Convex ordering criteria for Lévy processes

Jan Bergenthum, Ludger Rüschendorf[1]

Department of Mathematical Stochastics, University of Freiburg, Eckerstr. 1, 79104 Freiburg, Germany; E-Mail: ruschen@stochastik.uni-freiburg.de

Summary Modelling financial and insurance time series with Lévy processes or with exponential Lévy processes is a relevant actual practice and an active area of research. It allows qualitatively and quantitatively good adaptation to the empirical statistical properties of asset returns. Due to model incompleteness it is a problem of considerable interest to determine the dependence of option prices in these models on the choice of pricing measures and to establish nontrivial price bounds. In this paper we review and extend ordering results of stochastic and convex type for this class of models. We also extend the ordering results to processes with independent increments (PII) and present several examples and applications as to α-stable processes, NIG-processes, GH-distributions, and others. Criteria are given for the Lévy measures which imply corresponding comparison results for European type options in (exponential) Lévy models.

Key words: convex ordering, Lévy measure, Lévy process

1 Introduction

Since the proposition of α-stable Lévy processes by Mandelbrot (1960) in the early sixties as models for cotton prices it took until the nineties that these models were investigated thoroughly as adequate models that are able to reproduce the stylized properties of asset prices. This class of models allows to deal with the complete spectrum of pricing, of statistical analysis and risk measurement. An impressive presentation of stable modelling in finance and econometrics and corresponding empirical and statistical analysis is given in the comprehensive volume of Rachev and Mittnik (2000). Later on beginning in the mid nineties further classes of Lévy processes and exponential Lévy processes were proposed as adequate models for financial and econometric data allowing a great flexibility concerning tail behaviour, jumps, diffusion, and unsymmetry. A comprehensive presentation of the importance of these models for financial time series and the relevant development of option pricing, hedging, and statistical analysis

[1] communicating author; E-Mail: ruschen@stochastik.uni-freiburg.de; Tel.: +49-(0)761-203-5665; Fax: +49-(0)761-203-5661

is given in the treatise of Cont and Tankov (2004). There one also finds an outlook to modelling beyond Lévy processes as to processes with independent increments (PII) or to stochastic volatility models with jumps.

A continuous time stochastic process $X = (X_t)_{t\geq 0}$ is called Lévy process if it has stationary, independent increments, stochastically continuous paths and $X_0 = 0$. The distribution of X_1 is infinitely divisible and the charactersitic function of X_t is given by $\Phi_t(s) = Ee^{isX_t} = e^{t\Psi(s)}$, where the characterstic exponent Ψ has the Lévy–Khintchine representation

$$\Psi(s) = ibs - \frac{\sigma^2}{2}s + \int \left(e^{isy} - 1 - isy1_{\{|y|\leq 1\}}\right)F(dy) \tag{1}$$

for $s \in \mathbb{R}$. Here b is a drift parameter, $\sigma^2 \geq 0$ is a diffusion parameter and F is the Lévy measure. Thus the distribution of the process is uniquely characterized by the triplet of local characteristics (b, σ^2, F). A similar representation holds in the multivariate case. The truncation function $h(y) = y1_{\{|y|\leq 1\}}$ can be replaced by other versions of the truncation function. Only the drift term $b = b(h)$ depends on this choice.

Lévy type models typically are incomplete and thus it is a problem of interest to investigate the dependence of option prices or insurance risks on the choice of the pricing measure in the class of martingale measures which describe consistent prices and do not allow arbitrage opportunities. A particular problem in this context is to find relevant upper and lower bounds for the pricing interval $\{E_Q f(S_T); Q \in \mathcal{M}(P)\}$, where the option $Y = f(S_T)$ is a function of the price process at time T and Q runs through the class of P-continuous martingale measures. For the class of European style options as puts or calls these questions lead naturally to the consideration of the increasing and the (increasing or decreasing) convex ordering of the price processes as these options often are (increasing or decreasing) convex functions of terminal values of the underlying processes. It is also of interest to consider path dependent options which are (increasing or decreasing) convex functions of the path of the underlyings.

Since for Lévy processes typically (marginal) distributions or densities are not available in explicit form it is of interest to determine ordering conditions on the Lévy measures respectively the local characteristics of the processes which describe the drift part, the diffusion part and the jump part. The triplet $(b, \sigma^2, F)_h$ of local characteristics typically is known and it characterizes by the Lévy–Khintchine formula the distribution of the Lévy process. Here F denotes the Lévy measure describing the jumps, b, $b = b(h)$, σ^2 the drift and diffusion parameter, the truncation function h. An extensive statistical theory has been developed to estimate these spectral parameters from financial data (see Rachev and Mittnik (2000), Cont and Tankov (2004)).

Exponential Lévy models are given by

$$S_t = S_0 \exp(X_t), \tag{2}$$

where (X_t) is a Lévy process, i.e. a process with stationary independent increments. For financial modeling exponential Lévy models are typically better fitting the price processes than Lévy processes, even if the original models of Bachelier (Brownian motion) and Mandelbaum (α-stable processes) were Lévy models. The reason is that usually the relative price changes $(S_{t+\delta} - S_t)/S_t$ are empirically found to be approximatively independent identically distributed (i.i.d.). This is by approximation essentially equivalent to i.i.d. log returns and thus one gets as continuous time models exponential Lévy processes $S_t = S_0 \exp(X_t)$. For the statistical analysis and the analytical properties it is however common and largely equivalent to analyze the Lévy process $X_t = \log(S_t/S_0)$. This is also the point of view in this paper. The analysis of X allows in many respects to induce corresponding properties of S. Also it is pointed out throughout this paper that Lévy processes themselves are important models in various applications as in the insurance risk processes or in queuing systems which justify to investigate their ordering propertis also with respect to general physical (not necessarily martingale) measures. A more general class of processes are the PII-processes with independent increments or the corresponding exponential PIIs, which are still analytically tractable but allow an even better adaptation to empirical data.

In our paper we derive ordering criteria for Lévy type models X. In particular we consider stochastic and (increasing) convex ordering. The results imply also ordering criteria for the corresponding exponential Lévy models as the exponential function itself is convex and increasing. In fact by so-called 'duality arguments' also ordering of further classes of options can be reduced to these cases. For example the calculation of the price of a put option $E(K - S_T)_+$ in exponential Lévy model can be equivalently formulated as price of a call option $E(S'_T - K')_+$ of the 'dual' process, which is again an exponential Lévy process. Thus ordering of decreasing convex type for Lévy processes can be reduced to ordering of increasing convex type for a related dual Lévy process.

Some interesting convex comparison results for exponential stochastic models have been developed in recent papers in financial mathematics (see El Karoui, Jeanblanc-Picqué, and Shreve (1998), Hobson (1998), Bellamy and Jeanblanc (2000), Gushchin and Mordecki (2002), Henderson and Hobson (2003), Bergenthum and Rüschendorf (2006)). The main aim in these papers is to derive sharp upper or lower bounds for option prices in incomplete market models. The methods used in these papers are based on stochastic calculus (Itô formula) and the propagation of convexity property (see El Karoui et al. (1998), Bellamy and Jeanblanc (2000), Gushchin and Mordecki (2002), Bergenthum and Rüschendorf (2006)) as well as on the coupling method (see Henderson and Hobson (2003), Hobson (1998)). Extensions to the comparison of multivariate semimartingales and Lévy processes are given in Bergenthum and Rüschendorf (2007) (abbreviated as BR (2007) in the following) on which this paper is based.

The basic results in the above mentioned papers state that ordering properties of the Lévy triplets imply under some regularity conditions ordering of the finite dimensional distributions of the Lévy processes. In this paper we give a short review of these results and in particular we establish in a systematic way several cut criteria for one dimensional Lévy measures which allow to verify the ordering criteria for the Lévy triplets which are postulated in the above mentioned papers. As a consequence this allows to establish various explicit ordering results for the stochastic ordering and the (increasing) convex ordering.

The main aim of this paper is to give easy to verify ordering criteria which allow us to apply these results to concrete models and further to extend these ordering results to the class of PII processes because of their particular relevance for modelling. We consider orderings w.r.t. one of the following order generating function classes $\mathcal{F}$

$$\begin{aligned}
\mathcal{F}_{\mathrm{st}} &:= \{f : \mathbb{R}^d \to \mathbb{R}, f \text{ is increasing}\},\\
\mathcal{F}_{\mathrm{cx}} &:= \{f : \mathbb{R}^d \to \mathbb{R}, f \text{ is convex}\},\\
\mathcal{F}_{\mathrm{dcx}} &:= \{f : \mathbb{R}^d \to \mathbb{R}, f \text{ is directionally convex}\},\\
\mathcal{F}_{\mathrm{sm}} &:= \{f : \mathbb{R}^d \to \mathbb{R}, f \text{ is supermodular}\},\\
\mathcal{F}_{\mathrm{icx}} &:= \mathcal{F}_{\mathrm{cx}} \cap \mathcal{F}_{\mathrm{st}}, \quad \mathcal{F}_{\mathrm{idcx}} := \mathcal{F}_{\mathrm{dcx}} \cap \mathcal{F}_{\mathrm{st}}, \quad \mathcal{F}_{\mathrm{ism}} := \mathcal{F}_{\mathrm{sm}} \cap \mathcal{F}_{\mathrm{st}},
\end{aligned} \tag{3}$$

where for d-dimensional random vectors X, Y we define

$$X \leq_{\mathcal{F}} Y \text{ if } Ef(X) \leq Ef(Y) \tag{4}$$

for all $f \in \mathcal{F}$ such that $f(X), f(Y)$ are integrable. While the class of increasing convex functions is well motivated considering convex claims, the directionally convex order and the supermodular order are particulary well suited to describe risk arising from positive dependence (for definition and properties see Müller and Stoyan (2002)) Note that for the classes of ordering functions in (3) it is sufficient to establish the inequality in (4) for some suitable *generating* subclass of $\mathcal{F}$, as for example for smooth functions etc.

In this paper we consider orderings of infinite divisible distributions and of the finite-dimensional distributions of the corresponding Lévy processes. Thereto, we introduce the class of functions $\mathcal{F}^{(m)} := \{f := (\mathbb{R}^d)^m \to \mathbb{R} : f(s_1, \dots, s_{i-1}, \cdot, s_{i+1}, \dots, s_m) \in \mathcal{F}, s_i \in \mathbb{R}^d, i \leq m\}$, $m, d \in \mathbb{N}$, that are componentwise in $\mathcal{F}$. A d-dimensional process $S^{(1)}$ is said to have smaller finite-dimensional distributions with respect to the product ordering induced by $\mathcal{F}$ than a d-dimensional process $S^{(2)}$, if for every $m \in \mathbb{N}$ and all $0 \leq t_1 < \cdots < t_m \leq T$ it holds true that

$$Eg\big(S_{t_1}^{(1)}, \dots, S_{t_m}^{(1)}\big) \leq Eg\big(S_{t_1}^{(2)}, \dots, S_{t_m}^{(2)}\big), \tag{5}$$

for all $g \in \mathcal{F}^{(m)}$. We denote this ordering by

$$\big(S_t^{(1)}\big) \leq_{\mathcal{F}} \big(S_t^{(2)}\big). \tag{6}$$

For time-homogeneous Markov processes the existence of a $\leq_{\mathcal{F}}$-monotone transition kernel that separates the transition kernels of $S^{(i)}$ is sufficient to establish ordering of the finite-dimensional distributions. Here a kernel Q is $\leq_{\mathcal{F}}$-monotone if $f \in \mathcal{F}$ implies that $Qf \in \mathcal{F}$. A useful tool is the following separation result (see BR (2007)).

Lemma 1 (Separation lemma)**.** *Two time-homogeneous Markov processes $(S_t^{(1)})_{t\in[t_1,T]}$ and $(S_t^{(2)})_{t\in[t_1,T]}$ with transition kernels $Q_t^{(1)}$ and $Q_t^{(2)}$ satisfy*

$$\big(S_t^{(1)}\big) \leq_{\mathcal{F}} \big(S_t^{(2)}\big),$$

if $S_{t_1}^{(1)} \leq_{\mathcal{F}} S_{t_1}^{(2)}$ and if a family (Q_t) of $\leq_{\mathcal{F}}$-monotone transition kernels exists such that

$$Q_t^{(1)}(x, \cdot) \leq_{\mathcal{F}} Q_t(x, \cdot) \leq_{\mathcal{F}} Q_t^{(2)}(x, \cdot), \quad \text{for all } x \text{ and all } t > 0.$$

(The ordering between probability measures is defined as in (4)*.)*

In Section 2 we develop in detail several cut criteria for the Lévy measures of compound Poisson processes. These criteria allow to verify the ordering conditions for Lévy measures as postulated in the general comparison results in BR (2007). As consequence we obtain stochastic and (increasing) convex ordering results for the corresponding compound Poisson processes. In Section 3 we extend these ordering results to Lévy processes with infinite Lévy measures. For the proof we use approximation by processes with finite Lévy measures obtained by truncation of the infinite Lévy measures of the processes that we want to compare. These results need a careful consideration of the consequences of truncating the Lévy measures on the corresponding cutting criteria. Several of the more technical proofs in Sections 2 and 3 are defered to the appendix.

Section 4 is concerned with extensions of the ordering results to PII processes. This class of models has been suggested in several recent papers on financial modelling. (Exponential) Lévy models do not allow for time inhomogeneity. As a consequence Lévy models yield strong scaling properties for marginal distributions of returns. They allow to calibrate to implied volatility patterns for single maturity but fail to reproduce option prices correctly over a range of different maturities. The PII models in particular allow to take into account inhomogeneities described by deterministic local time behaviour. At the same time much of the analytical tractability of Lévy processes is preserved for the PII processes. In the final part of our paper we give several applications to comparison results for α-stable processes, NIG processes and GH distributions. In this part and throughout the first part of the paper we also include some examples and review and apply some general results of our previous paper BR (2007) .

2 Ordering results for compound Poisson processes

The first section of this paper is concerned with ordering results for compound Poisson processes. This is the most simple class of Lévy processes with finitely many jumps in finite time corresponding to a finite Lévy measure. On the other hand it is a rich class of models allowing to approximate general Lévy processes. In particular in insurance compound Poisson processes are common models for the insurance risk process, describing the accumulated premiums minus the claims of some insurance contract. The corresponding traded price process is of the form

$$X_t = X_0 + \sum_{j=1}^{N_t} Y_j - \kappa t = X_0 + U_t - \kappa t,$$

where N is a homogeneous Poisson process, (Y_i) are iid claim size variables with U_t the corresponding compound Poisson process and κ is the premium rate. This leads to an incomplete market model consisting of the savings account and the price process X. The relevance of financial (no-arbitrage and martingale) pricing concepts in insurance and conversely the importance of insurance premium principles in financial pricing has been detailed in several papers (see Delbaen and Haezendonck (1989), Møller (2002), Embrechts (2000)). Contingent claims, equivalently reinsurance contracts cannot be priced uniquely by no arbitrage theory. Different martingale measures lead to different prices. There are basically two main principles how to choose a martingale measure. One is to minimize some distance to the underlying physical market measure, the other one is based on utility principles. Both types or principles are essentially equivalent (see Goll and Rüschendorf (2001)). Thus it is of interest for dynamic reinsurance markets to establish comparison results for different martingale measures. Møller (2004) has given in this context several comparison results between some specific martingale measures, as the minimal martingale measure, the minimal entropy martingale measure, and others. In this section we extend some of Møller's results and establish general comparison criteria for compound Poisson models.

