# Computation of sharp bounds on the distribution of a function of dependent risks

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**Abstract**

We propose a new algorithm to compute numerically sharp lower and upper bounds on the distribution of a function of $d$ dependent random variables having fixed marginal distributions. Compared to the existing literature, the bounds are wide applicable, more accurate and more easily obtained.

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## 1. Introduction and preliminaries

Let $X_1, \dots, X_d$ be $d$ real-valued random variables on some probability space $(\Omega, \mathfrak{A}, P)$. Given a measurable function $\psi : \mathbb{R}^d \to \mathbb{R}$, we calculate numerical bounds on $P(\psi(X_1, \dots, X_d) \geq s)$, when we assume that each $X_j$ has known distribution $F_j(x) = P(X_j \leq x)$, $1 \leq j \leq d$, but the dependence structure of the vector $(X_1, \dots, X_d)'$ is unknown. Thus, for a fixed $s \in \mathbb{R}$, we look for

$$M_\psi(s) = \sup\left\{P(\psi(X_1, \dots, X_d) \geq s) : X_j \sim F_j, 1 \leq j \leq d\right\}, \tag{1.1a}$$

$$m_\psi(s) = \inf\left\{P(\psi(X_1, \dots, X_d) > s) : X_j \sim F_j, 1 \leq j \leq d\right\}. \tag{1.1b}$$

Note that the probabilities in (1.1a) and (1.1b) are defined differently in order to guarantee that the infimum and the supremum are attained. From mass transportation theory a dual representation is known for problems of type (1.1) (see Rü (1982)). This dual representation, however, is

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typically difficult to evaluate. For the case that $\psi = +$ is the sum operator, problems (1.1) are of particular interest in risk analysis (see EP (2006a)) and reliable dual bounds related to the dual representation were given in EP (2006b) for the case of homogeneous marginals. These were extended to the non homogenous case and to overlapping marginals in EP (2006a, 2010a). While these bounds are well computable for any dimension $d$ in the homogeneous case, the numerical evaluation in the non homogeneous case poses serious problems. For the case that $\psi = \max$, the sharp upper bound $M_\psi(s)$ has been given analytically in Lai and Robbins (1978) for the case of homogeneous marginals. Note that, in general, the sharp bound $M_\psi(s)$ is not attained when the structure of dependence of the vector $(X_1, \ldots, X_d)^{'}$ is comonotonic, that is when each risk is a.s. an increasing function of any of the others. Analogously, the sharp bound $m_\psi(s)$ is not attained when the risks are countermonotonic.

In this paper, we propose a new method to approximate numerically the sharp bounds $m_\psi(s)$ and $M_\psi(s)$ for certain classes of functions $\psi$ which include in particular the sum, min, max and product operators. This method is based on rearrangement arguments and on a simple rearrangement algorithm introduced in special cases in Rü (1983a,b). In comparison to the method of dual bounds, our method is easy and fast. It can handle reasonable dimensions $d$ and, in particular, also the inhomogeneous case. A numerical evaluation and comparison is given in Section 5 of this paper. The fact that this algorithm is computationally less demanding should be relevant for practical applications. It is interesting to note that, in the homogeneous examples considered, the approximate sharp bounds calculated by this method provide evidence for the sharpness of the analytical dual bounds in EP (2006b).

## 2. Assumptions on the function $\psi$

Given a vector $\boldsymbol{x} \in \mathbb{R}^d$, let $\boldsymbol{x}_{-j}$ be the vector in $\mathbb{R}^{d-1}$ obtained by deleting the $j$-th component of $\boldsymbol{x}$. Troughout the paper, we assume that the function $\psi : \mathbb{R}^d \to \mathbb{R}$ is defined recursively. We assume that:

- $\psi$ is measurable, symmetric and increasing in each coordinate;
- $\psi = \psi^d$, where the functions $\psi^i : \mathbb{R}^i \to \mathbb{R}$, $3 \le i \le d$ can be iteratively defined as

$$\psi^i(x_j, \boldsymbol{x}_{-j}) = \psi^2(\psi^1(x_j), \psi^{i-1}(\boldsymbol{x}_{-j})), \text{ for all } j = 1, \ldots i,$$

for some measurable, symmetric and increasing functions $\psi^2 : \mathbb{R}^2 \to \mathbb{R}$ and $\psi^1 : \mathbb{R} \to \mathbb{R}$. Typical examples of such $\psi$ are the sum, product, min and max operators.

