Optimal consumption strategies under model uncertainty

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Summary: In this paper we consider the problem of finding optimal consumption strategies in an incomplete semimartingale market model under model uncertainty. The quality of a consumption strategy is measured by not only one probability measure but as common in risk theory by a class of scenario measures. We formulate a dual version of the optimization problem and prove the existence of a saddle point and give a characterization of an optimal consumption strategy in terms of solutions of the dual problem. This generalizes results of Karatzas and Zitkovic (2003) for the optimal consumption problem under a fixed probability measure.

1 Introduction

One of the main problems in mathematical finance is how to invest in assets $S = (S^1, \dots, S^d)$ in an optimal way, where optimality is measured with an utility function $U : \mathbb{R} \longrightarrow \mathbb{R}$. Admissible investment or trading strategies are predictable processes $H = (H^1, \dots, H^d)$ for which the stochastic integral $H \cdot S$ is well defined and uniformly bounded below. Then the problem of optimal investment can be formulated as follows: For a given initial capital $x \in \mathbb{R}^+$ find an admissible trading strategy H^* such that the mean utility at maturity $T \in \mathbb{R}^+$ is maximal, that is, H^* maximizes the expression

$$EU(x + H \cdot S_T) \tag{1.1}$$

with respect to all admissible trading strategies H. There are essentially two approaches in the literature for solving problem (1.1). The first one establishes a deterministic partial differential equation and then shows that its solution is an optimal strategy. For this approach one needs the assumption that S is a markov process. A more general approach is the so called martingale method which uses the duality theory of convex analysis. Therefore the starting point is an equivalent description of admissible trading strategies by supermartingale measures. This allows the formulation of a dual problem which can be solved more easily. By a connection with the primal problem (1.1) one gets an optimal trading strategy (for a comprehensive overview see Schachermayer (2004)).

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In general one does not only want to invest but also to consume some of the gained capital. This is described by consumption strategies c which are optional nonnegative processes. The goal is the determination of an optimal consumption strategy for which the mean utility

$$E\left(\int_0^T U(c_t)\,dt\right) \tag{1.2}$$

is maximal with respect to an admissible class of consumption strategies. Under a suitable definition of admissibility the consumption problem (1.2) can be solved with the martingale method similar to the investment problem (1.1) (see Karatzas and Zitkovic (2003)).

In the following we consider a generalization of problem (1.2): Let ρ be a continuous coherent risk measure and let L be a loss function. We are searching for an admissible consumption strategy c^* which minimizes the risk of a loss, that is

$$\rho\left(-\int_0^T L(c_t^*)\,dt\right) = \inf_c \tag{1.3}$$

with respect to all admissible consumption strategies c. Since continuous coherent risk measures can be represented by sets of probability measures $\mathcal{P}$ in the form

$$\rho(X) = \sup_{Q\in\mathcal{P}} E_Q(-X)$$

problem (1.3) is equivalent to the minimization of

$$\sup_{Q\in\mathcal{P}} E_Q\left(\int_0^T L(c_t)\,dt\right) \tag{1.4}$$

with respect to all admissible consumption strategies c. In contrast to problem (1.2) there is not only one underlying probability measure but a whole class of these. Therefore, one speaks of model uncertainty. The correct model is unknown, so one chooses a class of models which, from a risk managers point of view, describes the reality in a sufficient precise way.

The related generalized version of problem (1.1) of maximizing the robust utility function $\inf_{Q\in\mathcal{P}} E_Q(u(X))$ over all admissible claims for incomplete market models has been considered in a recent paper by Gundel (2003) extending the nonrobust version of this problem from Goll and Rüschendorf (2001). A related variational problem was also considered in Schied (2004) where for law invariant risk measures explicit solutions could be given by Neyman-Pearson theory for robust testing problems. In a recent paper the portfolio optimization problem (1.1) has been studied in the robust case by Quenez (2004) extending a result from Cvitanic and Karatzas (1999) and the nonrobust version of this problem by Kramkov and Schachermayer (1999) where the solutions are characterized by duality based methods.