For the ordering of two compound Poisson processes we begin with some normalization. Our ordering criterion for compound Poisson processes with Lévy measures $F^{(k)}$ of different finite total mass, is given in terms of the *modified Lévy measures* $\tilde{F}^{(k)}$ which for $k \in \{1,2\}$ such that $\|F^{(k)}\| \le \|F^{(3-k)}\|$ are defined as

$$\tilde{F}^{(k)}(dx) = F^{(k)}(dx) + \big(\|F^{(3-k)}\| - \|F^{(k)}\|\big)^+ \delta_{\{0\}}(dx), \tag{7}$$

where $a^+ = \max\{0,a\}$ is the positive part of $a \in \mathbb{R}$ and $\delta_{\{0\}}$ denotes the Dirac measure in the origin. Thus we add some point mass in zero to the Lévy measure with the smaller mass such that the modified measures $\tilde{F}^{(k)}$ have equal total mass. By the Lévy–Khintchine formula $S^{(i)}$ has Lévy triplet $S^{(i)} \sim (b^{(i)}(0), 0, \tilde{F}^{(i)})_0$, $i = 1,2$. Here $(b, \sigma^2, F)_h$, $b = b(h)$ denotes generally the triplet of local characteristics with respect to the truncation function h, $(b, \sigma^2, F)_0$ the triplet for the case of no truncation, i.e. $h \equiv 0$. Even though the standard convention that the jump kernel has no point mass in the origin is left aside, the modified Lévy measures $\tilde{F}^{(i)}$ uniquely characterize the distributions of $S^{(i)}$ modulo this point mass. The results in this section are based on the following comparison result for multivariate compound Poisson processes in BR (2007, Lemma 3.2).

Proposition 2 (Ordering of compound Poisson processes)**.** *Let* $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$*,* $i = 1,2$*, be d-dimensional compound Poisson processes and assume that the Lévy measures* $F^{(i)}$ *satisfy* $\int_{\{|x|>1\}} |x| F^{(i)}(dx) < \infty$*. Let the modified Lévy measures* $\tilde{F}^{(i)}$ *be given by (7).*

If $\tilde{F}^{(1)} \le_{\mathcal{F}} \tilde{F}^{(2)}$ *holds true for*

1. $\mathcal{F} \in \{\mathcal{F}_{\mathrm{st}}, \mathcal{F}_{\mathrm{icx}}, \mathcal{F}_{\mathrm{idcx}}, \mathcal{F}_{\mathrm{ism}}\}$ *and, additionally,* $b^{(1)}(0) \le b^{(2)}(0)$*,*

or

2. $\mathcal{F} \in \{\mathcal{F}_{\mathrm{cx}}, \mathcal{F}_{\mathrm{dcx}}, \mathcal{F}_{\mathrm{sm}}\}$ *and, additionally,* $b^{(1)}(0) = b^{(2)}(0)$*,*

then $(S_t^{(1)}) \le_{\mathcal{F}} (S_t^{(2)})$*.*

The conditions in Proposition 2 on the ordering of the drift and Lévy measures are natural. The proof of Proposition 2 uses a natural coupling representation and some wellknown closure properties of the involved orders under mixtures and convolutions.

In the following we consider one-dimensional compound Poisson processes $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$. In order to establish $\big(S_t^{(1)}\big) \le_{\mathcal{F}} \big(S_t^{(2)}\big)$ for $\mathcal{F} \in \{\mathcal{F}_{\mathrm{st}}, \mathcal{F}_{\mathrm{icx}}, \mathcal{F}_{\mathrm{cx}}\}$ we derive sufficient conditions for the drifts $b^{(i)}$ and Lévy measures $F^{(i)}$ in order to establish the comparison criteria of Proposition 2 for the modified Lévy measures $\tilde{F}^{(i)}$. This needs a careful discussion of the consequences of adding some mass to the origin as in (7). First we establish three versions of a cut criterion that is parallel to the classical cut criterion for probability distribution functions due to Karlin and Novikoff (1963). If a Lévy measure $F^{(2)}$ has more mass in the tails than a Lévy measure $F^{(1)}$, and less mass near the center, then it is bigger with respect to the (increasing) convex order. In our case of Lévy measures with different finite total mass the comparison depends on the locus where the ordering of the Lévy measures changes. This is due to the fact that the corresponding modified Lévy measure $\tilde{F}^{(k)}$ may have some point mass in the origin.

In the first case where the two order changes of the Lévy measures $F^{(i)}$ take place on the negative half axis, we obtain the following result.

Proposition 3 (Cut criterion for compound Poisson processes, $\mathbf{k}_\ell < \mathbf{k_r} \leq \mathbf{0}$)**.** *Let $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$, $i = 1, 2$, be one-dimensional compound Poisson processes and assume that $\int_{\{|x|>1\}} |x| F^{(i)}(dx) < \infty$. For $k_\ell < k_r \leq 0$ assume that*

$$F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)), \tag{8}$$
$$F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_\ell, k_r)), \tag{9}$$
$$F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_r, \infty)). \tag{10}$$

In the case $\|F^{(1)}\| < \|F^{(2)}\|$ additionally assume that

$$F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-), \quad \textit{if } F^{(1)}((-\infty, k_r]) > F^{(2)}((-\infty, k_r]).$$

1. *If $b^{(1)}(0) \leq b^{(2)}(0)$ and $\int xF^{(1)}(dx) \leq \int xF^{(2)}(dx)$, then $(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)})$.*

2. *If $b^{(1)}(0) = b^{(2)}(0)$ and $\int xF^{(1)}(dx) = \int xF^{(2)}(dx)$, then $(S_t^{(1)}) \leq_{\mathrm{cx}} (S_t^{(2)})$.*

For the proof see the Appendix.

Remark 4. We did not make any assumptions on the ordering of $F^{(i)}$ in the points of order changes k_ℓ, k_r. Note also that the proof does not make use of the full strength of conditions (8)–(10). In fact weaker conditions can be given in terms of the increase of the distribution function of the Lévy measures $F^{(i)}$. But the formulation of the conditions in (8)–(10) is more intuitive and simpler.

If the pointwise ordering of the Lévy measures changes once on the negative and once on the positive half axis, we obtain a similar result. The proof is similar to the proof of Proposition 3. (For details see Bergenthum (2005, Theorem 2.1.4).)

Proposition 5 (Cut criterion for compound Poisson processes, $\mathbf{k}_\ell \leq \mathbf{0} \leq \mathbf{k_r}$)**.** *Let $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$, $i = 1, 2$, be one-dimensional compound Poisson processes and assume that $\int_{\{|x|>1\}} |x| F^{(i)}(dx) < \infty$. For $k_\ell \leq 0 \leq k_r$ assume that*

$$F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)),$$
$$F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_\ell, k_r)),$$
$$F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_r, \infty)).$$

In the case $\|F^{(1)}\| > \|F^{(2)}\|$ *additionally assume that*

$$F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-) + \|F^{(1)}\| - \|F^{(2)}\|,$$

if there is a $\kappa \in [k_\ell, 0)$ *s.th.* $F^{(1)}((-\infty, k]) \leq F^{(2)}((-\infty, k])$*, for all* $k < \kappa$ *and* $F^{(1)}((-\infty, k]) \geq F^{(2)}((-\infty, k])$*, for all* $k \in [\kappa, 0)$*.*

1. *If* $b^{(1)}(0) \leq b^{(2)}(0)$ *and* $\int xF^{(1)}(dx) \leq \int xF^{(2)}(dx)$*, then* $(S_t^{(1)}) \leq_{\text{icx}} (S_t^{(2)})$*.*
2. *If* $b^{(1)}(0) = b^{(2)}(0)$ *and* $\int xF^{(1)}(dx) = \int xF^{(2)}(dx)$*, then* $(S_t^{(1)}) \leq_{\text{cx}} (S_t^{(2)})$*.*

A third version of this assertion is given in the case when the dominance changes of the Lévy measures take place on the positive half axis. The proof is parallel to the proof of Proposition 3 (cf. Bergenthum (2005, Theorem 2.1.5)).

Proposition 6 (Cut criterion for compound Poisson processes, $\mathbf{0 \leq k_\ell < k_r}$)**.** *Let* $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$*,* $i = 1, 2$*, be one-dimensional compound Poisson processes and assume that* $\int_{\{|x|>1\}} |x| F^{(i)}(dx) < \infty$*. For* $0 \leq k_\ell < k_r$ *assume that*

$$\begin{aligned} F^{(1)}(A) &\leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)), \\ F^{(1)}(A) &\geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_\ell, k_r)), \\ F^{(1)}(A) &\leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_r, \infty)). \end{aligned}$$

In the case $\|F^{(1)}\| < \|F^{(2)}\|$ *additionally assume that*

$$F^{(2)}(\mathbb{R}_-) \geq F^{(1)}(\mathbb{R}_-) + \|F^{(2)}\| - \|F^{(1)}\|,$$

if $F^{(1)}((-\infty, k_\ell]) + \|F^{(2)}\| - \|F^{(1)}\| < F^{(2)}((-\infty, k_\ell])$*.*

1. *If* $b^{(1)}(0) \leq b^{(2)}(0)$ *and* $\int xF^{(1)}(dx) \leq \int xF^{(2)}(dx)$*, then* $(S_t^{(1)}) \leq_{\text{icx}} (S_t^{(2)})$*.*
2. *If* $b^{(1)}(0) = b^{(2)}(0)$ *and* $\int xF^{(1)}(dx) = \int xF^{(2)}(dx)$*, then* $(S_t^{(1)}) \leq_{\text{cx}} (S_t^{(2)})$*.*

Remark 7. 1. The second part of Proposition 6 is a generalization of Theorem 6.1 in Møller (2004), which is the main comparison tool for compound Poisson processes of that paper. The condition $b^{(1)}(0) = b^{(2)}(0)$ is implicitly assumed there, as Møller considers $S^{(i)}$ under martingale measures and assumes $\int xF^{(1)}(dx) = \int xF^{(2)}(dx) =: d$, hence $b^{(i)}(0) = -\int xF^{(i)}(dx) = d$, $i = 1, 2$. In addition to condition $\int xF^{(1)}(dx) = \int xF^{(2)}(dx)$, Møller assumes that $\frac{1}{\|F^{(1)}\|} \int xF^{(1)}(dx) \leq \frac{1}{\|F^{(2)}\|} \int xF^{(2)}(dx)$, hence $\|F^{(1)}\| \geq \|F^{(2)}\|$, to obtain ordering of the terminal values $S_T^{(1)} \leq_{\text{cx}} S_T^{(2)}$.

2. Similar to the results in Møller (2004) as mentioned in the introduction of this section our comparison results imply ordering criteria for several well-established martingale measures like the minimal martingale measure and the minimal entropy martingale measure in incomplete compound Poisson models (see Bergenthum (2005)). These measures correspond to the so-called market price of jump risk w.r.t. the physical measure. This risk parameter is identical to the density of the Lévy measure under the martingale measure w.r.t. the underlying physical measure and is obtained by the Girsanov theorem.

The following domination criterion is a corollary of Proposition 5 with $k_\ell = k_r = 0$. If a finite Lévy measure $F^{(2)}$ dominates a Lévy measure $F^{(1)}$ in every jump height in the sense that $F^{(1)}(A) \leq F^{(2)}(A)$ for all $A \in \mathcal{B}$ then (increasing) convex ordering of the corresponding compound Poisson processes is implied. We denote this ordering of $F^{(1)}, F^{(2)}$ by $F^{(1)} \leq F^{(2)}$.

Corollary 8 (Domination criterion for compound Poisson processes)**.** *Let $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$, $i = 1, 2$, be one-dimensional compound Poisson processes and assume that the Lévy measures $F^{(i)}$ satisfy $\int_{\{|x|>1\}} |x| F^{(i)}(dx) < \infty$ and $F^{(1)} \leq F^{(2)}$.*

1. *If $b^{(1)}(0) \leq b^{(2)}(0)$ and $\int x F^{(1)}(dx) \leq \int x F^{(2)}(dx)$, then $(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)})$.*
2. *If $b^{(1)}(0) = b^{(2)}(0)$ and $\int x F^{(1)}(dx) = \int x F^{(2)}(dx)$, then $(S_t^{(1)}) \leq_{\mathrm{cx}} (S_t^{(2)})$.*

Next, we establish three versions of a domination criterion for finite Lévy measures $F^{(i)}$, $i = 1, 2$, that imply stochastic ordering of the corresponding compound Poisson processes $S^{(i)}$. If $F^{(1)}$ has more mass on small values than $F^{(2)}$ and less mass on big values, then under a suitable drift condition stochastic ordering $(S_t^{(1)}) \leq_{\mathrm{st}} (S_t^{(2)})$ is implied, i.e. the ordering with respect to the class $\mathcal{F}_{\mathrm{st}}$ of increasing functions. Again, the statement of this comparison result depends on the locus of the dominance change. If the dominance change takes place on the negative half axis, the result is as follows.

Proposition 9 (Criterion for stochastic ordering of compound Poisson processes, $\mathbf{k \leq 0}$)**.** *Let $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$, $i = 1, 2$, be one-dimensional compound Poisson processes. Assume that $\int_{\{|x|>1\}} |x| F^{(i)}(dx) < \infty$ and that for $k \leq 0$ it holds true that*

$$\begin{aligned} F^{(1)}(A) &\geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k)), \\ F^{(1)}(A) &\leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k, \infty)), \end{aligned}$$

and $b^{(1)}(0) \leq b^{(2)}(0)$. In the case $\|F^{(1)}\| < \|F^{(2)}\|$ additionally assume that $F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-)$. Then

$$(S_t^{(1)}) \leq_{\mathrm{st}} (S_t^{(2)}).$$

For the proof see the Appendix.

If the dominance change of the Lévy measures takes place on the positive half axis, we obtain the following result. The proof is similar to the proof of the previous theorem (see Bergenthum (2005, Theorem 2.1.9)).