## 3. A combinatorial problem

In this section, we describe a combinatorial problem which will turn out to be strictly connected to (1.1). Let $X = (x_{i,j}), x_{i,j} \in \mathbb{R} \cup \{-\infty, +\infty\}$, $1 \le i \le n, 1 \le j \le d$, be a matrix $n \times d$. Let $X_{-j}$ the matrix $n \times (d-1)$ obtained by $X$ by deleting its $j$-th column. Denote by $\boldsymbol{\psi}(X)$ (respectively, $\boldsymbol{\psi}_{-j}(X)$) the $N$-dimensional vectors obtained by applying the function $\psi$ (resp., $\psi^{d-1}$), to each row of $X$ (resp., $X_{-j}$). Formally,

$$\boldsymbol{\psi}(X) = \begin{pmatrix} \psi(x_{1,1}, \ldots, x_{1,d}) \\ \vdots \\ \psi(x_{i,1}, \ldots, x_{i,d}) \\ \vdots \\ \psi(x_{n,1}, \ldots, x_{n,d}) \end{pmatrix}, \boldsymbol{\psi}_{-j}(X) = \begin{pmatrix} \psi^{d-1}(x_{1,1}, \ldots, x_{1,j-1}, x_{1,j+1}, \ldots, x_{1,d}) \\ \vdots \\ \psi^{d-1}(x_{i,1}, \ldots, x_{i,j-1}, x_{i,j+1}, \ldots, x_{i,d}) \\ \vdots \\ \psi^{d-1}(x_{n,1}, \ldots, x_{n,j-1}, x_{n,j+1}, \ldots, x_{n,d}) \end{pmatrix}.$$

Analogously, we set $\boldsymbol{\psi}_j(X) = (\psi^1(x_{1,j}), \ldots, \psi^1(x_n, j))^t$ the vector obtained applying the function $\psi^1$ to the $j-$th column of $X$. Using the assumptions on $\psi$, note that, for $1 \leq j \leq d$, we have

$$\boldsymbol{\psi}(X)_i = \psi^2(\boldsymbol{\psi}_j(X)_i, \boldsymbol{\psi}_{-j}(X)_i), \ 1 \leq i \leq n.$$

Let $\mathcal{P}(X)$ be the set of all matrices $(n \times d)$ obtained by $X$ by rearranging a number of its columns in a different order, that is

$$\mathcal{P}(X) = \left\{\tilde{X} = (\tilde{x}_{i,j}) : \tilde{x}_{i,j} = x_{i,\pi_j(i)}, \pi_1, \ldots, \pi_d \text{ are permutations of } \{1, \ldots, n\}\right\}.$$

Now, we study the problem of how to rearrange the columns of $X$ such that the minimal component of $\boldsymbol{\psi}(X)$ is maximized. Using the notation introduced above, this problem can be written as

$$G_\psi(X) = \max_{\tilde{X} \in \mathcal{P}(X)} \ \min_{1 \leq i \leq n} \ \boldsymbol{\psi}(\tilde{X})_i. \tag{3.1}$$

Similarly, we consider the problem of how to rearrange the columns of $X$ such that the maximal component of $\boldsymbol{\psi}(X)$ is minimized, that is

$$H_\psi(X) = \min_{\tilde{X} \in \mathcal{P}(X)} \ \max_{1 \leq i \leq n} \ \boldsymbol{\psi}(\tilde{X})_i. \tag{3.2}$$

Given two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, we denote by $\boldsymbol{a}_{[i]}$ the $i$-largest component of $\boldsymbol{a}$ ($\boldsymbol{a}_{[n]}$ is the minimal) and we write $\boldsymbol{a} \perp \boldsymbol{b}$ to indicate that the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ are oppositely ordered. Let

$$O_\psi(X) = \left\{X^* \in \mathcal{P}(X) \text{ is such that } \boldsymbol{\psi}_j(X^*) \perp \boldsymbol{\psi}_{-j}(X^*), 1 \leq j \leq d\right\}$$

be the set of those permutation matrices $X^*$ such that $\boldsymbol{\psi}_j(X^*)$ is oppositely ordered to $\boldsymbol{\psi}_{-j}(X^*)$.

**Theorem 3.1** *It is possible to rewrite the problems* (3.1) *and* (3.2) *as*

$$G_\psi(X) = \max_{X^* \in O_\psi(X)} \ \boldsymbol{\psi}(\tilde{X})_{[n]} \quad \textit{and} \quad H_\psi(X) = \min_{X^* \in O_\psi(X)} \ \boldsymbol{\psi}(\tilde{X})_{[1]}.$$

*Proof.* Take $\tilde{X} \in \mathcal{P}(X) \setminus O_\psi(X)$. Then, it is possible to find an index $j \in \{1, \ldots, d\}$ such that $\boldsymbol{a} = \boldsymbol{\psi}_j(\tilde{X})$ is not oppositely ordered to $\boldsymbol{b} = \boldsymbol{\psi}_{-j}(\tilde{X})$. Therefore, there exist two indexes $i_1, i_2 \in \{1, \ldots, n\}$ such that $a_{i_1} \leq a_{i_2}$ and $b_{i_1} \leq b_{i_2}$. Since $\psi^2$ is increasing in each coordinate, we have that

$$\psi^2(a_{i_1}, b_{i_1}) \leq \psi^2(a_{i_1}, b_{i_2}) \text{ and } \psi^2(a_{i_1}, b_{i_1}) \leq \psi^2(a_{i_2}, b_{i_1}).$$