The outline of the paper is as follows: In section 2 we specify our market model and formulate the problem of optimal consumption under model uncertainty. In section 3 we give some conditions under which problem (1.4) has a saddle point. As a result we obtain similar as in the nonrobust version of this problem in Karatzas and Zitkovic (2003) an optimal consumption strategy.

2 The model and the consumption problem

In the following let $S = (S^1, \dots, S^d)$ be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where $T \in \mathbb{R}^+$. S represents a discounted asset price process.

Trading strategies $H = (H^1, \dots, H^d)$ are predictable S-integrable processes, where H^i denotes the amount invested in S^i. Let $\mathcal{X}$ be the set of all stochastic integrals $H \cdot S$ which are bounded from below by some constant. We assume that the set $\mathcal{S}$ of probability measures Q equivalent with respect to P such that each process $X \in \mathcal{X}$ is a supermartingale with respect to Q is not empty. The set $\mathcal{D}$ denotes the weak*-closure of $\mathcal{S}$ in the space of finitely additive measures $\mathrm{ba}(P)$ absolutely continuous with respect to P. Apart from trading strategies let an optional nonnegative endowment process $(e_t)_{t\in[0,T]}$ be given, which describes additional income. We assume that $(e_t)_{t\in[0,T]}$ is essentially bounded. In addition there is the possibility to consume gained capital. This is modeled by optional nonnegative processes c. We call such processes c consumption strategies.

The following definitions and properties are taken from Karatzas and Zitkovic (2003).

Definition 2.1 *A consumption strategy $(c_t)_{t\in[0,T]}$ is called admissible with respect to the initial capital $x \in \mathbb{R}^+$ if there exists a trading strategy H such that*

$$x + \int_0^T H_t \, dS_t + \int_0^T (e_t - c_t) \, dt \geq 0.$$

The set of all admissible trading strategies with respect to the initial capital $x \in \mathbb{R}^+$ is denoted by $\mathcal{A}(x)$.

It is easy to see that $\mathcal{A}(x)$ is convex and closed under convergence in probability with respect to $P \otimes \lambda$.

For a finitely additive measure $Q \in \mathcal{D}$ we define a density process as follows: Let Q^r be the regular part of Q and let $L_t^Q := \frac{d(Q|\mathcal{F}_t)^r}{d(P|\mathcal{F}_t)}$. Then the density process Y^Q of Q with respect to P is defined by

$$Y_t^Q := \liminf_{\substack{q \downarrow t \\ q \in \mathbb{Q}}} L_q^Q.$$

This implies the following characterization of admissible consumption strategies (see Karatzas and Zitkovic (2003)).

Lemma 2.2 *c is an admissible consumption strategy with initial capital x, that is $c \in \mathcal{A}(x)$, if and only if*

$$E\left(\int_0^T c_t Y_t^Q \, dt\right) \leq x + \left\langle Q, \int_0^T e_t \, dt \right\rangle \quad \text{for all } Q \in \mathcal{D}.$$

Our goal is to find optimal admissible consumption strategies, that is, those strategies for which the risk of a loss is minimal. Losses are measured by some strictly decreasing, strictly convex, continuously differentiable loss function $L : \mathbb{R}^+ \longrightarrow \mathbb{R}$ which fulfills the Inada-conditions

$$L'(0) = -\infty \quad \text{and} \quad L'(\infty) = 0.$$

The risk of a loss is determined by some relevant coherent L^1-risk measure $\rho : L^1(P) \to \mathbb{R}$ having the Fatou-property. These risk measures are given by the following axioms: Let $X, Y \in L^1(P)$, $m \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$ then

1) If $X \geq Y$ then $\rho(X) \leq \rho(Y)$ (monotonicity).

2) $\rho(X + m) = \rho(X) - m$ (translation invariance).

3) $\rho(\lambda X) = \lambda\rho(X)$ and $\rho(X + Y) \leq \rho(X) + \rho(Y)$ (coherence).

4) For all $A \in \mathcal{F}$ with $P(A) > 0$ holds $\rho(-\mathbb{1}_A) > 0$ (relevance).