Proposition 10 (Criterion for stochastic ordering of compound Poisson processes, $\mathbf{k \geq 0}$)**.** *Let $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$, $i = 1, 2$, be one-dimensional compound Poisson processes. Assume that $\int_{\{|x|>1\}} |x| F^{(i)}(dx) < \infty$ and that for $k \geq 0$ it holds true that*

$$\begin{aligned} F^{(1)}(A) &\geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k)), \\ F^{(1)}(A) &\leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k, \infty)), \end{aligned}$$

and $b^{(1)}(0) \leq b^{(2)}(0)$. In the case $\|F^{(2)}\| < \|F^{(1)}\|$ additionally assume that $F^{(1)}(\mathbb{R}_+) \leq F^{(2)}(\mathbb{R}_+)$. Then

$$(S_t^{(1)}) \leq_{\mathrm{st}} (S_t^{(2)}).$$

Proposition 9 and 10 imply stochastic ordering without additional conditions, if the dominance change takes place in $k = 0$.

Corollary 11 (Criterion for stochastic ordering of compound Poisson processes, $\mathbf{k} = \mathbf{0}$**).** *Let* $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$, $i = 1, 2$, *be one-dimensional compound Poisson processes and let* $\int_{\{|x|>1\}} |x| F^{(i)}(dx) < \infty$. *If* $b^{(1)}(0) \le b^{(2)}(0)$ *and*

$$\begin{aligned} F^{(1)}(A) &\ge F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, 0)), \\ F^{(1)}(A) &\le F^{(2)}(A), \quad \forall A \in \mathcal{B}((0, \infty)), \end{aligned}$$

then

$$(S_t^{(1)}) \le_{\mathrm{st}} (S_t^{(2)}).$$

3 Lévy processes with infinite Lévy measures

More general price models containing a jump part and a diffusion part have already been considered in Merton (1976). In Merton's model jumps occur at the event times of a Posson process and the jump sizes are normally distributed. Options were priced in these incomplete models by assuming that the investors attitude to jump risk is risk neutral or that the jump risk is unpriced and only the diffusion risk is taken into account as in the classical Black–Scholes model. Later on these models were generalized to jump diffusion models which are defined as solutions of the stochastic differential equation

$$\frac{dS_t}{S_{t^-}} = b_t dt + \sigma_t dW_t + \int_{[0,1]} \phi_t(y) \tilde{v}(dt, dy), \tag{11}$$

where W_t is a Brownian motion, $\tilde{v}(dt, dy) = v(dt, dy) - q(dt, dy)$ is a compensated Poisson random measure on $[0, T] \times [0, 1]$ with intensity measure $q(dt, dy) = \lambda_t dt dy$.

There are many equivalent martingale measures for this model. Each of these measures corresponds to a pair of choices for the market price of diffusion risk and the market price of jump risk. Interesting questions then are to establish whether option prices are monotone increasing in the market price of jump risk or if they are monotone in the volatility. See El Karoui et al. (1998), Bellamy and Jeanblanc (2000), and Henderson and Hobson (2003) for results in this direction.

The same question of price comparison also arises for Lévy process models with infinite Lévy measures as for stable models (see Rachev and Mittnik (2000)), variance Gamma models (see Madan and Seneta (1990)), the hyperbolic model (see Eberlein and Keller (1995), the normal inverse Gaussian model (see Barndorff-Nielsen (1998)), or the CGMY-model (see Carr et al. (2003)). In this section we derive comparison results for Lévy processes with infinite Lévy measures.

We extend the cut and domination criteria for compound Poisson processes to Lévy measures with infinite total mass. We obtain two variants of the assertions depending on the regularity of the paths of the processes. For processes with paths of bounded variation we can state comparison results under relaxed conditions on the Lévy mesures.

The proofs of the comparison results make use of the following steps. We truncate the Lévy measures $F^{(i)}$ around the origin by sequences $\underline{\varepsilon}_n^{(i)} \uparrow 0, \overline{\varepsilon}_n^{(i)} \downarrow 0$ and obtain *truncated* Lévy measures

$$F_n^{(i)}(dx) := \mathbb{1}_{(\underline{\varepsilon}_n^{(i)},\overline{\varepsilon}_n^{(i)})^c}(x)F^{(i)}(dx), \quad \underline{\varepsilon}_n^{(i)} \uparrow 0, \overline{\varepsilon}_n^{(i)} \downarrow 0, \tag{12}$$

that have finite total mass $\|F_n^{(i)}\|$. If the total mass of the truncated Lévy measures is unequal, $\|F_n^{(1)}\| \neq \|F_n^{(2)}\|$, then we add some mass at zero in order to equalize the total mass. We define the *modified truncated* Lévy measures by

$$\tilde{F}_n^{(k)}(dx) := F_n^{(k)}(dx) + (\|F_n^{(3-k)}\| - \|F_n^{(k)}\|)^+\delta_{\{0\}}(dx). \tag{13}$$

Then we establish ordering of the corresponding compound Poisson processes $S_n^{(i)} \sim (ES_1^{(i)}, 0, \tilde{F}_n^{(i)})_{\mathrm{id}}$ by the cut and domination criteria of the previous section. Establishing functional weak convergence $S_n^{(i)} \xrightarrow{\mathcal{L}} S^{(i)}$ yields finite-dimensional ordering of the limit processes, if the ordering $\leq_{\mathcal{F}}$ is stable under weak convergence (property (W)). Property (W) is satisfied by the orders $\leq_{\mathrm{st}}, \leq_{\mathrm{ism}}, \leq_{\mathrm{sm}}$, whereas the orders generated by $\mathcal{F}_{\mathrm{cx}}, \mathcal{F}_{\mathrm{icx}}, \mathcal{F}_{\mathrm{dcx}}, \mathcal{F}_{\mathrm{idcx}}$ are not stable with respect to weak convergence. If for these orders additionally convergence of the expectations $ES_{n,t}^{(i)} \to ES_t^{(i)}$ holds true, then the ordering $(S_{n,t}^{(1)}) \leq_{\mathcal{F}} (S_{n,t}^{(2)})$ is propagated to the limit processes (cp. Müller and Stoyan (2002, Theorems 3.4.6 and 3.12.8)). A suitable functional weak convergence result is the following lemma which is a consequence of Jacod and Shiryaev (2003, Corollary VII.3.6). For the proof see the Appendix.

Lemma 12 (Functional weak convergence)**.** *Let $S \sim (b(h), 0, F)_h$ be a d-dimensional Lévy process whose Lévy measure F has infinite total mass and for $\underline{\varepsilon}_n \uparrow 0$, $\overline{\varepsilon}_n \downarrow 0$ let F_n be the corresponding truncated Lévy measure given in (12). If $b_n(h) \to b(h)$ then for the compound Poisson processes $S_n \sim (b_n(h), 0, F_n)_h$ functional weak convergence*

$$S_n \xrightarrow{\mathcal{L}} S$$

holds true.

For a Lévy process S with Lévy measure F the existence of the first moments ES_t is equivalent to the statement $\int_{\{|x|>1\}} |x|F(dx) < \infty$, which we will assume for all of the next assertions. In this case S has the representation $S \sim (ES_1, 0, F)_{\mathrm{id}}$. The fact that a Lévy process S has paths of finite variation, is characterized in terms of the Lévy measure by $\int_{\{|x|<1\}} |x|F(dx) < \infty$. In this case, S has the representation $S \sim (b(0), 0, F)_0$, and we implicitly assume that S has paths of finite variation, if we use this representation. In the sequel we make use of the following corollary of Lemma 12. For the proof see the Appendix.

Corollary 13 (Functional weak convergence)**.** *Let F be a Lévy measure with infinite total mass and for sequences $\underline{\varepsilon}_n \uparrow 0$, $\overline{\varepsilon}_n \downarrow 0$ let F_n be the corresponding truncated Lévy measure as in (12).*

1. *If $\int_{\{|x|>1\}} |x|^2F(dx) < \infty$, then for $S \sim (ES_1, 0, F)_{\mathrm{id}}$ and $S_n \sim (ES_1, 0, F_n)_{\mathrm{id}}$ it holds true that $S_n \xrightarrow{\mathcal{L}} S$.*

2. *If for some continuous truncation function h it holds true that $\int |h(x)|F(dx) < \infty$, then for $S \sim (b(0), 0, F)_0$ and $S_n \sim (b(0), 0, F_n)_0$ it follows that $b_n(h) \to b(h)$ and thus $S_n \xrightarrow{\mathcal{L}} S$.*

Our results make essential use of the following theorem, from BR (2007, Theorems 3.3 and 3.6), which gives a general result for orderings of Lévy processes with infinite Lévy measures in terms of their modified truncated Lévy measures, of the additional drift and of moment conditions on the Lèvy measures.

Theorem 14 (Ordering of Lévy processes with infinite Lévy measures)**.** *Let $S^{(i)} \sim (ES_1^{(i)}, 0, F^{(i)})_{\mathrm{id}}$, $i = 1, 2$, be d-dimensional Lévy processes with Lévy measures $F^{(i)}$ that have infinite total mass. Let $\underline{\varepsilon}_n^{(i)} \uparrow 0$, $\overline{\varepsilon}_n^{(i)} \downarrow 0$ be sequences such that for the modified truncated Lévy measures $\tilde{F}_n^{(i)}$ it holds true that*

$$\tilde{F}_n^{(1)} \leq_{\mathcal{F}} \tilde{F}_n^{(2)}.$$

Additionally assume in the case

1. *$\mathcal{F} \in \{\mathcal{F}_{\mathrm{st}}, \mathcal{F}_{\mathrm{icx}}, \mathcal{F}_{\mathrm{idcx}}, \mathcal{F}_{\mathrm{ism}}\}$ that $0 \leq \int xF_n^{(2)}(dx) - \int xF_n^{(1)}(dx) \leq ES_1^{(2)} - ES_1^{(1)}$,*

and

2. *$\mathcal{F} \in \{\mathcal{F}_{\mathrm{cx}}, \mathcal{F}_{\mathrm{dcx}}, \mathcal{F}_{\mathrm{sm}}\}$ that $ES_1^{(1)} = ES_1^{(2)}$ and $\int xF_n^{(1)}(dx) = \int xF_n^{(2)}(dx)$.*

Then it follows that

$$\big(S_t^{(1)}\big) \leq_{\mathcal{F}} \big(S_t^{(2)}\big).$$

Remark 15 (Finite variation)**.** For Lévy processes with infinite Lévy measure that have paths of finite variation a variant of the previous theorem is given in BR (2007, Theorem 3.7). In this case, the drift components $b^{(i)}(0)$ w.r.t. zero truncation exist and the drift and the Lévy moment condition for the monotone convex type orders are given by $b^{(1)}(0) \leq b^{(2)}(0)$ and $\int xF_n^{(2)}(dx) - \int xF_n^{(1)}(dx) \geq 0$, respectively. Accordingly no upper bound for the difference of the Lévy moments is required. For the convex type orders the drift and Lévy moment condition are $b^{(1)}(0) = b^{(2)}(0)$ and $\int xF_n^{(1)}(dx) = \int xF_n^{(2)}(dx)$, respectively, which in this case are equivalent to the drift and Lévy moment condition of Theorem 14.

Based on Proposition 5 and an approximation argument we establish the following cut criterion for Lévy processes with infinite Lévy measures.

Proposition 16 (Cut criterion for Lévy processes)**.** *Let $S^{(i)} \sim (ES_1^{(i)},\, 0, F^{(i)})_{\mathrm{id}}$, $i = 1, 2$, be one-dimensional Lévy processes and assume that $\|F^{(1)}\| = \infty$. For $k_\ell < 0 < k_r$ assume that*

$$\begin{aligned}
F^{(1)}(A) &\leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)),\\
F^{(1)}(A) &\geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_\ell, 0)),\\
F^{(1)}(A) &\geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((0, k_r)),\\
F^{(1)}(A) &\leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_r, \infty)),
\end{aligned} \tag{14}$$

Assume that $ES_1^{(1)} \leq ES_1^{(2)}$ and that there are sequences $\underline{\varepsilon}_n^{(i)} \uparrow 0$, $\overline{\varepsilon}_n^{(i)} \downarrow 0$, such that *(a)* $\underline{\varepsilon}_n^{(2)} \leq \underline{\varepsilon}_n^{(1)} < 0 < \overline{\varepsilon}_n^{(1)} < \overline{\varepsilon}_n^{(2)}$,

(b) $0 \leq \int x F_n^{(2)}(dx) - \int x F_n^{(1)}(dx) \leq ES_1^{(2)} - ES_1^{(1)}$, *and further*

(c) in case that $\|F_n^{(1)}\| > \|F_n^{(2)}\|$ *and if* $F_n^{(1)}((-\infty,k]) \leq F_n^{(2)}((-\infty,k])$, *for all* $k < \kappa$ *for some* $\kappa \in [k_\ell, 0)$, *and* $F_n^{(1)}((-\infty,k]) \geq F_n^{(2)}((-\infty,k])$, *for all* $k \in [\kappa, 0)$, *additionally assume that* $F_n^{(1)}(\mathbb{R}_-) \geq F_n^{(2)}(\mathbb{R}_-) + \|F_n^{(1)}\| - \|F_n^{(2)}\|$.

Then $(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)})$.

For the proof see the Appendix.

Remark 17. 1. As in Proposition 16 we do not require $F^{(2)}$ to have infinite total mass. The assertion also includes the case where $S^{(1)}$ is a Lévy process with infinite Lévy measure and $S^{(2)}$ has paths of finite variation or is a compound Poisson process.

2. The previous Proposition does not include the case where $S^{(1)}$ has paths of finite variation and at the same time $S^{(2)}$ has paths of infinite variation. In this case, condition (14) is violated as $F^{(2)} > F^{(1)}$ around the origin.

In the case where both Lévy processes $S^{(i)}$ have paths of finite variation, we obtain a variant of the cut criterion with a relaxed Lévy moment condition. We omit the proof, which is similar to the proof of Proposition 16.

Proposition 18 (Cut criterion for Lévy processes with paths of finite variation)**.** *Let* $S^{(i)} \sim (b^{(i)}(0), 0, F^{(i)})_0$, $i = 1, 2$, *be one-dimensional Lévy processes and assume that* $\|F^{(1)}\| = \infty$. *For* $k_\ell < 0 < k_r$ *assume that*

$$\begin{aligned}
F^{(1)}(A) &\leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)),\\
F^{(1)}(A) &\geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_\ell, 0)),\\
F^{(1)}(A) &\geq F^{(2)}(A), \quad \forall A \in \mathcal{B}((0, k_r)),\\
F^{(1)}(A) &\leq F^{(2)}(A), \quad \forall A \in \mathcal{B}((k_r, \infty)),
\end{aligned}$$

Assume that $b^{(1)}(0) \leq b^{(2)}(0)$ *and that there are sequences* $\underline{\varepsilon}_n^{(i)} \uparrow 0$, $\overline{\varepsilon}_n^{(i)} \downarrow 0$, *such that*

(a) $\underline{\varepsilon}_n^{(2)} \leq \underline{\varepsilon}_n^{(1)} < 0 < \overline{\varepsilon}_n^{(1)} < \overline{\varepsilon}_n^{(2)}$,

(b) $\int x F_n^{(1)}(dx) \leq \int x F_n^{(2)}(dx)$, *and*

(c) if $\|F_n^{(1)}\| > \|F_n^{(2)}\|$ *and if there is a* $\kappa \in [k_\ell, 0)$ *such that* $F_n^{(1)}((-\infty,k]) \leq F_n^{(2)}((-\infty,k])$, *for all* $k < \kappa$ *and* $F_n^{(1)}((-\infty,k]) \geq F_n^{(2)}((-\infty,k])$, *for all* $k \in [\kappa, 0)$, *then additionally assume that* $F_n^{(1)}(\mathbb{R}_-) \geq F_n^{(2)}(\mathbb{R}_-) + \|F_n^{(1)}\| - \|F_n^{(2)}\|$.