Then, we obtain

$$\min\left\{\psi^2(a_{i_1}, b_{i_1}), \psi^2(a_{i_2}, b_{i_2})\right\} = \psi^2(a_{i_1}, b_{i_1}) \leq \min\left\{\psi^2(a_{i_1}, b_{i_2}), \psi^2(a_{i_2}, b_{i_1})\right\}.$$

Thus, if we rearrange $\boldsymbol{a}$ by switching the indexes $i_1$ and $i_2$, $\min_{1 \leq i \leq n} \psi^2(\boldsymbol{a}_i, \boldsymbol{b}_i)$ is increased. Repeating this procedure on the columns of $\tilde{X}$, we can pass from the matrix $\tilde{X}$ to a matrix $X^* \in O_\psi(X)$ in a finite number of times, having that

$$\min_{1 \leq i \leq n} \boldsymbol{\psi}(X^*)_i = \min_{1 \leq i \leq n} \psi^2(\boldsymbol{\psi}_j(X^*)_i, \boldsymbol{\psi}_{-j}(X^*)_i) \geq \min_{1 \leq i \leq n} \psi^2(\boldsymbol{\psi}_j(\tilde{X})_i, \boldsymbol{\psi}_{-j}(\tilde{X})_i) = \min_{1 \leq i \leq n} \boldsymbol{\psi}(\tilde{X})_i.$$

As a consequence, we can restrict the domain of the max in (3.1) to the set $O_\psi(X)$. A similar proof shows that the domain of the min in (3.2) can be reduced to the set $O_\psi(X)$. □

The proof of Theorem (3.1) indicates a simple algorithm to find elements in $O_\psi(X)$ and, hence, possible solutions to (3.1) and (3.2). This algorithm is a more general version of the algorithm described in Section 3 in Rü (1983a).

**Rearrangement algorithm** to find elements in $O_\psi(X)$. Start with any $\tilde{X} \in \mathcal{P}(X)$. Define $\tilde{X}_1$ by iteratively rearranging its $j-th$ column $\tilde{\boldsymbol{x}}_j$ such that $\tilde{\boldsymbol{x}}_j \perp \boldsymbol{\psi}_{-j}(\tilde{X})$, for $1 \le j \le d$. Then, repeat using $\tilde{X}_1$ as the initial matrix until an element $X^* \in O_\psi(X)$ is found.

## 4. Numerical approximation

In the remainder, let $F_j^{-1}(\alpha) := \sup\left\{x \in \mathbb{R} : F_j(x) \le \alpha\right\}$, $\alpha \in [0,1]$ be the generalized inverse of $F_j$, $1 \le j \le d$. For a subset $A \subset [0,1]$, we denote by $F_j^{-1}|A$ the restriction of $F_j^{-1}$ to $A$. We write $f_j \sim F_j^{-1}|A$ to indicate that the function $f_j : A \to \mathbb{R}$ is a rearrangement of $F_j^{-1}|A$. We refer to Rü (1983b) for a basic introduction to the theory of rearrangements. The following representation of (1.1) is given in Theorem 2 in Rü (1983b).

If $U$ is a random variable uniformly distributed in $[0,1]$, then

$$M_\psi(s) = \sup\left\{P(\psi(f_1(U),\dots,f_d(U)) \ge s) : f_j \sim F_j^{-1}, 1 \le j \le d\right\}, \tag{4.1a}$$

$$m_\psi(s) = \inf\left\{P(\psi(f_1(U),\dots,f_d(U)) > s) : f_j \sim F_j^{-1}, 1 \le j \le d\right\}. \tag{4.1b}$$

In order to establish a link between (4.1) and the rearrangement algorithm described in Section 3 we need the following theorem.

**Theorem 4.1** *If the function $\psi$ is increasing in each coordinate, then, for all real threshold $s$, we have that*

$$M_\psi(s) = 1 - \inf\left\{\alpha : \text{ there exist } f_j^\alpha \sim F_j^{-1}|[\alpha,1], 1 \le j \le d \text{ s.t. } \psi(f_1^\alpha,\dots,f_d^\alpha) \ge s\right\}, \tag{4.2a}$$

$$m_\psi(s) = 1 - \sup\left\{\alpha : \text{ there exist } f_j^\alpha \sim F_j^{-1}|[0,\alpha], 1 \le j \le d \text{ s.t. } \psi(f_1^\alpha,\dots,f_d^\alpha) \le s\right\}. \tag{4.2b}$$