5) Let $(X_n)_{n\in\mathbb{N}}$ be an uniformly bounded sequence in $L^1(P)$ with $X_n \overset{L^1(P)}{\longrightarrow} X$ then

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n) \quad \text{(Fatou-property).}$$

By a modification of Delbaens (2002) representation theorem, by Nakano (2004) there exists a weak*-closed convex set $\mathcal{P}$ of probability measures Q equivalent with respect to P with the properties

$$\sup_{Q\in\mathcal{P}} \left\| \frac{dQ}{dP} \right\|_\infty < \infty$$

and

$$\rho(X) = \sup_{Q\in\mathcal{P}} E_Q(-X) \quad \text{for all } X \in L^1(P). \tag{2.1}$$

Examples for coherent L^1-risk measures are given by

a) $\mathcal{P} = \{ Q \}$, where $Q \sim P$ has a bounded P-density.

b) $\mathcal{P} = \left\{ Q \sim P : \frac{dQ}{dP} \leq \frac{1}{\lambda} \right\}$ for some $\lambda \in (0, 1]$.

The set $\mathcal{P}$ in b) generates the coherent risk measure

$$\text{AVaR}_\lambda(f) := \frac{1}{\lambda}\int_0^\lambda \text{VaR}_\alpha(f)\, d\alpha,$$

where VaR_α denotes the Value at Risk at level α. Under appropriate assumptions (see Föllmer and Schied (2002)) AVaR is equal to the worst conditional expectation WCE_λ which is given by

$$\text{WCE}_\lambda(f) := \sup\{ E(-f \mid A) : A \in \mathcal{F},\ \ P(A) > \lambda \}.$$

AVaR_λ is in some sense the smallest coherent risk measure which dominates VaR_λ. In addition it can be shown that AVaR_λ generates the class of all law invariant risk measures (see Inoue (2003)).

For $x \in \mathbb{R}^+$ let

$$\mathcal{A}_L^1(x) := \left\{ c \in \mathcal{A}(x) : \int_0^T L(c_t)\, dt \in L^1(P) \right\}.$$

We are searching for a solution of the optimization problem

$$\rho\left(-\int_0^T L(c_t)\, dt\right) = \inf_{c \in \mathcal{A}_L^1(x)}. \tag{2.2}$$

By the representation (2.1) of ρ problem (2.2) has the form

$$\inf_{c \in \mathcal{A}_L^1(x)} \rho\left(-\int_0^T L(c_t)\, dt\right) = \inf_{c \in \mathcal{A}_L^1(x)} \sup_{Q \in \mathcal{P}} E_Q\left(\int_0^T L(c_t)\, dt\right).$$

One can formulate the problem equivalently by utility functions. Let $U := -L$. Then we are searching for a solution of

$$\sup_{c \in \mathcal{A}_L^1(x)} \inf_{Q \in \mathcal{P}} E_Q\left(\int_0^T U(c_t)\, dt\right). \tag{2.3}$$

In this case our problem is a generalization of Karatzas and Zitkovic (2003) where $\mathcal{P} = \{P\}$.

In the following we give some conditions under which (2.3) has a saddle point which allows to construct an optimal consumption strategy analogously to Karatzas and Zitkovic (2003).

3 A solution of the consumption problem

The optimal consumption problem in (2.3) is defined by the functional

$$u_*(x) := \sup_{c \in \mathcal{A}_L^1(x)} \inf_{Q \in \mathcal{P}} E_Q\left(\int_0^T U(c_t)\, dt\right), \quad x > 0. \tag{3.1}$$

Let u^* denote the corresponding functional

$$u^*(x) := \inf_{Q \in \mathcal{P}} \sup_{c \in \mathcal{A}_L^1(x)} E_Q\left(\int_0^T U(c_t)\, dt\right), \quad x > 0. \tag{3.2}$$

The dual problem to (3.1) is given by

$$v(y) := \inf_{\widetilde{Q} \in \mathcal{D}} \inf_{Q \in \mathcal{P}} \left\{ E_Q\left(\int_0^T V\left(y \frac{Y_t^{\widetilde{Q}}}{Z_t^Q}\right) dt\right) + y \left\langle \widetilde{Q}, \int_0^T e_t\, dt \right\rangle \right\},$$