Then $(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)})$.

The following majorization criterion is an extension of Corollary 8 for compound Poisson processes to Lévy processes with infinite Lévy measures $F^{(i)}$. In this case, majorization $F^{(1)} \leq F^{(2)}$ is defined by $F^{(1)}(A) \leq F^{(2)}(A)$ for all $A \in \mathcal{B}((-\infty, 0))$ and for all $A \in \mathcal{B}((0, \infty))$. Thus the Lévy measure $F^{(2)}$ dominates the Lévy measure $F^{(1)}$ in every jump height. Under some conditions on the moments, the cutting sequences, and the masses, we obtain that $(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)})$.

Proposition 19 (Majorization criterion for Lévy processes)**.** *Let* $S^{(i)} \sim (ES_1^{(i)}, 0, F^{(i)})_{\mathrm{id}}$, $i = 1, 2$, *be one-dimensional Lévy processes. Assume that the Lévy measure* $F^{(2)}$ *has infinite total mass and that* $F^{(2)}$ *majorizes* $F^{(1)}$, $F^{(1)} \leq F^{(2)}$. *Assume that* $ES_1^{(1)} \leq ES_1^{(2)}$ *and that there are sequences* $\underline{\varepsilon}_n^{(i)} \uparrow 0$, $\overline{\varepsilon}_n^{(i)} \downarrow 0$, $i = 1, 2$, *such that*

(a) $0 \leq \int x F_n^{(2)}(dx) - \int x F_n^{(1)}(dx) \leq ES_1^{(2)} - ES_1^{(1)}$.

(b) In the case $\underline{\varepsilon}_n^{(2)} < \underline{\varepsilon}_n^{(1)} < 0 < \overline{\varepsilon}_n^{(1)} < \overline{\varepsilon}_n^{(2)}$ *and* $\|F_n^{(1)}\| > \|F_n^{(2)}\|$ *further assume that* $F_n^{(1)}(\mathbb{R}_-) \geq F_n^{(2)}(\mathbb{R}_-) + \|F_n^{(1)}\| - \|F_n^{(2)}\|$ *if* $F_n^{(1)}((-\infty, k]) \leq F_n^{(2)}((-\infty, k])$, $\forall k < \kappa_n$ *for some* $\kappa_n \in [\underline{\varepsilon}_n^{(2)}, 0)$ *and* $F_n^{(1)}((-\infty, k]) \geq F_n^{(2)}((-\infty, k])$, $\forall k \in [\kappa_n, 0)$.

Then $(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)})$.

For the proof see the Appendix.

Remark 20. Also for this assertion a variant for Lévy processes $S^{(i)}$, $i = 1, 2$, that have paths of finite variation holds true. This is similar to the relationship between Propositions 16 and 18 (see Bergenthum (2005)).

Proposition 19 implies a domination criterion for Lévy processes that have paths of infinite variation and whose Lévy measures are absolutely continuous w.r.t. the Lebesgue measure (see also BR (2007, Corollary 3.4)). Here the conditions reduce to some easy check conditions on the densities of $F^{(i)}$.

Corollary 21 (Domination criterion for Lévy processes with paths of infinite variation)**.** *Let* $S^{(i)} \sim (ES_1^{(i)}, 0, F^{(i)})_{\mathrm{id}}$, $i = 1, 2$, *be one-dimensional Lévy processes and assume that* $\int_A |x| F^{(i)}(dx) = \infty$ *for* $A = (-1, 0)$ *and for* $A = (0, 1)$. *Let* $F^{(i)}$ *be absolutely continuous with densities* $f^{(i)}$. *If* $ES_1^{(1)} \leq ES_1^{(2)}$ *and*

$$0 < f^{(1)}(x) \leq f^{(2)}(x), \quad \forall x \in \mathbb{R}, \tag{15}$$

then $(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)})$.

Remark 22. The integrability condition $\int_A |x| F^{(i)}(dx) = \infty$ for $A = (-1, 0)$ and $A = (0, 1)$ implies that the paths of the corresponding Lévy processes have infinite variation.

The following result is an extension of the stochastic ordering result in Corollary 11 to Lévy processes with infinite activity.

Proposition 23 (Criterion for stochastic ordering of Lévy processes with infinite Lévy measures)**.** *Let* $S^{(i)} \sim (ES_1^{(i)}, 0, F^{(i)})_{\mathrm{id}}$, $i = 1, 2$, *be one-dimensional Lévy processes. Assume that the Lévy measures* $F^{(i)}$ *have infinite total mass and are ordered as*

$$\begin{aligned} &F^{(1)}(A) \geq F^{(2)}(A), \quad \forall A \in \mathcal{B}\big((-\infty, 0)\big), \text{ and} \\ &F^{(1)}(A) \leq F^{(2)}(A), \quad \forall A \in \mathcal{B}\big((0, \infty)\big). \end{aligned}$$

Assume that $ES_1^{(1)} \leq ES_1^{(2)}$ *and that there are sequences* $\underline{\varepsilon}_n^{(i)} \uparrow 0$, $\overline{\varepsilon}_n^{(i)} \downarrow 0$, $i = 1, 2$, *s.th.*

(a) $0 \leq \int x F_n^{(2)}(dx) - \int x F_n^{(1)}(dx) \leq ES_1^{(2)} - ES_1^{(1)}$.

(b) Not both of the conditions $\underline{\varepsilon}_n^{(1)} < \underline{\varepsilon}_n^{(2)}$ *and* $\overline{\varepsilon}_n^{(1)} < \overline{\varepsilon}_n^{(2)}$ *hold simultaneously.*

(c) In case $\underline{\varepsilon}_n^{(1)} < \underline{\varepsilon}_n^{(2)} < 0 < \overline{\varepsilon}_n^{(2)} \leq \overline{\varepsilon}_n^{(1)}$ *and* $\|F_n^{(1)}\| < \|F_n^{(2)}\|$ *it holds true that* $F_n^{(1)}(\mathbb{R}_-) \geq F_n^{(2)}(\mathbb{R}_-)$,

(d) In case $\underline{\varepsilon}_n^{(2)} \leq \underline{\varepsilon}_n^{(1)} < 0 < \overline{\varepsilon}_n^{(1)} < \overline{\varepsilon}_n^{(2)}$ *and* $\|F_n^{(1)}\| > \|F_n^{(2)}\|$ *it holds true that* $F_n^{(1)}(\mathbb{R}_+) \leq F_n^{(2)}(\mathbb{R}_+)$.

Then $(S_t^{(1)}) \leq_{\mathrm{st}} (S_t^{(2)})$.

For the proof see the Appendix.

Remark 24. 1. If $\int_{(-1,0)} |x| F^{(i)}(dx) = \infty$ and $\int_{(0,1)} x F^{(i)}(dx) = \infty$ and $F^{(i)}$ are absolutely continuous, then one can construct sequences $\underline{\varepsilon}_n^{(i)} \uparrow 0$, $\overline{\varepsilon}_n^{(i)} \downarrow 0$ such that condition (a) of the previous theorem is satisfied. At first, we fix $\underline{\varepsilon}_n^{(1)}$ and $\overline{\varepsilon}_n^{(2)}$. If

$$\int_{\mathbb{R}\setminus(\underline{\varepsilon}_n^{(1)},\overline{\varepsilon}_n^{(2)})} x F^{(2)}(dx) - \int_{\mathbb{R}\setminus(\underline{\varepsilon}_n^{(1)},\overline{\varepsilon}_n^{(2)})} x F^{(1)}(dx) \leq ES_1^{(2)} - ES_1^{(1)}, \tag{16}$$

then condition (a) is satisfied with $\underline{\varepsilon}_n^{(2)} := \underline{\varepsilon}_n^{(1)}$ and $\overline{\varepsilon}_n^{(1)} := \overline{\varepsilon}_n^{(2)}$. If (16) is not satisfied, we choose $\underline{\varepsilon}_n^{(2)} > \underline{\varepsilon}_n^{(1)}$ and $\overline{\varepsilon}_n^{(1)} := \overline{\varepsilon}_n^{(2)}$ or $\overline{\varepsilon}_n^{(1)} < \overline{\varepsilon}_n^{(2)}$ and $\underline{\varepsilon}_n^{(2)} := \underline{\varepsilon}_n^{(1)}$ (but not simultaneously, cp. condition (b)) such that condition (a) holds. This is possible, due to the assumptions on $F^{(i)}$. Additionally, we have to check if conditions (c) and (d), respectively, are satisfied. In the next step we fix $\underline{\varepsilon}_{n+1}^{(1)} > \underline{\varepsilon}_n^{(2)}$ and $\overline{\varepsilon}_{n+1}^{(2)} < \overline{\varepsilon}_n^{(1)}$ and restart the algorithm.

2. In BR (2007, Corollary 3.8) a variant for Lévy processes with infinite Lévy measures and paths of finite variation is given:
If $F^{(i)}$ have densities $f^{(i)}$ monotonically increasing to infinity as x tends to zero, if $b^{(1)}(0) \leq b^{(2)}(0)$ and $f^{(1)}(x) \geq f^{(2)}(x)$ for all $x \in \mathbb{R}_-$ and $f^{(1)}(x) \leq f^{(2)}(x)$ for all $x \in \mathbb{R}_+$, then $\left(S_t^{(1)}\right) \leq_{st} \left(S_t^{(2)}\right)$.

Hitherto we have considered the comparison of pure jump Lévy processes. Next we incorporate the Gaussian part. Firstly, we state a comparison result for continuous Lévy processes. This is a corollary of well-known ordering results for normally distributed random variables (cp. Müller and Stoyan (2002, Section 3.13) and Corollary 2.11 in BR (2007)).

Proposition 25 (Ordering of Gaussian Lévy processes)**.** *Let* $S^{(i)} \sim (ES_1^{(i)}, c^{(i)}, 0)_{\mathrm{id}}$, $i = 1, 2$, *be* d*-dimensional continuous Lévy processes.*

1. *If* $ES_1^{(1)} \leq ES_1^{(2)}$ *and* $c^{(1)} = c^{(2)}$, *then* $(S_t^{(1)}) \leq_{\mathrm{st}} (S_t^{(2)})$.

2. *If* $ES_1^{(1)} \leq ES_1^{(2)}$ *and* $c^{(1)} \leq_{\mathrm{psd}} c^{(2)}$, *then* $(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)})$.

3. *If* $ES_1^{(1)} \leq ES_1^{(2)}$ *and* ${c^{(1)}}^{ij} \leq {c^{(2)}}^{ij}$, $\forall i, j \leq d$, *then* $(S_t^{(1)}) \leq_{\mathrm{idcx}} (S_t^{(2)})$.

4. If $ES_1^{(1)} \leq ES_1^{(2)}$ *and* ${c^{(1)}}^{ij} \leq {c^{(2)}}^{ij}$, $\forall i,j \leq d$, $i \neq j$, *then* $(S_t^{(1)}) \leq_{\mathrm{ism}} (S_t^{(2)})$.

The corresponding convex type orderings hold true if $ES_1^{(1)} = ES_1^{(2)}$.

As the Gaussian and the jump part of a Lévy process are independent, we obtain ordering of Lévy processes that incorporate both, the Gaussian part and of the jump part.

Proposition 26 (Ordering of Lévy processes)**.** *Let* $\mathcal{F}$ *be one of the order generating function classes in* (3)*. Let the jump processes* $S_{\mathrm{J}}^{(i)} \sim (ES^{(i)}, 0, F^{(i)})_{\mathrm{id}}$ *and the continuous process* $S_{\mathrm{C}}^{(i)} \sim (0, c^{(i)}, 0)_{\mathrm{id}}$ *be independent. If*

$$(S_{\mathrm{J},t}^{(1)}) \leq_{\mathcal{F}} (S_{\mathrm{J},t}^{(2)}) \quad \textit{and} \quad (S_{\mathrm{C},t}^{(1)}) \leq_{\mathcal{F}} (S_{\mathrm{C},t}^{(2)}),$$

then $S^{(i)} := S_{\mathrm{C}}^{(i)} + S_{\mathrm{J}}^{(i)} \sim (ES^{(i)}, c^{(i)}, F^{(i)})_{\mathrm{id}}$ *and* $(S_t^{(1)}) \leq_{\mathcal{F}} (S_t^{(2)})$.

Proof. As the orders generated by $\mathcal{F}$ that are considered in (3) satisfy the convolution property and due to the fact that $S^{(i)} \stackrel{d}{=} S_{\mathrm{J}}^{(i)} + S_{\mathrm{C}}^{(i)}$, the result follows from the convolution closedness property (C) (see Müller and Stoyan (2002)). □

Remark 27. Propositions 25 and 26 imply also comparison results for exponential Lévy processes for $\mathcal{F} \in \{\mathcal{F}_{\mathrm{st}}, \mathcal{F}_{\mathrm{icx}}, \mathcal{F}_{\mathrm{idcx}}, \mathcal{F}_{\mathrm{ism}}\}$. This also holds true for the finite-dimensional orderings w.r.t. $\mathcal{F}^{(m)}$. For further implications of these ordering results via 'duality' arguments see the introduction.

4 Extension to PII

While Lévy processes resp. exponential Lévy models are analytically nice and allow to calibrate to implied volatility patterns for single maturity they fail to reproduce option prices and the time pattern over a range of different maturities. The problem is that (exponential) Lévy models do not allow for time inhomogeneity. The class of processes with independent increments (PII) is still analytically tractable and takes into account the time inhomogeneities. PII processes have a Lévy–Khintchine representation of the form $E\exp(iuX_t) = \exp\Psi_t(u)$ with

$$\Psi_t(u) = iu \cdot b(t) - \frac{1}{2}u^\top C(t)u + \int \big(e^{iux} - 1 - iuh(x)\big)K(t, dx) \tag{17}$$

with local characteristics $(b(t), \mathrm{C}(t), K(t, dx))$.