*Proof.* First, we prove (4.2a). If there exist such $f_j^\alpha$'s, we can easily extend them to rearrangements of $F_j^{-1}|[0,1]$ and, by (4.1), $M_\psi(s) \ge 1-\alpha$, hence (4.2a) holds with $\ge$. For the $\le$ inequality, we use a similar argument as in Proposition 3(c) in Rü (1982). Let $f_j^* \sim F_j^{-1}$ be solutions of (4.1a) and define the set

$$A = \{u \in [0,1] : \psi(f_1^*(u),\dots,f_d^*(u)) \ge s\}.$$

Then, the Lebesgue measure of $A$ is $\lambda(A) = M_\psi(s)$. With $\alpha = 1 - M_\psi(s)$, there exists a $\lambda$-preserving transformation $\phi : [0,1] \to [0,1]$ such that $A = \phi([\alpha,1])$. Therefore, we can assume w.l.g. that $A = [\alpha,1]$. Moreover, there exist $\phi_j : [0,1] \to [0,1]$, $\phi_j \sim F_j^{-1}$, $1 \le j \le d$, such that $f_j^\alpha = \phi_j|[\alpha,1] \sim F_j^{-1}|[\alpha,1]$ and $f_j^\alpha(u) \ge f_j^*(u), u \in [\alpha,1]$. Define, for example,

$$A_j^* = \{u \in [\alpha,1] : f_j^*(u) \ge F_j^{-1}(\alpha)\},$$

and $f_j^\alpha|[\alpha,1] = f_j^* 1_{\{A_j^*\}} + F_j^{-1} 1_{\{[\alpha,1]\setminus A_j^*\}}$, $1 \le j \le d$. For the functions $\phi_j$, we can use an extension of $f_j^\alpha|[\alpha,1]$ to $[0,1]$ such that $f_j^\alpha \sim F_j^{-1}$. This implies, by monotonicity of $\psi$, and since $A = [\alpha,1]$, that, for $u \in [\alpha,1]$, we have

$$\psi(f_1^\alpha(u),\dots,f_d^\alpha(u)) \ge \psi(f_1^*(u),\dots,f_d^*(u)) \ge s.$$

The proof for (4.2b) is analogous, considering that $m_\psi(s) = 1 - \sup\{P(\psi(X_1, \dots, X_d) \le s) : X_j \sim F_j, 1 \le j \le d\}$. □

The rearrangement algorithm can be applied to find solutions of (4.2) when the marginal distributions $F_j$ are (rational) discrete. Assume that each $F_j^{-1}|[\alpha, 1]$ takes only the $n$ real values $\boldsymbol{x}_j^\alpha = \{x_{i,j}^\alpha, 1 \le i \le n\}$, for $1 \le j \le d$. We may assume, by using repetitions, that all the $x_{i,j}^\alpha$'s have the same probability $1/n$. The rearrangements $f_j^\alpha$ of $F_j^{-1}|[\alpha, 1]$ are then replaced by the rearrangements of the components of $\boldsymbol{x}_j^\alpha$. For instance, if the componentsof each $\boldsymbol{x}_j^\alpha$ are arranged in increasing order, the columns of the matrix $X^\alpha = (x_{i,j}^\alpha)$ represent the increasing rearrangement $(F_1^{-1}|[\alpha, 1], \dots, F_d^{-1}|[\alpha, 1])$. Since in the following we only consider order induced rearrangements like increasing or decreasing order rearrangements, the transition to rearrangements of the components of the discrete vectors is justified.

Therefore, all the possible rearrangements $(f_1^\alpha, \dots, f_d^\alpha)$ in (4.2a) can be represented by a rearrangement of the columns of $X^\alpha$, that is, using the notation introduced in Section 3 by a matrix $\tilde{X}^\alpha \in \mathcal{P}(X^\alpha)$. Thus, the condition $\psi(f_1^\alpha, \dots, f_d^\alpha) \ge s$ becomes $\boldsymbol{\psi}(\tilde{X}^\alpha)_{[n]} \ge s$. In conclusion, denoting by $X^\alpha$ the matrix having as columns the increasing rearrangements of the points of the domains $F_j^{-1}([\alpha, 1])$, $1 \le j \le d$ we can rewrite (4.2a) as

$$\begin{aligned} M_\psi(s) &= 1 - \inf\left\{\alpha : \text{ there exist } \tilde{X}^\alpha \in \mathcal{P}(X^\alpha) \text{ s.t. } \boldsymbol{\psi}(\tilde{X}^\alpha)_{[n]} \ge s\right\} \\ &= 1 - \inf\left\{\alpha : G_\psi(X^\alpha) \ge s\right\}. \end{aligned} \tag{4.3}$$

