψ_α, but the differential equations allow only numerical solutions and the uniqueness of solutions is difficult to analyse.

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