The local characteristics of PII processes are time dependent and deterministic. This extension allows a good adaptation to inhomogeneities and gives similar freedom of modelling as in local volatility models. Typical examples of PII-processes are time inhomogeneous jump diffusions

$$X_t = \int_0^t \sigma(s)dW_s + \sum_{i=1}^{N_{\Lambda(t)}} Y_i \tag{18}$$

where W is a Brownian motion, σ is a deterministic volatility, $N_{\Lambda(t)}$ is a Poisson process with intensity Λ and (Y_i) are iid jump variables. But also Lévy volatility models with deterministic volatility $\sigma(s)$

$$X_t = \int_0^t \sigma(s) dL_s \tag{19}$$

and time changed Lévy processes

$$X_t = L_{\nu(t)} \tag{20}$$

are interesting examples. For general results on PII processes we refer to Sato (1999) and Jacod and Shiryaev (2003). Modelling aspects, calibration to prices and option pricing is presented in a nice way in Cont and Tankov (2004).

For the ordering results for PII processes we begin with the case of finite Lévy kernels $K(s, dy)$ and drift and diffusion characteristics equal to zero. Then further on similarly to the case of Lévy processes we consider infinite Lévy kernels by approximation and include also diffusion and drift characteristics.

The nonstationarity makes the PII models more flexible and allows to adapt them to essentially similar situations as possible by means of stochastic volatility models. Thus the extension of ordering results to this class is of considerable importance.

We make use of the following representation result of Norberg (1993, Theorem 1).

Proposition 28 (Representation for finite Lévy kernels). *Let $S \sim (0, 0, K(s; dy))_0$ be a d-dimensional PII with $\lambda(s) := K(s, \mathbb{R}^d) < \infty$ and let Y_t be a random sum process that is defined by*

$$Y_t = \sum_{j=1}^{\hat{N}_t} \hat{X}_{t,j}, \quad t \in [0, T],$$

where the extended Poisson process $\hat{N}_t \sim \mathcal{P}(\Lambda(t))$, $\Lambda(t) := \int_0^t \lambda(s) ds$, is independent of the iid sequence $(\hat{X}_{t,j}) \sim \hat{R}_t$, $\hat{R}_t(dy) = \frac{1}{\Lambda(t)} \int_0^t K(s; dy) ds$.

Then for all $t \in [0, T]$ it holds true that $S_t \stackrel{d}{=} Y_t$.

To obtain a generalization of Proposition 2 that implies $S_t^{(1)} \leq_{\mathcal{F}} S_t^{(2)}$, for all $t \in [0, T]$, for PII $S^{(i)} \sim \big(b^{(i)}(t; 0), 0, K^{(i)}(t; \cdot)\big)_0$, we make use of the *modified Lévy kernels* $\tilde{K}^{(i)}$, which for $t \in [0, T]$ and $k \in \{1, 2\}$ such that $K^{(k)}(t; \mathbb{R}^d) \leq K^{(3-k)}(t; \mathbb{R}^d)$ are defined by

$$\tilde{K}^{(k)}(t; dx) = K^{(k)}(t; dx) + \big(K^{(3-k)}(t; \mathbb{R}^d) - K^{(k)}(t; \mathbb{R}^d)\big)^+ \delta_{\{0\}}(dx). \tag{21}$$

Then by adding mass to the point zero the total masses of $\tilde{K}^{(k)}$ are equal and similarly to the Lévy-case $S^{(i)} \sim \big(b^{(i)}(t; 0),\ 0, \tilde{K}^{(i)}(t; \cdot)\big)_0$, $i = 1, 2$. We again do not use the convention $\tilde{K}(t; \{0\}) = 0$, and the kernels are unique modulo the point mass in the origin.

Proposition 29 (Ordering of PII with finite Lévy kernels). *Let $S^{(i)} \sim \big(b^{(i)}(t; 0), 0, K^{(i)}(t; \cdot)\big)_0$, $i = 1, 2$, be d-dimensional PII with $K^{(i)}(t, \mathbb{R}^d) < \infty$, for all $t \in [0, T]$, and* *assume that* $\int_{\{|x|>1\}} |x| K^{(i)}(t;dx) < \infty$*, for all* $t \in [0,T]$*. Let the modified Lévy kernels* $\tilde{K}^{(i)}$ *be given by (21). Assume that for all* $t \in [0,T]$ *it holds true that* $\tilde{K}^{(1)}(t;\cdot) \leq_{\mathcal{F}} \tilde{K}^{(2)}(t;\cdot)$*.*

1. *If* $\mathcal{F} \in \{\mathcal{F}_{\mathrm{st}}, \mathcal{F}_{\mathrm{icx}}, \mathcal{F}_{\mathrm{idcx}}, \mathcal{F}_{\mathrm{ism}}\}$ *then assume additionally* $b^{(1)}(t;0) \leq b^{(2)}(t;0)$*, and*
2. *if* $\mathcal{F} \in \{\mathcal{F}_{\mathrm{cx}}, \mathcal{F}_{\mathrm{dcx}}, \mathcal{F}_{\mathrm{sm}}\}$ *then assume additionally* $b^{(1)}(t;0) = b^{(2)}(t;0)$*.*

Then $S_t^{(1)} \leq_{\mathcal{F}} S_t^{(2)}$ *for all* $t \in [0,T]$*.*

Proof. Let $t \in [0,T]$ and define $\lambda(s) := \tilde{K}^{(i)}(s, \mathbb{R}^d)$, $s \leq t$ and $\Lambda(t) := \int_0^t \lambda(s) ds$. Proposition 28 implies that the random vectors $S_t^{(i)}$ are distributed as random sums with additional drift

$$S_t^{(i)} \stackrel{d}{=} \int_0^t b^{(i)}(s;0) ds + \sum_{j=1}^{\hat{N}_t} \hat{X}_{t,j}^{(i)},$$

where $\hat{N}_t \sim \mathcal{P}(\Lambda(t))$ is independent of the iid sequences $(\hat{X}_{t,j}^{(i)}) \sim \hat{R}_t^{(i)}$, $\hat{R}_t^{(i)}(dy) = \frac{1}{\Lambda(t)} \int_0^t \tilde{K}^{(i)}(s;dy) ds$. Hence $S^{(1)}$ and $S^{(2)}$ are naturally coupled by the same extended Poisson process $\hat{N}$. As $\hat{R}_t^{(i)}$ are mixtures of $\tilde{K}^{(i)}(s;\cdot)$, $s \leq t$ with mixing distribution $\frac{1}{\Lambda(t)} \lambda\!\!\lambda|_{[0,t]}$, it follows from $\tilde{K}^{(1)}(s;\cdot) \leq_{\mathcal{F}} \tilde{K}^{(2)}(s;\cdot)$, for all $s \in [0,T]$, and the stability under mixtures property that $\hat{R}_t^{(1)} \leq \hat{R}_t^{(2)}$. As $b^{(1)}(s;0) \leq b^{(2)}(s;0)$ for all $s \in [0,T]$ implies $\int_0^t b^{(1)}(s;0) ds \leq \int_0^t b^{(2)}(s;0) ds$, the result follows similarly to the proof of Proposition 2. $\square$

Remark 30. 1. If one is only interested in the ordering of $S^{(i)}$ at a specific time point $\bar{t} \in [0,T]$, the conditions in the first part of the previous lemma can be reduced to $\int_0^{\bar{t}} b^{(1)}(s;0) ds \leq \int_0^{\bar{t}} b^{(2)}(s;0) ds$ and $\int_0^{\bar{t}} \tilde{K}^{(1)}(s;\cdot) ds \leq_{\mathcal{F}} \int_0^{\bar{t}} \tilde{K}^{(2)}(s;\cdot) ds$; similarly for the conditions of the second part.

2. Proposition 29 allows to derive several cut and domination criteria for one-dimensional PII $S^{(i)} \sim (b^{(i)}(t;0), 0, K^{(i)}(s;\cdot))_0$, $i = 1,2$, in terms of the corresponding Lévy kernels $K^{(i)}(t;\cdot)$ under appropriate drift conditions. These results are parallel to the cut and domination criteria in Section 2.

To derive ordering of the finite dimensional distributions $(S^{(1)}) \leq_{\mathcal{F}} (S^{(2)})$ we can as in Section 2 apply the separation lemma. In fact for PII processes the separation lemma simplifies essentially.

Lemma 31. *If* $S_0^{(1)} \leq_{\mathcal{F}} S_0^{(2)}$ *for* $\mathcal{F}$ *as in* (3) *and for* $0 \leq t_1 < t_2$

$$S_{t_2}^{(1)} - S_{t_1}^{(1)} \leq_{\mathcal{F}} S_{t_2}^{(2)} - S_{t_1}^{(2)} \tag{22}$$

then $(S^{(1)}) \leq_{\mathcal{F}} (S^{(2)})$*.*

Proof. Let $0 \leq t_1 < t_2$ then $(S_{t_1}^{(i)}, S_{t_2}^{(i)}) = (S_{t_1}^{(i)}, S_{t_1}^{(i)}) + (0, S_{t_2}^{(i)} - S_{t_1}^{(i)})$ is a sum of independent variables. Thus by the convolution property of the $\leq_{\mathcal{F}}$ ordering it is enough to ensure the $\leq_{\mathcal{F}}$ ordering of both summands in order to conclude that $(S_{t_1}^{(1)}, S_{t_2}^{(1)}) \leq_{\mathcal{F}^{(2)}} (S_{t_1}^{(2)}, S_{t_2}^{(2)})$. This ordering condition however is obtained from our assumptions. The general finite dimensional case follows from induction. $\square$

Assuming (eventually after modification as above) that the kernels $K^{(1)}(t,\cdot)$ and $K^{(2)}(t,\cdot)$ have the same total mass we obtain for the differences $S_{t_2}^{(i)} - S_{t_1}^{(i)}$ by Proposition 28 a representation as compound Poisson distribution

$$S_{t_2}^{(i)} - S_{t_1}^{(i)} = \sum_{j=1}^{\hat{N}_{t_1,t_2}^{(i)}} \hat{X}_j^{(i)} \tag{23}$$

where $\hat{N}_{t_1,t_2}^{(1)} \stackrel{d}{=} \hat{N}_{t_1,t_2}^{(2)} \sim \mathcal{P}(\Lambda(t_1,t_2))$, $\Lambda(t_1,t_2) = \int_{t_1}^{t_2} \lambda(s)ds$ and $(X_j^{(i)}))_j \sim \hat{R}_{t_1,t_2}^{(i)}$ with

$$\hat{R}_{t_1,t_2}^{(i)}(dy) = \frac{1}{\Lambda(t_1,t_2)} \int_{t_1}^{t_2} K^{(i)}(s,dy)ds. \tag{24}$$

Since $\leq_{\mathcal{F}}$ ordering is preserved under mixtures we obtain as corollary.

Corollary 32. *Let $S^{(i)} \sim (b^{(i)}(t,0), 0, K^{(i)}(t,\cdot))_0$ be d-dimensional PII with $K^{(1)}(t,\mathbb{R}^d) = K^{(2)}(t,\mathbb{R}^d) < \infty$ for all $t \geq 0$. Then under the assumptions of Propositon 29 we have that*

$$(S^{(1)}) \leq_{\mathcal{F}} (S^{(2)}). \tag{25}$$

To obtain ordering results for PII with infinite Lévy kernels that correspond to the results of Section 3, for $t \in [0,T]$ we introduce the *truncated Lévy kernels* $K_n(t;\cdot)$, similarly to the truncated Lévy measures given in (12), by

$$K_n(t;dx) := \mathbb{1}_{(\underline{\varepsilon}_{t,n},\overline{\varepsilon}_{t,n})^c}(x)K(t;dx), \quad \underline{\varepsilon}_{t,n} \uparrow 0, \overline{\varepsilon}_{t,n} \downarrow 0. \tag{26}$$

As in Section 3, ordering of the truncated Lévy kernels $K_n^{(i)}(t;\cdot)$ implies ordering of the limit processes that have Lévy kernels $K^{(i)}(t;\cdot)$, if an appropriate weak convergence holds true. The proof of the following lemma is given in the Appendix. Let B, C, ν denote the characteristics of a PII process w.r.t. some specified truncation function h, while b, c, K denote the differential characteristics.

Lemma 33 (Functional weak convergence, PII case)**.** *For a continuous truncation function h let $S \sim \big(b(t;h), 0, K(t;\cdot)\big)_h$ be a d-dimensional PII process whose Lévy kernel K has infinite total mass and for $\underline{\varepsilon}_{t,n} \uparrow 0$, $\overline{\varepsilon}_{t,n} \downarrow 0$ let K_n be the corresponding truncated Lévy kernel introduced in (26). If $\sup_{s\leq t} |B_n(s)-B(s)| \to 0$, for all $t \in [0,T]$, then for $S_n \sim \big(b_n(t), 0, K_n(t;\cdot)\big)_h$ functional weak convergence*

$$S_n \xrightarrow{\mathcal{L}} S$$

holds true.

Similar to Corollary 13 we obtain a weak convergence result that fits to our situation. Observe that the first part is formulated in terms of the characteristics of S, while part b) is stated in terms of the differential characteristics. The proof is similar to that of Corollary 13. It is based on the square-integrable version of the convergence theorem in Jacod and Shiryaev (2003, VII.3.7). For details of the proof see Bergenthum (2005, Corollary 2.3.5).

Corollary 34 (Functional weak convergence, PII case)**.** *Let $K(t;\cdot)$ be a Lévy kernel with infinite total mass for all $t \in [0,T]$, and for sequences $\underline{\varepsilon}_{t,n} \uparrow 0$, $\overline{\varepsilon}_{t,n} \downarrow 0$ let $K_n(t;\cdot)$ be the corresponding truncated Lévy kernel given in (26).*

1. *If $|x|^2 * \nu_t < \infty$, then for $S \sim \big(ES_t, 0, \nu_t\big)_{\mathrm{id}}$ and $S_n \sim \big(ES_t, 0, \nu_{n,t}\big)_{\mathrm{id}}$, it holds true that $S_n \xrightarrow{\mathcal{L}} S$.*
2. *If for some continuous truncation function h it holds true that $\int |h(x)| K(t;dx) < \infty$, for all $t \in [0,T]$, then for $S \sim \big(b(t;0), 0, K(t;\cdot)\big)_0$ and $S_n \sim \big(b(t;0), 0,\ K_n(t;\cdot)\big)_0$ it follows that $S_n \xrightarrow{\mathcal{L}} S$.*

To obtain an extension of Theorem 14 to the PII case we introduce the *modified truncated Lévy kernels* $\tilde{K}_n^{(k)}$, $k = 1,2$, by

$$\tilde{K}_n^{(k)}(t;dx) := K_n^{(k)}(t;dx) + \big(K_n^{(3-k)}(t;\mathbb{R}^d) - K^{(k)}(t,\mathbb{R}^d)\big)^+ \delta_{\{0\}}(dx).$$

The comparison result for d-dimensional PII with infinite Lévy kernels is stated as follows.