Analogously, denoted by $X_\alpha$ the matrix having as columns the increasing rearrangements of the points of the domains $F_j^{-1}([0, \alpha])$, $1 \le j \le d$, we can rewrite (4.2b) as

$$\begin{aligned} m_\psi(s) &= 1 - \sup\left\{\alpha : \text{ there exist } \tilde{X}_\alpha \in \mathcal{P}(X_\alpha) \text{ s.t. } \boldsymbol{\psi}(\tilde{X}_\alpha)_{[1]} \le s\right\} \\ &= 1 - \sup\left\{\alpha : H_\psi(X_\alpha) \le s\right\}. \end{aligned} \tag{4.4}$$

The representations (4.3) and (4.4) hold only when the $F_j$'s are discrete, yet they are useful also in the case of arbitrary marginals. Indeed, it is always possible to define two discrete distribution functions which approximate any $F_j$ from below and from above. For instance, we define the discrete dfs $\underline{F}_j$ and $\overline{F}_j$ as

$$\underline{F}_j(x) = \frac{1}{n}\sum_{r=0}^{n-1} 1_{[q_r, +\infty)}(x) \quad \text{and} \quad \overline{F}_j(x) = \frac{1}{n}\sum_{r=1}^{n} 1_{[q_r, +\infty)}(x), \tag{4.5}$$

where the jump points $q_r$ are defined by $q_r := F_j^{-1}(r/n), 0 \le r \le n$. Since $\underline{F}_j \le F_j \le \overline{F}_j$ and $\psi$ is non-decreasing, it follows that, for every real $s$,

$$\begin{aligned} \underline{M}_\psi(s) \le M_\psi(s) \le \overline{M}_\psi(s), \\ \underline{m}_\psi(s) \le m_\psi(s) \le \overline{m}_\psi(s). \end{aligned}$$

where $\underline{m}_\psi(s)$ (respectively $\overline{m}_\psi(s)$) is the analogous of (1.1b) when $F_j = \underline{F}_j$ (resp. $F_j = \overline{F}_j$). Analogously, $\underline{M}_\psi(s)$ (resp. $\overline{M}_\psi(s)$) is the analogous of (1.1a) when$F_j = \underline{F}_j$(resp. $F_j = \overline{F}_j$).

Note that we can always choose a different number of points $n_j$ in the support of the discrete distributions $\underline{F}_j$ and $\overline{F}_j$ so that, for $1 \le j \le d$, the increasing rearrangement of the supports $\underline{F}_j^{-1}([\alpha, 1])$ and $\overline{F}_j^{-1}([\alpha, 1])$ have all the same number $n$ of components. Using (4.3), it is possible to write

$$\underline{M}_{\psi}(s) = 1 - \inf\{\alpha : G_{\psi}(\underline{X}^{\alpha}) \geq s\},$$
$$\overline{M}_{\psi}(s) = 1 - \inf\{\alpha : G_{\psi}(\overline{X}^{\alpha}) \geq s\},$$

where $\underline{X}^{\alpha}$, respectively $\overline{X}^{\alpha}$, are the matrix $(n \times d)$ having as columns the increasing rearrangements of the supports $\underline{F}_j^{-1}([\alpha, 1])$, respectively $\overline{F}_j^{-1}([\alpha, 1])$, for $1 \leq j \leq d$.

Similarly,

$$\underline{m}_{\psi}(s) = 1 - \sup\{\alpha : H_{\psi}(\underline{X}_{\alpha}) \leq s\},$$
$$\overline{m}_{\psi}(s) = 1 - \sup\{\alpha : H_{\psi}(\overline{X}_{\alpha}) \leq s\},$$

where $\underline{X}_{\alpha}$, respectively $\overline{X}_{\alpha}$, are the matrix $(n \times d)$ having as columns the increasing rearrangements of the supports $\underline{F}_j^{-1}([0, 1])$, respectively $\overline{F}_j^{-1}([0, 1])$, for $1 \leq j \leq d$.

At this point, the algorithm described at the end of Section 3 can be used to find numerical ranges for the sharp bounds $M_{\psi}$ and $m_{\psi}$. Define

$$\mathcal{G}_{\psi}(X) = \left\{\boldsymbol{\psi}(X^*)_{[n]} : X^* \in O_{\psi}(X)\right\} \quad \text{and} \quad \mathcal{H}_{\psi}(X) = \left\{\boldsymbol{\psi}(X^*)_{[1]} : X^* \in O_{\psi}(X)\right\},$$

the set of possible values for the max in (3.1) and, respectively, the min in (3.2). First, we illustrate how to obtain a range on $M_{\psi}(s)$. Start selecting randomly a matrix $\tilde{X}^{\alpha} \in \mathcal{P}(\underline{X}^{\alpha})$. Define $\tilde{X}_1^{\alpha}$ by rearranging its $j-th$ column $\tilde{\boldsymbol{x}}_j^{\alpha}$ such that $\tilde{\boldsymbol{x}}_j^{\alpha} \perp \boldsymbol{\psi}_{-j}(\tilde{X}^{\alpha})$, for all $j = 1, \ldots, d$. Then, repeat using $\tilde{X}_1^{\alpha}$ as the initial matrix, until an element $\underline{g}(\alpha) \in \mathcal{G}_{\psi}(\underline{X}^{\alpha})$ is found. Denote by

$$\underline{\alpha}(s) = \inf\{\alpha \in [0, 1] : \underline{g}(\alpha) \geq s\}.$$