$y > 0$, where $Z_t^Q := E\left(\frac{dQ}{dP}\,\middle|\,\mathcal{F}_t\right)$ is the density process of $Q \in \mathcal{P}$ with respect to P and V is the conjugate of U, that is

$$V(y) := \sup_{x\in\mathbb{R}^+} (U(x) - xy).$$

For $Q \in \mathcal{P}$ let u_Q the mean value of the optimal consumption with respect to Q,

$$u_Q(x) := \sup_{c\in\mathcal{A}_L^1(x)} E_Q\left(\int_0^T U(c_t)\,dt\right) \tag{3.3}$$

and let v_Q denote the related dual functional

$$v_Q(y) := \inf_{\widetilde{Q}\in\mathcal{D}} \left\{ E_Q\left(\int_0^T V\left(y\frac{Y_t^{\widetilde{Q}}}{Z_t^Q}\right)\,dt\right) + y\left\langle \widetilde{Q}, \int_0^T e_t\,dt \right\rangle \right\}.$$

We assume the following conditions:

A1) $\int_0^T U(e_t)\,dt$ is P-a.s. bounded below.

A2) The utility function U fulfills the asymptotic elasticity condition, that is

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1.$$

Assumption A1) is fulfilled if $(e_t)_{t\in[0,T]}$ is bounded below by a strictly positive constant. Assumption A2) was introduced by Kramkov and Schachermayer (1999). It implies

Lemma 3.1 *For all $y \in \mathbb{R}^+$ and all $Q \in \mathcal{P}$ the function $y \longmapsto v_Q(y)$ is finite and continuously differentiable.*

The proof is analogous to the proof of Lemma 3.10 in Kramkov and Schachermayer (1999).

In the following we identify probability measures $Q \in \mathcal{P}$ with their P-densities. In the following theorem existence of a saddle point is stated and an optimal consumption strategy is constructed in terms of the dual problem.

Theorem 3.2 *1) Let $x \in \mathbb{R}^+$ and let the assumptions $A1)$ and $A2)$ be satisfied. Then there exists a saddle point $(Z^*, c^*) \in \mathcal{P} \times \mathcal{A}_L^1(x)$ for (2.3). As consequence*

$$u^*(x) = u_*(x) = E\left(Z^* \int_0^T U(c_t^*)\,dt\right) \tag{3.4}$$

and c^* is an optimal consumption strategy.

2) The following duality relations hold:

$$\begin{aligned} u_*(x) &= \inf_{y\in\mathbb{R}^+}(v(y)+xy), \\ v(y) &= \sup_{x\in\mathbb{R}^+}(u_*(x)-xy). \end{aligned} \tag{3.5}$$

3) u_* *is strictly concave and continuously differentiable.*

4) For $x \in \mathbb{R}^+$ *let* $y := u'_*(x)$ *and let* $Q^* \in \mathcal{D}$ *be the solution of the dual problem* $v(y)$. *Then the optimal consumption strategy* c^* *is given by*

$$c_t^* = I\left(y\frac{Y_t^{Q^*}}{Z_t^*}\right), \quad t\in[0,T], \tag{3.6}$$

where $I := (U')^{-1}$ *and* $Z_t^* = E(Z^* \mid \mathcal{F}_t)$.

In the following we assume without loss of generality that $P \in \mathcal{P}$. Otherwise we replace the a priori measure P by some equivalent $Q \in \mathcal{P}$. The proof of the theorem is based on some lemmata.

Lemma 3.3 *For all* $Q \in \mathcal{P}$ *holds* $\lim_{x\to\infty}\frac{u_Q(x)}{x} = 0$.