Theorem 35 (Ordering of PII with infinite Lévy kernels)**.** *Let $S^{(i)} \sim (ES_t^{(i)}, 0, \nu_t)_{\mathrm{id}}$, $i = 1,2$, be d-dimensional PII with Lévy kernels $K^{(i)}$ that satisfy $K^{(i)}(t;\mathbb{R}^d) = \infty$ and $|x|^2 * \nu_t < \infty$, for all $t \in [0,T]$. Let $\underline{\varepsilon}_{t,n} \uparrow 0$, $\overline{\varepsilon}_{t,n} \downarrow 0$ be sequences such that for the modified truncated Lévy kernels $\tilde{K}_n^{(i)}$ it holds true that*

$$\tilde{K}_n^{(1)}(t;\cdot) \leq_{\mathcal{F}} \tilde{K}_n^{(2)}(t;\cdot), \quad \forall t \in [0,T], n \in \mathbb{N}.$$

1. *If $\mathcal{F} \in \{\mathcal{F}_{\mathrm{st}}, \mathcal{F}_{\mathrm{icx}}, \mathcal{F}_{\mathrm{idcx}}, \mathcal{F}_{\mathrm{ism}}\}$ then additionally, assume that*
$$0 \leq \int x K_n^{(2)}(t;dx) - \int x K_n^{(1)}(t;dx) \leq ES_t^{(2)} - ES_t^{(1)}, \ \ \forall t \in [0,T], n \in \mathbb{N},$$
2. *if $\mathcal{F} \in \{\mathcal{F}_{\mathrm{cx}}, \mathcal{F}_{\mathrm{dcx}}, \mathcal{F}_{\mathrm{sm}}\}$ then additionally assume that $ES_t^{(1)} = ES_t^{(2)}$ and $\int x K_n^{(1)}(t;dx) = \int x K_n^{(2)}(t;dx)$, for all $t \in [0,T]$, $n \in \mathbb{N}$.*

Then $(S^{(1)}) \leq_{\mathcal{F}} (S^{(2)})$ and Corollary 32 follows.

Proof. The claim follows from Proposition 29 and Corollary 34 in the same manner as in the proof of Theorem 14. ☐

Remark 36. Theorem 35 can be used to derive cut and domination criteria that are parallel to the results obtained in Section 3.

Similarly as in Proposition 25 we also obtain orderings for continuous PII. This is due to the fact that a PII $S \sim (0, c, 0)$ is a Gaussian process with deterministic covariance function c (see Jacod and Shiryaev (2003, Theorems II.4.4 and II.4.15)). As in Proposition 25 the orderings follow from the comparison results for multivariate normal distributions (see Müller and Stoyan (2002, Section 1.13) or BR (2007)).

Lemma 37 (Ordering of Gaussian PII)**.** *Let $S^{(i)} \sim (ES_t^{(i)}, c^{(i)}(t),\ 0)_{\mathrm{id}}$, $i = 1,2$, be d-dimensional PII.*

1. *If* $ES_t^{(1)} \leq ES_t^{(2)}$ *and* $c^{(1)}(t) = c^{(2)}(t)$, *for all* $t \in [0,T]$, *then* $(S^{(1)}) \leq_{\mathrm{st}} (S^{(2)})$.

2. *If* $ES_t^{(1)} \leq ES_t^{(2)}$ *and* $c^{(1)}(t) \leq_{\mathrm{psd}} c^{(2)}(t)$, *for all* $t \in [0,T]$, *then* $(S^{(1)}) \leq_{\mathrm{icx}} (S^{(2)})$.

3. *If* $ES_t^{(1)} \leq ES_t^{(2)}$ *and* ${c^{(1)}}^{ij}(t) \leq {c^{(2)}}^{ij}(t)$, *for all* $t \in [0,T]$, $\forall i,j \leq d$, *then* $(S^{(1)}) \leq_{\mathrm{idcx}} (S^{(2)})$.

4. *If* $ES_t^{(1)} \leq ES_t^{(2)}$ *and* ${c^{(1)}}^{ij}(t) \leq {c^{(2)}}^{ij}(t)$, *for all* $t \in [0,T]$, $\forall i,j \leq d$, $i \neq j$, *then* $(S^{(1)}) \leq_{\mathrm{ism}} (S^{(2)})$.

The corresponding convex type orderings hold true if $ES_t^{(1)} = ES_t^{(2)}$, *for all* $t \in [0,T]$.

Finally, similarly to Proposition 26 the stability under convolutions property (C) implies an ordering result for PII that incorporate continuous martingale and jump parts.

Corollary 38 (Ordering of PII). *Let* $\mathcal{F}$ *be one of the order generating function classes in (3) and assume that* $S^{(i)} \sim (ES_t^{(i)}, c^{(i)}(t), \nu^{(i)})_t)_{\mathrm{id}}$, $i = 1,2$, *are* d*-dimensional PII. Let the jump process* $S_{\mathrm{J}}^{(i)} \sim (ES_t^{(i)}, 0, \nu_t^{(i)})_{\mathrm{id}}$ *and the continuous process* $S_{\mathrm{C}}^{(i)} \sim (0, c^{(i)}(t), 0)_{\mathrm{id}}$ *be independent and assume that*

$$\big(S_{\mathrm{J}}^{(1)}\big) \leq_{\mathcal{F}} \big(S_{\mathrm{J}}^{(2)}\big) \quad \text{and} \quad \big(S_{\mathrm{C}}^{(1)}\big) \leq_{\mathcal{F}} \big(S_{\mathrm{C}}^{(2)}\big), \quad \forall t \in [0,T],$$

then $(S^{(1)}) \leq_{\mathcal{F}} (S^{(2)})$.

5 Examples

In this section we derive increasing convex ordering results for finite-dimensional distributions of univariate α-stable processes for $\alpha \in (1,2)$ and for univariate and multivariate normal inverse Gaussian (NIG) processes. The ordering criteria are formulated in terms of parameters of the models. For the proofs of the univariate ordering results we apply the cut criteria for Lévy measures of Sections 2, 3. We end this section by stating an increasing convex ordering result for generalized hyperbolic (GH) distributions. This ordering result is based on a representation of GH distributions as variance mixtures of multivariate normal distributions with a generalized inverse Gaussian (GIG) distribution as mixing distribution (see BR (2007)). This approach also works for some further classes of particular interest for financial mathematical models as e.g. multivariate t distributions or elliptically contoured distributions.

We start with the ordering result for univariate α-stable processes.

Theorem 39 (Comparison of α-stable processes). *For* $c^{(i)}, d^{(i)} > 0$, *and* $\alpha^{(i)} \in (1,2)$, $i = 1,2$, *let*

$$S^{(i)} \sim \big(ES^{(i)}, 0, (c^{(i)} x^{-1-\alpha^{(i)}} \mathbb{1}_{\mathbb{R}_+}(x) + d^{(i)} |x|^{-1-\alpha^{(i)}} \mathbb{1}_{\mathbb{R}_-}(x)) dx\big)_{\mathrm{id}}$$

be one-dimensional α*-stable Lévy processes. If* $\alpha^{(2)} \leq \alpha^{(1)}$ *and* $ES^{(1)} \leq ES^{(2)}$, *then*

$$(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)}).$$

Proof. For $x < 0$ it holds true that $d^{(1)}|x|^{-1-\alpha^{(1)}} < d^{(2)}|x|^{-1-\alpha^{(2)}}$ if and only if $x < -\left(\frac{d^{(2)}}{d^{(1)}}\right)^{\frac{1}{\alpha^{(2)}-\alpha^{(1)}}}$. For $x > 0$ we have $c^{(1)}x^{-1-\alpha^{(1)}} < c^{(2)}x^{-1-\alpha^{(2)}}$ if and only if $x > \left(\frac{c^{(2)}}{c^{(1)}}\right)^{\frac{1}{\alpha^{(2)}-\alpha^{(1)}}}$. Hence the cut criterion of Proposition 16 is satisfied with $k^{(1)} = -\left(\frac{d^{(2)}}{d^{(1)}}\right)^{\frac{1}{\alpha^{(2)}-\alpha^{(1)}}}$ and $k^{(2)} = \left(\frac{c^{(2)}}{c^{(1)}}\right)^{\frac{1}{\alpha^{(2)}-\alpha^{(1)}}}$. It remains to verify the conditions stated in the second part of that theorem. We choose the sequences $\underline{\varepsilon}_n^{(i)} \uparrow 0$, $\overline{\varepsilon}_n^{(i)} \downarrow 0$ such that

$$\begin{aligned} \int_{-\infty}^{\underline{\varepsilon}_n^{(1)}} xF_-^{(1)}(dx) &\stackrel{!}{=} \int_{-\infty}^{\underline{\varepsilon}_n^{(2)}} xF_-^{(2)}(dx) \quad \text{and} \\ \int_{\overline{\varepsilon}_n^{(1)}}^{\infty} xF_+^{(1)}(dx) &\stackrel{!}{=} \int_{\overline{\varepsilon}_n^{(2)}}^{\infty} xF_+^{(2)}(dx). \end{aligned} \tag{27}$$

This is possible as $\int_A xF^{(i)}(dx) = \infty$ for $A = (-1, 0)$ and $A = (0, 1)$. From (27) we obtain

$$\begin{aligned} |\underline{\varepsilon}_n^{(1)}| &= \left(\frac{d^{(2)}}{d^{(1)}}\left(\frac{1-\alpha^{(1)}}{1-\alpha^{(2)}}\right)\right)^{\frac{1}{1-\alpha^{(1)}}} |\underline{\varepsilon}_n^{(2)}|^{\frac{1-\alpha^{(2)}}{1-\alpha^{(1)}}}, \\ \overline{\varepsilon}_n^{(1)} &= \left(\frac{c^{(2)}}{c^{(1)}}\left(\frac{1-\alpha^{(1)}}{1-\alpha^{(2)}}\right)\right)^{\frac{1}{1-\alpha^{(1)}}} \overline{\varepsilon}_n^{(2)\,\frac{1-\alpha^{(2)}}{1-\alpha^{(1)}}}. \end{aligned}$$

As $\frac{1-\alpha^{(2)}}{1-\alpha^{(1)}} > 1$ and $|\underline{\varepsilon}_n^{(2)}|, \overline{\varepsilon}_n^{(2)} \downarrow 0$, it follows that there are $\underline{N}, \overline{N} \in \mathbb{N}$, such that $\overline{\varepsilon}_n^{(1)} < \overline{\varepsilon}_n^{(2)}$, for all $n \geq \overline{N}$ and $|\underline{\varepsilon}_n^{(1)}| < |\underline{\varepsilon}_n^{(2)}|$. Hence $\underline{\varepsilon}_n^{(1)} > \underline{\varepsilon}_n^{(2)}$, for all $n \geq \underline{N}$. Therefore, the conditions of the second part of Proposition 16 are satisfied and it follows that $(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)})$. □

Remark 40. Despite the fact that we can establish the domination criterion also for the Lévy measures of α-stable processes with stability parameters $0 < \alpha^{(1)} \leq \alpha^{(2)} < 1$, an analogous ordering result does not hold true in this case. This is due to the fact that the first moments do not exist, cp. Sato (1999, Proposition 3.14.5).

Next, we establish ordering of one-dimensional normal inverse Gaussian processes in the shape and scaling parameters α and δ. The Lévy density of an $\mathrm{NIG} = \mathrm{NIG}(\alpha, \beta, \delta, \mu)$ distributed random variable S is given by

$$f_{\alpha,\beta,\delta}(x) = \frac{\delta\alpha K_1(\alpha|x|)e^{\beta x}}{\pi|x|}, \tag{28}$$

where K_1 denotes the modified Bessel function of third kind with index 1, $\alpha > 0$, $0 \leq |\beta| \leq \alpha$ and $\delta > 0$ and S has expectation $ES = \mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}$.

Theorem 41 (Ordering of NIG processes)**.** *Let $S^{(i)} \sim (ES^{(i)}, 0, f^{(i)}(x)dx)_{\mathrm{id}}$, $i = 1, 2$, be one-dimensional NIG processes.*

1. *Ordering in α. Let $f^{(i)}(x) := f_{\alpha^{(i)},\beta,\delta}(x)$, $i = 1, 2$. If $|\beta| \leq \alpha^{(2)} \leq \alpha^{(1)}$ and $ES^{(1)} \leq ES^{(2)}$, then $(S_t^{(1)}) \leq_{\mathrm{icx}} (S_t^{(2)})$.*

2. *Ordering in δ. Let $f^{(i)}(x) := f_{\alpha,\beta,\delta^{(i)}}(x)$, $i = 1, 2$. If $\delta^{(1)} \le \delta^{(2)}$ and $ES^{(1)} \le ES^{(2)}$, then $(S_t^{(1)}) \le_{\text{icx}} (S_t^{(2)})$.*

Proof. 1. Let $\delta > 0$ and β, $|\beta| \le \alpha^{(2)}$ be given. For fixed $x > 0$ and $\alpha \ge \alpha^{(2)}$ we consider $g(\alpha) := f_{\alpha,\beta,\delta}(x)$. As $g'(\alpha) = -\frac{\delta e^{\beta x}}{\pi} \alpha K_0(\alpha x) \le 0$, it follows that $f^{(1)}(x) \le f^{(2)}(x)$, for all $x \in \mathbb{R}_+$. For fixed $x < 0$ we similarly obtain $g'(\alpha) = -\frac{\delta e^{\beta x}}{\pi} \alpha K_0(-\alpha x) \le 0$, thus $f^{(1)}(x) \le f^{(2)}(x)$, for all $x \in \mathbb{R}_-$. As NIG processes have paths of infinite variation, Theorem 21 implies $(S_t^{(1)}) \le_{\text{icx}} (S_t^{(2)})$.

2. As $\delta^{(1)} \le \delta^{(2)}$ implies $f^{(1)}(x) \le f^{(2)}(x)$, for all $x \in \mathbb{R}$, Theorem 21 yields $(S_t^{(1)}) \le_{\text{icx}} (S_t^{(2)})$. □

Remark 42. If in the previous theorem it holds true that $\mu^{(1)} = \mu^{(2)}$, then the ordering condition for the expectations is satisfied.