$\underline{\alpha}(s)$ can be computed numerically in several ways, as for example by iteratively bisecting the interval $[0, 1]$ and checking the condition $\underline{g}(\alpha) \geq s$. From (3.1), we have that $G_{\psi}(\underline{X}^{\alpha}) \geq \underline{g}(\alpha)$. Therefore, it follows that $\inf\{\alpha : G_{\psi}(\underline{X}^{\alpha}) \geq s\} \leq \underline{\alpha}(s)$ and, finally,

$$M_{\psi}(s) \geq \underline{M}_{\psi}(s) \geq 1 - \underline{\alpha}(s). \tag{4.6}$$

In order to find an upper bound on $M_{\psi}$, we proceed analogously by finding an element $\overline{g}(\alpha) \in \mathcal{G}_{\psi}(\overline{X}^{\alpha})$. Denote by

$$\overline{\alpha}(s) = \inf\{\alpha \in [0, 1] : \overline{g}(\alpha) \geq s\}.$$

If $\overline{g}(\alpha)$ is optimal, that is $\overline{g}(\alpha) = G_{\psi}(\overline{X}^{\alpha})$, we obtain

$$M_{\psi}(s) \leq \overline{M}_{\psi}(s) = 1 - \overline{\alpha}(s). \tag{4.7}$$

Note that, while (4.6) is always satisfied, (4.6) may fail to hold if $\overline{g}(\alpha)$ is not optimal. However, if $\overline{g}(\alpha)$ is a good approximation for $G_{\psi}(\overline{X}^{\alpha})$, $\overline{\alpha}(s)$ represents a good approximation for $\overline{M}_{\psi}(s)$. In conclusion, combining (4.6) and (4.7), we obtain

$$1 - \underline{\alpha}(s) \leq M_{\psi}(s) \simeq 1 - \overline{\alpha}(s). \tag{4.8}$$

In order to find a range for the sharp bound $m_{\psi}(s)$, we proceed analogously. Applying the algorithm to some matrices $\tilde{X}_{\alpha} \in \mathcal{P}(\underline{X}_{\alpha})$ and $\tilde{X}_{\alpha} \in \mathcal{P}(\overline{X}_{\alpha})$, we find elements $\underline{h}(\alpha) \in \mathcal{H}_{\psi}(\underline{X}_{\alpha})$ and $\underline{h}(\alpha) \in \mathcal{H}_{\psi}(\overline{X}_{\alpha})$. Defining

$$\underline{\beta}(s) = \sup\{\alpha \in [0, 1] : \underline{h}(\alpha) \leq s\},$$
$$\overline{\beta}(s) = \sup\{\alpha \in [0, 1] : \overline{h}(\alpha) \leq s\},$$

it follows that

$$1 - \underline{\beta}(s) \simeq m_{\psi}(s) \leq 1 - \overline{\beta}(s). \tag{4.9}$$

For a fixed function $\psi$ and marginals $F_j$, $1 \leq j \leq d$, the accuracy of the approximations given in (4.8) and in (4.9) can be increased by choosing:

– a larger value of $n$, so that the approximation to $F_j$ given by the discrete distributions $\underline{F}_j$ and $\overline{F}_j$ is more accurate;
– a number of different random starting matrices in order to find different elements in the sets $\mathcal{G}_\psi(X)$ and $\mathcal{H}_\psi(X)$.

In the application to follow, we always find that any element in $\mathcal{G}_\psi(X)$ and $\mathcal{H}_\psi(X)$ yields a very good approximation of the real solutions $G_\psi(X)$ and $H_\psi(X)$, and the algorithm works very well with a single starting point and a high value for $n$.

## 5. Applications

In this section, we compute the ranges defined in (4.8) and in (4.9) for different functionals $\psi$ and set of marginals $F_j, 1 \leq j \leq d$. In order to test the quality of the dual bound, EP (2006b) calculate a numerical range for $M_+(s)$ via two linear problems and using a discretization of the $F_j$'s identical to the one described in (4.5). Note that EP (2006b) obtain bounds on $P(X_1 + \cdots + X_d < s)$ instead of $P(X_1 + \cdots + X_d \geq s)$.

Being only based on the iterative rearrangements of the columns of a matrix, an operation which can efficiently performed with R, our algorithm turns out to be less demanding, in terms of computational time and memory, than the numerical procedure described in EP (2006b). Indeed, both methods use discrete versions of the marginals with $n$ points in their supports, and calculate $M_+(s)$ with an error that approximately decreases as $o(1/n)$. However, here we were able to use $n = 10^5$ with respect to $n = 180$ used in EP (2006b). In the case of $\psi = +$, using $n = 10^5$ reduces the range in (4.8) to a single value with an absolute error of about $10^{-4}$.