Proof: Because of assumption A1) there exists a constant $K > 0$ such that $u_Q(x) \geq -K$ for all $x \in \mathbb{R}^+$. By definition of v_Q holds

$$u_Q(x) \leq v_Q(y) + xy \quad \text{for all } y \in \mathbb{R}^+.$$

Hence $\frac{u_Q(x)}{x} \leq \frac{v_Q(y)}{x} + y$. By assumption A2) holds $v_Q(y) < \infty$ for all $y \in \mathbb{R}^+$ and so

$$\limsup_{x\to\infty}\frac{u_Q(x)}{x} \leq y \quad \text{for all } y \in \mathbb{R}^+,$$

that is $\lim_{x\to\infty}\frac{u_Q(x)}{x} = 0$. □

In the next step we show the existence of a saddle point. Therefore we choose a sequence $(c^n)_{n\in\mathbb{N}}$ in $\mathcal{A}_L^1(x)$ such that

$$\inf_{Q\in\mathcal{P}} E_Q\left(\int_0^T U(c_t^n)\,dt\right) \longrightarrow \sup_{c\in\mathcal{A}_L^1(x)}\inf_{Q\in\mathcal{P}} E_Q\left(\int_0^T U(c_t)\,dt\right). \tag{3.7}$$

Since c^n is nonnegative for all $n \in \mathbb{N}$, there exists a subsequence $(\widetilde{c}^n)_{n\in\mathbb{N}}$ of convex combinations and some $c^* \geq 0$ such that $\widetilde{c}^n \longrightarrow c^*$ $P\otimes\lambda$-a.s. (see Delbaen and Schachermayer (1994)). Since $\mathcal{A}(x)$ is convex and closed under convergence in probability, we have $c^* \in \mathcal{A}(x)$.

Lemma 3.4 *The consumption strategy c^* is maximin-optimal, that is $c^* \in \mathcal{A}_L^1(x)$ and*

$$\inf_{Q\in\mathcal{P}} E_Q\left(\int_0^T U(c_t^*)\,dt\right) = \sup_{c\in\mathcal{A}_L^1(x)} \inf_{Q\in\mathcal{P}} E_Q\left(\int_0^T U(c_t)\,dt\right). \tag{3.8}$$

Proof: With the concavity of U follows

$$\begin{aligned}\lim_{n\to\infty}\inf_{Q\in\mathcal{P}} E_Q\left(\int_0^T U(c_t^n)\,dt\right) &\le \lim_{n\to\infty}\inf_{Q\in\mathcal{P}} E_Q\left(\int_0^T U(\widetilde{c}_t^n)\,dt\right)\\ &\le \inf_{Q\in\mathcal{P}}\lim_{n\to\infty} E_Q\left(\int_0^T U(\widetilde{c}_t^n)\,dt\right).\end{aligned}$$

We have to show that

$$E_Q\left(\int_0^T U^+(\widetilde{c}_t^n)\,dt\right) \longrightarrow E_Q\left(\int_0^T U^+(\widetilde{c}_t^*)\,dt\right).$$

If $(U^+(\widetilde{c}^n))_{n\in\mathbb{N}}$ is uniformly integrable with respect to $\mu := Q\otimes\lambda$, then convergence holds. The proof of this uniform integrability is analogous to that in Kramkov and Schachermayer (2003). We assume without loss of generality that $U^+(\infty) > 0$. Assume that $(U^+(\widetilde{c}^n))_{n\in\mathbb{N}}$ is not uniformly μ-integrable. Then there exists a measurable partition $(A_n)_{n\in\mathbb{N}}$ of $\Omega\times[0,T]$ and a constant $K \ge 0$ such that

$$\int U^+(\widetilde{c}^n)\mathbb{1}_{A_n}\,d\mu \ge K \quad \text{for all } n\in\mathbb{N}.$$

Let

$$\begin{aligned} x_0 &:= \inf\{\, x>0 : U(x) \ge 0\,\},\\ d^n &:= x_0 + \sum_{k=1}^n \widetilde{c}^k \mathbb{1}_{A_k}.\end{aligned}$$

By concavity of U^+ holds

$$\int U^+(d^n)\,d\mu \ge \sum_{k=1}^n \int U^+(\widetilde{c}^k)\mathbb{1}_{A_k}\,d\mu \ge nK.$$