For some cases of interest it is possible to obtain comparison results by using mixing type representations. GH distributions are variance mixtures of multivariate normal distributions with a generalized inverse Gaussian distribution as mixing distribution. For $\mu^{(i)}, \beta^{(i)} \in \mathbb{R}^d$, $\Delta^{(i)} \in M(d, \mathbb{R})$ with $\det(\Delta^{(i)}) = 1$ $i = 1, 2$, and $N^{(i)} \sim \mathcal{N}(0, \Delta^{(i)})$ we consider the d-dimensional random variable

$$S^{(i)} = \mu^{(i)} + X^{(i)} \Delta^{(i)} \beta^{(i)} + \sqrt{X^{(i)}} N^{(i)}, \tag{29}$$

where $X^{(i)}$ are generalized inverse Gaussian random variables with densities

$$d_{\mathrm{GIG}(\lambda,\delta,\gamma)}(x) := \left(\frac{\gamma}{\delta}\right)^{\lambda} \frac{1}{2K_\lambda(\delta\gamma)} x^{\lambda-1} e^{-\frac{1}{2}\left(\frac{\delta^2}{x} + \gamma^2 x\right)} \mathbb{1}_{\mathbb{R}_+}(x), \tag{30}$$

where $\delta \ge 0$, $\alpha^2 > \beta \Delta \beta^T$ and $\gamma = \sqrt{\alpha^2 - \beta \Delta \beta^T}$. Then S is generalized hyperbolic distributed with parameters $d, \lambda, \alpha, \beta, \delta, \mu$ and covariance matrix Δ and we write $\mathrm{GH}(d, \lambda, \alpha, \beta, \delta, \Delta)$ (cp. Barndoff-Nielsen (1977)).

The following lemma from BR (2007) states a comparison result for GIG distributions with respect to the likelihood ratio order $\le_{\text{lr}}$, if the parameters λ, δ, γ are ordered.

Lemma 43 (Likelihood ratio ordering of GIG random variables)**.** *Let $X^{(i)}$ be GIG distributed with density $d_{\mathrm{GIG}(\lambda^{(i)},\delta^{(i)},\gamma^{(i)})}(x)$.*
If $\lambda^{(1)} \le \lambda^{(2)}$, $\delta^{(1)} \le \delta^{(2)}$ and $\gamma^{(1)} \ge \gamma^{(2)}$, then $X^{(1)} \le_{\text{lr}} X^{(2)}$.

As consequence this implies increasing convex ordering of multivariate GH processes using the mixing type representation (29) of GH distributions. We consider the following three cases for $\beta^{(i)}$ and $\Delta^{(i)}$:

$$0 \le \beta^{(1)} \le \beta^{(2)}, \quad \Delta^{(i)} = I, \tag{31}$$

$$\beta^{(i)} = 0, \quad \Delta^{(1)} \le_{\text{psd}} \Delta^{(2)}, \tag{32}$$

$$0 \le \beta^{(1)} \le \beta^{(2)}, \quad \Delta^{(1)} \le_{\text{psd}} \Delta^{(2)}, \quad 0 \le \Delta_{ij}^{(1)} \le \Delta_{ij}^{(2)}, \forall i, j \le d. \tag{33}$$

Theorem 44 (Increasing convex comparison of GH)**.**
Let $S^{(i)}$ be $\mathrm{GH}(d, \lambda^{(i)}, \alpha^{(i)}, \beta^{(i)}, \delta^{(i)}, \Delta^{(i)})$ *distributed. If*

$$\lambda^{(1)} \leq \lambda^{(2)}, \quad \delta^{(1)} \leq \delta^{(2)}, \quad \alpha^{(1)} \geq \alpha^{(2)}, \quad \mu^{(1)} \leq \mu^{(2)}$$

and one of the cases (31)–(33) holds true for $\beta^{(i)}$ and $\Delta^{(i)}$, then

$$S^{(1)} \leq_{\mathrm{icx}} S^{(2)} \quad \textit{for all } t > 0.$$

For the proof see BR (2007, Theorem 4.2). In the case $\lambda = -\frac{1}{2}$, $S^{(i)}$ is normally inverse Gaussian distributed. NIG distributed random variables are stable under convolutions,

$$NIG(d, \alpha, \beta, \delta, \mu, \Delta; t) = NIG(d, \alpha, \beta, t\delta, t\mu, \Delta; 1).$$

Therefore, Theorem 44 together with the separation lemma in Section 1 also implies an increasing convex comparison result of the finite-dimensional distributions of NIG processes. NIG processes have a mixing type representation

$$S_t^{(i)} := \mu^{(i)} t + X_t^{(i)} \Delta^{(i)} \beta^{(i)} + \sqrt{X_t^{(i)}} N^{(i)}, \tag{34}$$

where $X_t^{(i)} \sim GIG(-\frac{1}{2}, t\delta^{(i)}, \gamma^{(i)})$.

Corollary 45 (Increasing convex comparison of NIG processes)**.** *Let $S_t^{(i)}$ be $NIG(d, \alpha^{(i)}, \beta^{(i)}, \delta^{(i)}, \mu^{(i)}, \Delta^{(i)}; t)$ processes. If*

$$\delta^{(1)} \leq \delta^{(2)}, \quad \alpha^{(1)} \geq \alpha^{(2)}, \quad \mu^{(1)} \leq \mu^{(2)}$$

and one of the cases (31)–(33) holds true for $\beta^{(i)}$ and $\Delta^{(i)}$, then

$$\big(S^{(1)}\big) \leq_{\mathrm{icx}} \big(S^{(2)}\big).$$

Appendix

The Appendix contains some of the technical proofs of the paper.

Proof of Proposition 3

Proof. We consider the increasing convex comparison result, the convex case is treated similarly. Let $\tilde{F}^{(i)}$, $i = 1, 2$, be the modified Lévy measures introduced in (7). As $\|\tilde{F}^{(i)}\| = \lambda$, $i = 1, 2$, we are in the position to apply the classical cut criterion to the corresponding distribution functions $G^{(i)}(x) := \tilde{F}^{(i)}((-\infty, x])$. As on $\mathbb{R}_-$ it holds true that $\tilde{F}^{(i)} = F^{(i)}$, it follows from (8) that

$$G^{(2)} \geq G^{(1)} \text{ on } (-\infty, k_\ell).$$

For $x \in (0, \infty)$, condition (10) yields $\lambda - G^{(1)}(x) = F^{(1)}((x, \infty)) \leq F^{(2)}((x, \infty)) = \lambda - G^{(2)}(x)$, hence $G^{(1)} \geq G^{(2)}$ on $(0, \infty)$ and from right-continuity of $G^{(i)}$ it follows that

$$G^{(1)} \geq G^{(2)}, \text{ on } [0, \infty).$$

For the comparison of $G^{(1)}$ and $G^{(2)}$ on $[k_\ell, 0)$ we consider two different cases.

i. Assume that $\|F^{(2)}\| \leq \|F^{(1)}\|$. Then $\tilde{F}^{(2)}(dx) = F^{(2)}(dx) + (\|F^{(1)}\| - \|F^{(2)}\|)$ $\delta_{\{0\}}(dx)$ and $\tilde{F}^{(1)} = F^{(1)}$. This implies w.l.g. that $G^{(1)}(0_-) = G^{(1)}(0)$, thus $G^{(1)}(0_-) = G^{(1)}(0) \geq G^{(2)}(0) = G^{(2)}(0_-) + \|F^{(1)}\| - \|F^{(2)}\| \geq G^{(2)}(0_-)$ and from (10) it follows for $x \in (k_r, 0)$ that $G^{(1)}(0_-) - G^{(1)}(x) = F^{(1)}((x,0)) \leq F^{(2)}((x,0)) = G^{(2)}(0_-) - G^{(2)}(x)$, thus $G^{(1)}(x) \geq G^{(2)}(x)$. Right-continuity of $G^{(i)}$ implies that also $G^{(1)}(k_r) \geq G^{(2)}(k_r)$, hence

$$G^{(1)} \geq G^{(2)} \text{ on } [k_r, 0).$$

As $G^{(1)}(k_r) \geq G^{(2)}(k_r)$ and $G^{(1)}(k_{\ell-}) \leq G^{(2)}(k_{\ell-})$ it follows from (9) that there is a $\kappa \in [k_\ell, k_r)$ s.th. $G^{(1)}(k) \leq G^{(2)}(k)$, for all $k < \kappa$ and $G^{(1)}(k) \geq G^{(2)}(k)$, for all $k \in [\kappa, k_r]$. Consequently, $G^{(1)}$ and $G^{(2)}$ cross once and the sign sequence of the difference $G^{(1)} - G^{(2)}$ is $-, +$, hence the cut criterion implies $\tilde{F}^{(1)} \leq_{\text{icx}} \tilde{F}^{(2)}$, as $\int x\tilde{F}^{(1)}(dx) = \int xF^{(1)}(dx) \leq \int xF^{(2)}(dx) = \int x\tilde{F}^{(2)}(dx)$ by assumption.

ii. Assume that $\|F^{(1)}\| < \|F^{(2)}\|$. Then $\tilde{F}^{(1)}(dx) = F^{(1)}(dx) + (\|F^{(2)}\| - \|F^{(1)}\|)$ $\delta_{\{0\}}(dx)$ and $\tilde{F}^{(2)} = F^{(2)}$. Again, we denote the corresponding distribution functions by $G^{(i)}$.

First consider the case where $G^{(1)}(k_r) \leq G^{(2)}(k_r)$. In this case it follows from (9) and right-continuity of $G^{(i)}$ that

$$G^{(1)} \leq G^{(2)} \text{ on } [k_\ell, k_r].$$

As (10) then implies $G^{(1)} \leq G^{(2)}$ on $(k_r, 0)$ it follows that $G^{(1)}$ and $G^{(2)}$ cross once, namely in the origin. The cut criterion implies $\tilde{F}^{(1)} \leq_{\text{icx}} \tilde{F}^{(2)}$, as additionally it holds true that $\int x\tilde{F}^{(1)}(dx) \leq \int x\tilde{F}^{(2)}(dx)$.

In the case when $G^{(1)}(k_r) \geq G^{(2)}(k_r)$ we make the additional assumption that $F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-)$, which is $G^{(1)}(0_-) \geq G^{(2)}(0_-)$ in terms of $G^{(i)}$. From (10) and right-continuity of $G^{(i)}$ it then follows that

$$G^{(1)} \geq G^{(2)} \text{ on } [k_r, 0),$$

as for $x \in (k_r, 0)$ it holds true that $G^{(1)}(0-) - G^{(1)}(x) = F^{(1)}((x,0)) \leq F^{(2)}((x,0))$ $= G^{(2)}(0-) - G^{(2)}(x)$. As $G^{(1)}(k_r) > G^{(2)}(k_r)$ and $G^{(1)}(k_{\ell-}) \leq G^{(2)}(k_{\ell-})$ it follows from (9) that $G^{(1)}$ and $G^{(2)}$ cross once in $[k_\ell, k_r]$ with sign sequence $-, +$ for $G^{(1)} - G^{(2)}$. The cut criterion implies $\tilde{F}^{(1)} \leq_{\text{icx}} \tilde{F}^{(2)}$, as $\int x\tilde{F}^{(1)}(dx) \leq \int x\tilde{F}^{(2)}(dx)$ by assumption.

Then the drift condition $b^{(1)}(0) \leq b^{(2)}(0)$ and the Lévy moment condition $\int xF^{(1)}(dx)$ $\leq \int xF^{(2)}(dx)$ imply $(S_t^{(1)}) \leq_{\text{icx}} (S_t^{(2)})$, due to Proposition 2. □

Proof of Proposition 9

Proof. We establish that the distribution functions $G^{(i)}$ corresponding to the modified Lévy measures $\tilde{F}^{(i)}$, $i = 1, 2$, are ordered as $G^{(1)} \geq G^{(2)}$. Then $\tilde{F}^{(1)} \leq_{\text{st}} \tilde{F}^{(2)}$ holds true and the result follows from Proposition 2.

From $F^{(1)}(A) \geq F^{(2)}(A)$ for all $A \in \mathcal{B}((-\infty, k))$ it follows that

$$G^{(1)} \geq G^{(2)} \text{ on } (-\infty, k).$$

For the comparison of $G^{(i)}$ on $[k, \infty)$ we consider two cases.

1. Assume that $\|F^{(1)}\| < \|F^{(2)}\|$ and let $\tilde{F}^{(1)}(dx) = F^{(1)}(dx) + (\|F^{(2)}\| - \|F^{(1)}\|)\, \delta_{\{0\}}(dx)$, $\tilde{F}^{(2)} = F^{(2)}$ and $\lambda := \|\tilde{F}^{(i)}\|$. For $A := (x, \infty)$, $x \in (0, \infty)$, it holds true that $\lambda - G^{(2)}(x) = F^{(2)}(A) \geq F^{(1)}(A) = \lambda - G^{(1)}(x)$, hence $G^{(2)}(x) \leq G^{(1)}(x)$, $x \in (0, \infty)$, and from right-continuity of $G^{(i)}$ it follows that

$$G^{(2)} \leq G^{(1)} \text{ on } [0, \infty).$$

   Assumption $F^{(1)}(\mathbb{R}_-) \geq F^{(2)}(\mathbb{R}_-)$ is $G^{(1)}(0_-) \geq G^{(2)}(0_-)$ in terms of distribution functions and a consideration that is similar to the step that yielded $G^{(2)} \leq G^{(1)}$ on $[0, \infty)$ implies

$$G^{(2)} \leq G^{(1)} \text{ on } [k, 0);$$

   hence $G^{(2)} \leq G^{(1)}$ and thus $\tilde{F}^{(1)} \leq_{\mathrm{st}} \tilde{F}^{(2)}$.

2. Assume that $\|F^{(2)}\| \leq \|F^{(1)}\|$ and let $\tilde{F}^{(2)}(dx) = F^{(2)}(dx) + (\|F^{(1)}\| - \|F^{(2)}\|)\, \delta_{\{0\}}(dx)$ and $\tilde{F}^{(1)} = F^{(1)}$. Similar to the previous case it follows from $F^{(1)}(A) \leq F^{(2)}(A)$, for all $A \in \mathcal{B}((k, \infty))$, that

$$G^{(2)} \leq G^{(1)} \text{ on } [0, \infty)$$

   and $G^{(2)}(0_-) = G^{(2)}(0) - (\|F^{(1)}\| - \|F^{(2)}\|) \leq G^{(1)}(0) = G^{(1)}(0_-)$. A similar argument as in the previous case yields

$$G^{(2)} \leq G^{(1)} \text{ on } [k, 0)$$

   and the result follows. □

Proof of Lemma 12

Proof. We establish convergence of $\tilde{c}_n(h)$ and $F_n(g)$, $g \in C_2(\mathbb{R}^d)$, where the modified second characteristic $\tilde{c}_n(h)$ is given by $\tilde{c}_n^{kl}(h) = \int h^k(x) h^l(x) F_n(dx)$ and $C_2(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{R} : f \text{ is bounded, continuous and } 0 \text{ around } 0\}$. Then the result follows from Jacod and Shiryaev (2003, Corollary VII.3.6). From $h^k(x) h^l(x) \mathbb{1}_{(\underline{\varepsilon}_n, \overline{\varepsilon}_n)^c}(x) \to h^k(x) h^l(x)$ and $\int_{\{|x|<1\}} |x|^2 F(dx) < \infty$ it follows that $\tilde{c}_n \to \tilde{c}$, due to the Lebesgue Theorem. As $g \in C_2(\mathbb{R}^d)$ is zero around the origin, there is a $N \in \mathbb{N}$ such that $F_n(g) = \int g(x) \mathbb{1}_{(\underline{\varepsilon}_n, \overline{\varepsilon}_n)^c}(x) F(dx) = \int g(x) F(dx) = F(g)$, for all $n \geq N$. □

Proof of Corollary 13

Proof. 1. As $B_n(s; \mathrm{id}) = sES_1 = B(s; \mathrm{id})$, for all $s \leq T$, the result follows from Lemma 12 or from Jacod and Shiryaev (2003, Theorem VII.3.7). Observe that in this case $\tilde{C}_t^{ij} = x^i x^j * \nu_t$, therefore we need the additional integrability condition $\int |x|^2 F(dx) < \infty$.