In Figure 1, we plot the dual bound functional introduced in EP (2006b) for the sum of $d$ random variables being all Pareto($\theta$)-distributed, that is $P(X_j \leq x) = 1 - (1 + x)^{-\theta}$. We set $\theta = 2$ and $d = 3$ (Figure 1, left) and $d = 30$ (right). In the same figure, we provide the range for $M_+(s)$ obtained using (4.8), at some threshold of interest. Figure 1 seems to indicate that the dual bound introduced in EP (2006b) is actually sharp.

In Table 1, we report the numerical range for $m_+(s)$ and $M_+(s)$, obtained using (4.8) and (4.9), under the same marginal assumtpions, for $d = 3$. We used $n = 10^5$, and each figure is obtained within two minutes. We also check the accuracy of our rearrangement algorithm for the case $\psi = max$, where the sharp bound $M_\psi(s)$ has been given analytically in Lai and Robbins (1978) for the case of homogeneous marginals. In Table 2, we report the numerical range for $m_{\max}(s)$ and $M_{\max}(s)$, as well as the sharp bound calculated in Lai and Robbins (1978). Finally, in Table 3, we report the numerical range for $m_\times(s)$ and $M_\times(s)$, where $\psi = \times$ is the product operator. Here we use different marginal distributions, for $d = 5$ and $n = 10^5$.

The results obtained for $n = 10^5$ and $d = 3$ in a two-minute time can be considered reasonably accurate. However, an important feature of our algorithm is that it can handle larger values of $n$ and $d$ without memory issues. Indeed, changing $n$ and $d$ means changing the dimensions of the matrices representing the rearrangements of the discrete marginals used. If extra-accuracy is required, with $n = 10^6$ one can obtain an estimate for $M_\psi(s)$ in about forty minutes. On the other side, if one needs only two decimal digits for $M_\psi(s)$, using $n = 10^4$ provides one estimate in about 3 seconds. An analogous reasoning can be applied to an increase of the number of random variables $d$. With $n = 10^5$ we can handle up to $d = 30$ different marginals keeping the computational time under 40 minutes. If one needs to compute $M_\psi(s)$ and $m_\psi(s)$ at different thresholds $s$, the average computational time for a single estimate can be reduced by knowing the bounds calculated at a different threshold.

Though the dual bound given in EP (2006b) is mainly analytic and the rearrangement method in this paper is entirely numerical, it is useful to make a final comparison between the two. A dual upper bound on $M_\psi(s)$ has been given in EP (2006b) for homogeneous marginals ($F_j = F, 1 \leq j \leq d$), and extended to general marginal settings in EP (2006a) and EP (2010a). While the dual bound is stated for arbitrary marginals, its computational complexity increases with the number of different marginals used. It is easy to calculate the dual bound with an arbitrary number $d$ of homogeneus random variabes, while it is much more complicated to deal with a relatively small number $d \leq 10$ of non homogeneous marginals. Moreover, the dual bound functional has been introduced only for the sum operator. The algorithm introduced in this paper, on the contrary, can handle more general functionals $\psi$ and inhomogeneous marginals. It approximates the sharp upper and lower bounds $M_\psi(s), m_\psi(s)$ numerically while the dual bounds are constructed only as upper bounds for $M_\psi(s)$. However, the rearrangement method in this paper cannot be used to handle dimensions $d > 100$, where the computation of dual bounds is possible with homogeneous marginals. In the examples considered, the dual bounds in EP (2006b) for $M_\psi(s)$ seem to be sharp. It would be interesting to prove sharpness for certain classes of distributions.

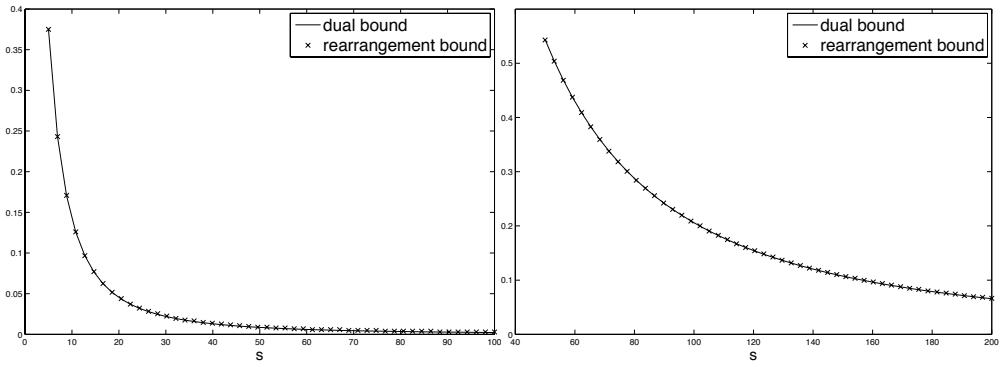


Fig. 1. Upper dual bound on $\mathbb{P}[X_1 + \cdots + X_d \geq s]$, calculated using Theorem 4.2 in EP (2006b), when the $X_j's$ are all Pareto(2)-distributed. We set $d = 3$ (left), and $d = 30$ (right). In both figures, the value of $M_+(s)$, calculated using (4.8), is provided at some threshold of interest.