Further, for $Q\in\mathcal{D}$ we have

$$\begin{aligned} E\int_0^T Y_t^Q d_t^n\,dt &\le x_0 + \sum_{k=1}^n E\int_0^T Y_t^Q \widetilde{c}_t^k\,dt\\ &\le x_0 + nx + n\left\langle Q, \int_0^T e_t\,dt\right\rangle\\ &\le x_0 + nx + (n-1)M + \left\langle Q, \int_0^T e_t\,dt\right\rangle,\end{aligned}$$

where M is chosen such that $\left\| \int_0^T e_t\,dt \right\|_\infty \le M$. Hence $d_n \in \mathcal{A}_L^1(x_0+nx+(n-1)M)$ and therefore

$$\begin{aligned}\limsup_{y\to\infty} \frac{u_Q(y)}{y} &= \limsup_{n\to\infty} \frac{u_Q(x_0+nx+(n-1)M)}{x_0+nx+(n-1)M} \\ &\ge \limsup_{n\to\infty} \frac{\int U^+(d_n)\,d\mu}{x_0+nx+(n-1)M} \\ &\ge \limsup_{n\to\infty} \frac{nK}{x_0+nx+(n-1)M} \\ &> 0.\end{aligned}$$

But this is a contradiction to $\lim\limits_{y\to\infty} \frac{u_Q(y)}{y} = 0$. So we have shown that $(U^+(\tilde{c}^n))_{n\in\mathbb{N}}$ is uniformly μ-integrable; in particular $\int_0^T U^+(c_t^*)\,dt \in L^1(P)$. The optimality of c^* implies $c^* \ge e$. By assumption A1) holds

$$\left(\int_0^T U(c_t^*)\,dt\right)^- \in L^1(P).$$

Therefore $\int_0^T U(c_t^*)\,dt \in L^1(P)$. □

Lemma 3.5 *There exists some $Z^* \in \mathcal{P}$ such that*

$$\sup_{c\in\mathcal{A}_L^1(x)} E\left(Z^* \int_0^T U(c_t)\,dt\right) = \inf_{Q\in\mathcal{P}} \sup_{c\in\mathcal{A}_L^1(x)} E_Q\left(\int_0^T U(c_t)\,dt\right).$$

Proof: Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}$ such that

$$\lim_{n\to\infty} \sup_{c\in\mathcal{A}_L^1(x)} E\left(Z_n \int_0^T U(c_t)\,dt\right) = \inf_{Q\in\mathcal{P}} \sup_{c\in\mathcal{A}_L^1(x)} E_Q\left(\int_0^T U(c_t)\,dt\right).$$

Since $\mathcal{P}$ is sequentially weak*-compact, there exists a subsequence of $(Z_n)_{n\in\mathbb{N}}$ denoted by (Z_m) and some $Z^* \in \mathcal{P}$ such that $\lim\limits_{m\to\infty} E(Z_m f) = E(Z^* f)$ for all $f \in L^1(P)$. Therefore

$$\begin{aligned}\sup_{c\in\mathcal{A}_L^1(x)} E\left(Z^* \int_0^T U(c_t)\,dt\right) &\le \lim_{m\to\infty} \sup_{c\in\mathcal{A}_L^1(x)} E\left(Z_m \int_0^T U(c_t)\,dt\right) \\ &= \inf_{Q\in\mathcal{P}} \sup_{c\in\mathcal{A}_L^1(x)} E_Q\left(\int_0^T U(c_t)\,dt\right).\end{aligned}$$

Hence $\sup\limits_{c\in\mathcal{A}_L^1(x)} E\left(Z^* \int_0^T U(c_t)\,dt\right) = \inf\limits_{Q\in\mathcal{P}} \sup\limits_{c\in\mathcal{A}_L^1(x)} E_Q\left(\int_0^T U(c_t)\,dt\right)$. □

For showing that (Z^*, c^*) is a saddle point, we use the Minimax-theorem in the following form: Let A, B be nonempty sets and let $f : A \times B \longrightarrow \overline{\mathbb{R}}$ be a mapping. The triple $\Gamma = (A, B, f)$ is called two-person zero-sum game. Γ is of concave-convex type if