2. An extension of Jacod and Shiryaev (2003, Proposition II.2.24) to the"truncation function" $h' \equiv 0$ implies $b(h) = b(0) + \int h(x)F(dx)$ and $\tilde{C}(h) = \tilde{C}(0) + (h^i h^j * \nu)$, similar for the truncated characteristics $\tilde{b}_n(h)$, $\tilde{C}_n(h)$. As $\tilde{C}(0) = C = 0$ it remains to establish $b_n(h) \to b(h)$, then the result follows from Lemma 12. As $b_n(0) = b(0)$ and $h(x)\mathbb{1}_{(\underline{\varepsilon}_n, \overline{\varepsilon}_n)^c}(x) \to h(x)$ this follows from the Lebesgue Theorem. □

Proof of Proposition 16

Proof. Let $F_n^{(i)}$ denote the truncated Lévy measures and let $S_n^{(i)} \sim (ES_1^{(i)}, 0, F_n^{(i)})_{\mathrm{id}}$ be the corresponding compound Poisson processes. From (14) and as $\underline{\varepsilon}_n^{(2)} \leq \underline{\varepsilon}_n^{(1)} < 0 < \overline{\varepsilon}_n^{(1)} < \overline{\varepsilon}_n^{(2)}$ there is a $N \in \mathbb{N}$ such that

$$\begin{aligned} F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, k_\ell)), \\ F_n^{(1)}(A) &\geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((k_\ell, k_r)), \\ F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((k_r, \infty)), \end{aligned}$$

for all $n \geq N$. Due to conditions (b) and (c), Proposition 5 implies $(S_{n,t}^{(1)}) \leq_{\mathrm{icx}} (S_{n,t}^{(2)})$. The result follows from Corollary 13 and the fact that $ES_{n,1}^{(i)} = ES_1^{(i)}$. □

Proof of Proposition 19

Proof. Let $\underline{\varepsilon}_n^{(i)} \uparrow 0$, $\overline{\varepsilon}_n^{(i)} \downarrow 0$ be sequences such that the conditions (a) and (b) hold true. Let $S_n^{(i)} \sim (ES_1^{(i)}, 0, F_n^{(i)})_{\mathrm{id}}$, $i = 1, 2$, be compound Poisson processes corresponding to the truncations of $F^{(i)}$. The main part of the proof is to establish that for any choices of the truncating sequences the conditions of Propositions 5, 6 are fulfilled for the truncated Lévy measures $F_n^{(i)}$ and as consequence $(S_{n,t}^{(1)}) \leq_{\mathrm{icx}} (S_{n,t}^{(2)})$. The result then follows by the weak convergence result in Corollary 13.

We consider several different cases depending on the relative location of $\overline{\varepsilon}_n^{(i)} \downarrow 0$.

If $\underline{\varepsilon}_n^{(1)} \leq \underline{\varepsilon}_n^{(2)} < 0 < \overline{\varepsilon}_n^{(2)} \leq \overline{\varepsilon}_n^{(1)}$ it holds true that $F_n^{(1)} \leq F_n^{(2)}$ and Corollary 8 implies $(S_{n,t}^{(1)}) \leq_{\mathrm{icx}} (S_{n,t}^{(2)})$.
In the case $\underline{\varepsilon}_n^{(1)} \leq \underline{\varepsilon}_n^{(2)} < 0 < \overline{\varepsilon}_n^{(1)} \leq \overline{\varepsilon}_n^{(2)}$ the domination of the truncated Lévy measures is as follows

$$\begin{aligned} F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}\big((-\infty, \underline{\varepsilon}_n^{(2)})\big), \\ 0 = F_n^{(1)}(A) &= F_n^{(2)}(A), \quad \forall A \in \mathcal{B}\big((\underline{\varepsilon}_n^{(2)}, \overline{\varepsilon}_n^{(1)})\big), \\ F_n^{(1)}(A) &\geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}\big((\overline{\varepsilon}_n^{(1)}, \overline{\varepsilon}_n^{(2)})\big), \\ F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}\big((\overline{\varepsilon}_n^{(2)} \infty)\big). \end{aligned}$$

If $\|F_n^{(1)}\| \leq \|F_n^{(2)}\|$, Proposition 5 yields $(S_{n,t}^{(1)}) \leq_{\mathrm{icx}} (S_{n,t}^{(2)})$, and if $\|F_n^{(1)}\| \geq \|F_n^{(2)}\|$ the result follows from Proposition 6.

The case $\underline{\varepsilon}_n^{(2)} \leq \underline{\varepsilon}_n^{(1)} < 0 < \overline{\varepsilon}_n^{(2)} \leq \overline{\varepsilon}_n^{(1)}$ follows from Proposition 3 and 5, depending on the ordering of the total masses of $F_n^{(i)}$.

If $\underline{\varepsilon}_n^{(2)} \leq \underline{\varepsilon}_n^{(1)} < 0 < \overline{\varepsilon}_n^{(1)} \leq \overline{\varepsilon}_n^{(2)}$ it holds true that

$$\begin{aligned}
F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}\big((-\infty, \underline{\varepsilon}_n^{(2)}\big),\\
F_n^{(1)}(A) &\geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}\big((\underline{\varepsilon}_n^{(2)}, \overline{\varepsilon}_n^{(2)})\big),\\
F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}\big((\overline{\varepsilon}_n^{(2)}, \infty)\big).
\end{aligned}$$

Proposition 5 implies $(S_{n,t}^{(1)}) \leq_{\text{icx}} (S_{n,t}^{(2)})$, due to condition (b). □

Proof of Proposition 23

Proof. Let $\underline{\varepsilon}_n^{(i)} \uparrow 0, \overline{\varepsilon}_n^{(i)} \downarrow 0$ be sequences s.th. the conditions (a)–(d) are satisfied. For the compound Poisson processes $S_n^{(i)} \sim (ES_1^{(i)}, 0, F_n^{(i)})_{\text{id}}$ we establish stochastic ordering of the finite dimensional distributions $(S_{n,t}^{(1)}) \leq_{\text{st}} (S_{n,t}^{(2)})$. Then the result follows from the weak convergence, result in Corollary 13.

For $\underline{\varepsilon}_n^{(2)} \leq \underline{\varepsilon}_n^{(1)} < 0 < \overline{\varepsilon}_n^{(2)} \leq \overline{\varepsilon}_n^{(1)}$ it follows from the ordering condition on $F^{(i)}$ that

$$\begin{aligned}
F_n^{(1)}(A) &\geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, \underline{\varepsilon}_n^{(1)})),\\
0 = F_n^{(1)}(A) &= F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\underline{\varepsilon}_n^{(1)}, \overline{\varepsilon}_n^{(2)})),\\
F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\overline{\varepsilon}_n^{(2)}, \infty)).
\end{aligned}$$

Therefore, Corollary 11 implies $(S_{n,t}^{(1)}) \leq_{\text{st}} (S_{n,t}^{(2)})$.
In case $\underline{\varepsilon}_n^{(1)} \leq \underline{\varepsilon}_n^{(2)} < 0 < \overline{\varepsilon}_n^{(2)} \leq \overline{\varepsilon}_n^{(1)}$ it holds true that

$$\begin{aligned}
F_n^{(1)}(A) &\geq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((-\infty, \underline{\varepsilon}_n^{(1)})),\\
F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\underline{\varepsilon}_n^{(1)}, \underline{\varepsilon}_n^{(2)})),\\
0 = F_n^{(1)}(A) &= F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\underline{\varepsilon}_n^{(2)}, \overline{\varepsilon}_n^{(2)})),\\
F_n^{(1)}(A) &\leq F_n^{(2)}(A), \quad \forall A \in \mathcal{B}((\overline{\varepsilon}_n^{(2)}, \infty)).
\end{aligned}$$

Hence Proposition 9 implies $(S_{n,t}^{(1)}) \leq_{\text{st}} (S_{n,t}^{(2)})$, due to the additional assumption in the case $\|F_n^{(1)}\| < \|F_n^{(2)}\|$. Similarly, the case $\underline{\varepsilon}_n^{(2)} \leq \underline{\varepsilon}_n^{(1)} < 0 < \overline{\varepsilon}_n^{(1)} \leq \overline{\varepsilon}_n^{(2)}$ follows from condition (d) and Proposition 10. □

Proof of Lemma 33

Proof. Similar to the proof of Lemma 12 we establish appropriate converence of the characteristics to obtain the result from Jacod and Shiryaev (2003, Theorem VII.3.4). Let $t \in [0,T]$ and $(x,s) \in \mathbb{R}^d \times [0,t]$. As $\underline{\varepsilon}_{s,n} \uparrow 0, \overline{\varepsilon}_{s,n} \downarrow 0$, it holds true that $|h^k(x)h^l(x)|\mathbb{1}_{(\underline{\varepsilon}_{s,n}, \overline{\varepsilon}_{s,n})}(x) \to 0$, hence it follows from the Lebesgue Theorem that

$$\begin{aligned}
\tilde{C}_{n,t}^{k,l} &= \int_0^t \int h^k(x)h^l(x)\mathbb{1}_{(\underline{\varepsilon}_{t,n}, \overline{\varepsilon}_{t,n})^c}(x)K(s;dx)ds\\
&\to \int_0^t \int h^k(x)h^l(x)K(s;dx)ds = \tilde{C}^{k,l}.
\end{aligned}$$

For $g \in C_1(\mathbb{R}^d) := \{f \in C_2(\mathbb{R}^d) : f \geq 0\}$, where the function classes $C_i(\mathbb{R}^d)$ are defined as in Jacod and Shiryaev (2003, VII.2.7), it holds true that g is zero around the origin. For $t \in [0, T]$ let $(x, s) \in \mathbb{R}^d \times [0, t]$. As $\underline{\varepsilon}_n \uparrow 0, \overline{\varepsilon}_{s,n} \downarrow 0$, it holds true that there is a $N \in \mathbb{N}$ such that $g(x)\mathbb{1}_{(\underline{\varepsilon}_{s,n}, \overline{\varepsilon}_{s,n})}(x) = 0$, hence $g(x)\mathbb{1}^c_{(\underline{\varepsilon}_{s,n}, \overline{\varepsilon}_{s,n})}(x) - g(x) = 0$, for all $n \geq \mathbb{N}$. As $g \in C_1(\mathbb{R}^d)$ is bounded, it follows from the Lebesgue Theorem that $g * \nu_{n,t} \to g * \nu_t$. □

References

O. Barndorff-Nielsen. Processes of normal inverse Gaussian type. *Finance & Stochastics*, 2:41–68, 1998.

O. E. Barndorff-Nielsen. Exponentially decreasing distributions for the logarithm of particle size. *Pro. R. Soc. London Ser. A*, 353:401–419, 1977.

N. Bellamy and M. A. Jeanblanc. Incompleteness of markets driven by a mixed diffusion. *Finance and Stochastics*, 4:209–222, 2000.

J. Bergenthum. *Comparison of semimartingales and Lévy processes with applications to finanical mathematics*. PhD thesis, University of Freiburg, 2005.

J. Bergenthum and L. Rüschendorf. Comparison of option prices in semimartingale models. *Finance and Stochastics*, 10(2):222–249, 2006.

J. Bergenthum and L. Rüschendorf. Comparison of semimartingales and Lévy processes. *Annals of Probability*, 35:228–254, 2007.

P. Carr, H. Geman, D. B. Madan, and M. Yor. Stochastic volatility of Lévy processes. *Mathematical Finance*, 13:345–382, 2003.

R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman & Hall/CRC Financial Mathematics Series, 2004.

F. Delbaen and J. Haezendonck. A martingale approach to premium calculation principles in an arbitrage free market. *Insurance: Mathematics and Economics*, 8:269–277, 1989.

E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, 1(3):281–299, 1995.

N. El Karoui, M. Jeanblanc-Picqué, and S. E. Shreve. Robustness of the black and scholes formula. *Mathematical Finance*, 8(2):93–126, 1998.

P. Embrechts. Actuarial versus financial pricing of insurance. *Risk Finance*, 1(4):17–26, 2000.

T. Goll and L. Rüschendorf. Minimax and minimal distance martingale measures and their relationship to portfolio optimization. *Finance Stoch.*, 5:557–581, 2001.

A. A. Gushchin and E. Mordecki. Bounds for option prices in the semimartingale market models. *Proceedings of the Steklov Mathematical Institute*, 237:73–113, 2002.

V. Henderson and D. Hobson. Coupling and option price comparisons in a jump diffusion model. *Stochastics and Stochastics Reports*, 75(3):79–101, 2003.

D. Hobson. Volatility misspecification, option pricing and superreplication via coupling. *Annals of Applied Probability*, 8(1):193–205, 1998.

J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes.* Springer, 2003.

S. Karlin and A. Novikoff. Generalized convex inequalities. *Pac. J. Math.*, 13:1251–1279, 1963.

D. Madan and E. Seneta. The vg model for share market returns. *Journal of Business*, 63:511–524, 1990.

B. Mandelbrot. The Pareto–Lévy law and the distribution of income. *Internat. Econ. Rev.*, 1:79–106, 1960.

R. C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144, 1976.

T. Møller. On valuation and risk management at the interface of insurance and finance. *British Actuarial Journal*, 8(IV):787–827, 2002.

T. Møller. Stochastic orders in dynamic reinsurance markets. *Finance Stoch.*, 8(4): 479–499, 2004.

A. Müller and D. Stoyan. *Comparison Methods for Stochastic Models and Risks.* John Wiley & Sons, Ltd, 2002.

R. Norberg. Prediction of outstanding liabilities in non-life insurance. *ASTIN Bulletin*, 23(1):95–115, 1993.

S. Rachev and S. Mittnik. *Stable Paretian Models in Finance.* Wiley, 2000.

K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, 1999.