| | $1 - \underline{\beta}(s)$ | $1 - \overline{\beta}(s)$ | | $1 - \underline{\alpha}(s)$ | dual bound | $1 - \overline{\alpha}(s)$ |
|---|---|---|---|---|---|---|
| s=0.5 | 0.5101929 | 0.51025391 | s=10 | 0.1419678 | 0.142011834319527 | 0.1420288 |
| s=1.0 | 0.2500000 | 0.25006104 | s=15 | 0.0740356 | 0.074074074074074 | 0.0740967 |
| s=1.5 | 0.1599731 | 0.16003418 | s=20 | 0.0453491 | 0.045368620037807 | 0.0454102 |
| s=2.0 | 0.1110840 | 0.11114502 | s=25 | 0.0305786 | 0.030612244897959 | 0.0306397 |
| s=2.5 | 0.0816040 | 0.08166504 | s=30 | 0.0220337 | 0.022038567493113 | 0.0220947 |

Table 1
Range for $m_+(s)$ and $M_+(s)$ for the sum of three Pareto(2) random variables. Values for the upper dual bound on $M_+(s)$, as defined in EP (2006b), is also provided.

| | $1-\underline{\beta}(s)$ | $1-\overline{\beta}(s)$ | | $1-\underline{\alpha}(s)$ | sharp bound | $1-\overline{\alpha}(s)$ |
|---|---|---|---|---|---|---|
| s=1 | 0.25000000 | 0.25000000 | s=1 | 0.75000000 | 0.75000000 | 0.75000000 |
| s=2 | 0.11035156 | 0.11132812 | s=2 | 0.33300781 | 0.33333333 | 0.33398438 |
| s=3 | 0.06250000 | 0.06250000 | s=3 | 0.18750000 | 0.18750000 | 0.18750000 |
| s=4 | 0.03906250 | 0.04003906 | s=4 | 0.11914062 | 0.12000000 | 0.12011719 |
| s=5 | 0.02734375 | 0.02832031 | s=5 | 0.08300781 | 0.08333333 | 0.08398438 |

Table 2
Range for $m_{\max}(s)$ and $M_{\max}(s)$ for the sum of three Pareto(2) random variables. Values for the sharp bound on $M_{\max}(s)$, as provided in Lai and Robbins (1978), is also provided.

| | $1-\underline{\beta}(s)$ | $1-\overline{\beta}(s)$ | | $1-\underline{\alpha}(s)$ | $1-\overline{\alpha}(s)$ |
|---|---|---|---|---|---|
| s=0.001 | 0.16113281 | 0.16210938 | s=100 | 0.2158203 | 0.2167969 |
| s=0.002 | 0.09852281 | 0.09863281 | s=200 | 0.1787109 | 0.1796875 |
| s=0.003 | 0.06347656 | 0.06445312 | s=300 | 0.1591797 | 0.1601562 |
| s=0.004 | 0.04101562 | 0.04199219 | s=400 | 0.1464844 | 0.1474609 |
| s=0.005 | 0.02441406 | 0.02539062 | s=500 | 0.1376953 | 0.1386719 |

Table 3
Range for $m_{\times}(s)$ and $M_{\times}(s)$ for the product of five Pareto($\theta_j$) random variables. We set $\boldsymbol{\theta} = (1.5, 1.8, 2.0, 2.2, 2.5)$.

## 6. Conclusions and forthcoming research

In this paper, we introduce the rerrangement algorithm to calculate numerically the sharp bounds $M_{\psi}(s)$ and $m_{\psi}(s)$ on the distribution of a function of dependent random variables having fixed marginals. This algorithm is accurate, fast and can be used to handle random variables with inhomogeneous marginals, in moderately high dimensions. It provides evidence that the dual bounds in EP (2006b) are sharp for some classes of homogeneous distributions.

Problems (1.1) have a wide range of application in quantitative risk management. For an overview of this kind of application we refer the reader to EP (2006a) and EP (2010b). In a forthcoming paper, we will describe how to use the rearrangement algorithm also in the case of overlapping marginals. Moreover, the authors propose to prove sharpness of the dual bounds in EP (2006b) in the case of the sum of risks, for some classes of homogeneous distributions.

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