1) For $b_1, b_2 \in B$ and $\alpha \in [0,1]$ exists some $b \in B$, such that for all $a \in A$ holds:

$$f(a,b) \le (1-\alpha) f(a,b_1) + \alpha f(a,b_2).$$

2) For $a_1, a_2 \in A$ and $\alpha \in [0,1]$ exists some $a \in A$, such that for all $b \in B$ holds:

$$f(a,b) \ge (1-\alpha) f(a_1,b) + \alpha f(a_2,b).$$

Minimax-Theorem: *Let $\Gamma = (A, B, f)$ be a concave-convex two-person zero-sum game and let $f < \infty$. If there exists a topology τ on A with the properties*

1) A is τ-compact,

2) For all $b \in B$ the mapping $f(\,\cdot\,, b) : A \longrightarrow \mathbb{R}$ is upper-semicontinuous, that is, for $a_0 \in A$ holds

$$\limsup_{a \to a_0} f(a,b) \le f(a_0, b),$$

then

$$\inf_{b \in B} \sup_{a \in A} f(a,b) = \sup_{a \in A} \inf_{b \in B} f(a,b).$$

The Minimax-theorem implies

Lemma 3.6 $(Z^*, c^*) \in \mathcal{P} \times \mathcal{A}_L^1(x)$ *is a saddle point and*

$$u^*(x) = u_*(x) = E\left(Z^* \int_0^T U(c_t^*)\, dt \right).$$

Proof: Apply the Minimax-theorem to $A := \mathcal{P}$ and $B := \mathcal{A}_L^1(x)$. Then

$$u^*(x) = u_*(x) = E\left(Z^* \int_0^T U(c_t^*)\, dt \right).$$

Since Z^* and c^* are by construction minimax strategies, it follows that (Z^*, c^*) is a saddle point. □

Similar to Karatzas and Zitkovic (2003) holds for $Q \in \mathcal{P}$ that

$$\begin{aligned} u_Q(x) &= \sup_{c \in \mathcal{A}_L^1(x)} E_Q\Big(\int_0^T u(c_t)\, dt \Big), \\ v_Q(y) &= \inf_{\widetilde{Q} \in \mathcal{D}} \Big\{ E_Q\Big(\int_0^T V\Big(y \frac{Y_t^{\widetilde{Q}}}{Z_t^Q} \Big)\, dt \Big) + y \Big\langle \widetilde{Q}, \int_0^T e_t\, dt \Big\rangle \Big\} \end{aligned}$$

are conjugate functions. For showing that this holds true also for u_* and v, we use again the Minimax-theorem.

Let $w_Q(x) := u_Q(x) - xy$. Then the mapping $Q \longmapsto w_Q(x)$ is convex and upper-semicontinuous above on $\mathbb{R}^+$, that is, for every sequence $(Q_n)_{n\in\mathbb{N}}$ in $\mathcal{P}$ which converges in the weak*-topology to some $Q \in \mathcal{P}$ holds

$$w_Q(x) \leq \liminf_{n\to\infty} w_{Q_n}(x).$$

Further, the mapping $x \longmapsto w_Q(x)$ is concave. An application of the Minimax-theorem to $A := \mathcal{P}$, $B := \mathbb{R}^+$ and $f(Q,x) := -w_Q(x)$ results in

$$\begin{aligned} v(y) &= \inf_{Q\in\mathcal{P}} v_Q(y) = \inf_{Q\in\mathcal{P}} \sup_{x\in\mathbb{R}^+} w_Q(x) \\ &= \sup_{x\in\mathbb{R}^+} \inf_{Q\in\mathcal{P}} w_Q(x) \\ &= \sup_{x\in\mathbb{R}^+} (\inf_{Q\in\mathcal{P}} u_Q(x) - xy) \\ &= \sup_{x\in\mathbb{R}^+} (u^*(x) - xy) \\ &= \sup_{x\in\mathbb{R}^+} (u_*(x) - xy). \end{aligned}$$

By standard arguments from convex analysis (see Rockafellar (1970)) follows

$$u_*(x) = \inf_{y\in\mathbb{R}^+} (v(y) + xy).$$

In the last step we show the representation of the optimal consumption strategy. Let $(Z^*, c^*) \in \mathcal{P} \times \mathcal{A}^1_L(x)$ be a saddle point. Then the dual problem $v(y)$ is given by

$$v(y) = \inf_{\widetilde{Q}\in\mathcal{D}} \left\{ E\left(Z^* \int_0^T V\Big(y \frac{Y_t^{\widetilde{Q}}}{Z_t^*}\Big)\, dt \right) + y \Big\langle\, \widetilde{Q}, \int_0^T e_t\, dt \,\Big\rangle \right\}. \tag{3.9}$$

This can be shown similar to Karatzas and Zitkovic (2003) by an application of the Minimax-theorem. By the weak*-closedness of $\mathcal{D}$ there exists a minimal element $Q^* \in \mathcal{D}$. This implies the differentiability of u_* and v. The mapping v' is given by

$$v'(y) = -E\left(Z^* \int_0^T I\Big(y \frac{Y_t^{Q^*}}{Z_t^*}\Big) \frac{Y_t^{Q^*}}{Z_t^*}\, dt \right) + \Big\langle\, Q^*, \int_0^T e_t\, dt \,\Big\rangle. \tag{3.10}$$

Let y be given by the equation $u_*'(x) = y$. Since $V(y) = U(I(y)) + yI(y)$, we get

$$\begin{aligned}
u_*(x) &= E\Big(Z^* \int_0^T U(c_t^*)\,dt\Big)\\
&\le E\Big(Z^* \int_0^T V\Big(y\frac{Y_t^{Q^*}}{Z_t^*}\Big)\,dt\Big) + E\Big(Z^* \int_0^T y\frac{Y_t^{Q^*}}{Z_t^*} I\Big(y\frac{Y_t^{Q^*}}{Z_t^*}\Big)\,dt\Big)\\
&= E\Big(Z^* \int_0^T V\Big(y\frac{Y_t^{Q^*}}{Z_t^*}\Big)\,dt\Big) + y\Big(- v'(y) + \Big\langle\, Q^*, \int_0^T e_t\,dt \Big\rangle\Big)\\
&= v(y) + xy\\
&= u_*(x).
\end{aligned}$$

Because of

$$E\Big(\int_0^T (Y_t^Q - Y_t^{Q^*}) I\Big(y\frac{Y_t^{Q^*}}{Z_t^*}\Big)\,dt\Big) \le \Big\langle\, Q - Q^*, \int_0^T e_t\,dt \Big\rangle$$

for all $Q \in \mathcal{D}$ and all $y \in \mathbb{R}^+$ (see Karatzas and Zitkovic (2003)), we have $I\Big(y\frac{Y^{Q^*}}{Z^*}\Big) \in \mathcal{A}_L^1(x)$. This implies that the optimal consumption process c^* has the representation

$$c_t^* = I\left(y\frac{Y_t^{Q^*}}{Z_t^*}\right) \quad \text{with } I = (U')^{-1},$$

where y is given by the equation $u_*'(x) = y$.

Remark 3.7 The arguments in this paper are also valid for the more general problem

$$\sup_{c\in\mathcal{A}_L^1(x)} \inf_{Q\in\mathcal{P}} E_Q\left(\int_0^T U(c_t)\,\mu(dt)\right)$$

with some weighting measure $\mu(dt)$ which allows to consider the problem of maximizing the robust expected utility from consumption and terminal wealth

$$\sup \inf_{Q\in\mathcal{P}} E_Q\left(\int_0^T U_1(c_t)\,dt + U_2(x + H\cdot S_T)\right)$$

where the $\sup$ is over all admissible investment and consumption strategies (c, H) (see Karatzas and Zitkovic (2003), Ex. 3.15). In particular for the robust optimization of terminal wealth the optimal wealth is of the form

$$x + H^*\cdot S_T = I\left(y\,\frac{Y_T^{Q^*}}{Z^*}\right)$$

with $Q^* \in \mathcal{D}$, $Z^* \in \mathcal{P}$ and $y = u_*'(x)$ as in Theorem 3.2.

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