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CONSERVATION OF THE UMP-RESP. MAXIMIN-PROPERTY OF STATISTICAL TESTS UNDER EXTENSIONS OF PROBABILITY MEASURES *

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1. INTRODUCTION

Let $(\mathscr{X}, \mathscr{A})$ be a measure space, let $\mathscr{A}_0 \subset \mathscr{A}$ be a sub σ-algebra and let $\mathscr{P}_0, \mathscr{P}_1$ be two sets of probability measures on $\mathscr{A}_0$. In the present paper we consider testproblems $\mathscr{P}'_0, \mathscr{P}'_1$, where $\mathscr{P}'_i$ are subsets of the extensions of elements of $\mathscr{P}_i$ to the larger σ-algebra $\mathscr{A}$. Especially we are interested in the question, in which cases optimality properties of tests for $\mathscr{P}_0, \mathscr{P}_1$ can be lifted to the testproblem $\mathscr{P}'_0, \mathscr{P}'_1$. (We consider maximin-tests and UMP-tests only.) Although this question looks a little bit artificial, testing problems of this kind occur in many practical situations. Some examples are the following:

(a) Let $\mathscr{P}'_0, \mathscr{P}'_1$ be sets of probability measures on $(\mathscr{X}, \mathscr{A})$ and suppose that for some reason one only can observe a function T for testing $\mathscr{P}'_0, \mathscr{P}'_1$. Then defining $\mathscr{P}_i$ to be the images of $\mathscr{P}'_i$ under T we answer

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the question whether optimal tests for $\mathscr{P}_0, \mathscr{P}_1$ (based on T) are optimal for $\mathscr{P}'_0, \mathscr{P}'_1$.

(b) Let G be a set of transformations on $(\mathscr{X}, \mathscr{A})$, let $\mathscr{A}_0$ be the σ-algebra of G-invariant sets, let $\mathscr{P}_0, \mathscr{P}_1$ be probability measures on $(\mathscr{X}, \mathscr{A})$ and consider the testproblem

$$\mathscr{P}'_0 = \{P_0^g \mid g \in G\}, \ \ \mathscr{P}'_1 = \{P_1^g \mid g \in G\}.$$

Thus especially the Hunt–Stein situation is covered by our model.

(c) Let $\mathscr{P}'_i = \{P_{(\theta,\eta)};\ \ \theta \in \Theta_i,\ \ \eta \in \Gamma\}$, $i = 0, 1$, be two sets of probability measures on $(\mathscr{X}, \mathscr{A})$. Consider the test problem $\Theta_0 : \Theta_1$; thus in this case $\eta \in \Gamma$ is a nuisance parameter. Assume that $\mathscr{A}_0 \subset \mathscr{A}$ is a sub σ-algebra such that the restriction of $P_{(\theta,\eta)}$ on $\mathscr{A}_0$ is independent of $\eta \in \Gamma$. So the question is whether $\mathscr{A}_0$ is "sufficient" for the test problem $\Theta_0 : \Theta_1$ in the presence of nuisance parameters.

In Section 2 of this paper we derive some properties of extensions of probability measures. In Section 3 we consider the question whether the maximin-property of tests for $\mathscr{P}_0, \mathscr{P}_1$ may be lifted. We also discuss this question in connection with the concept of least favourable distributions. In Section 4 we consider UMP-tests and finally in Section 5 we give some examples.

2. SOME PROPERTIES OF EXTENSIONS OF PROBABILITY MEASURES

Let $(\mathscr{X}, \mathscr{A})$ be a measure space, $M_1(\mathscr{X}, \mathscr{A})$ be the set of probability measures on $(\mathscr{X}, \mathscr{A})$ and $\mathscr{A}_0 \subset \mathscr{A}$ be a sub σ-algebra of $\mathscr{A}$. For $P \in M_1(\mathscr{X}, \mathscr{A}_0)$ let $E(P) \subset M_1(\mathscr{X}, \mathscr{A})$ denote the set of all extensions of P to the larger σ-algebra $\mathscr{A}$. Furthermore, for $\mathscr{P} \subset M_1(\mathscr{X}, \mathscr{A}_0)$ let $E(\mathscr{P}) = \bigcup_{P \in \mathscr{P}} E(P)$.

The aim of this section is to present some properties of the set $E(\mathscr{P})$ of extensions. In the following we shall always consider the relative weak *-topology on $M_1(\mathscr{X}, \mathscr{A})$. The first proposition is trivial.

Proposition 1.

(a) *If* $\mathscr{P} \subset M_1(\mathscr{X}, \mathscr{A}_0)$ *is convex, then* $E(\mathscr{P})$ *is convex.*

(b) $E(\bar{\mathscr{P}}) \supset \overline{E(\mathscr{P})}$ *($\bar{A}$ denoting the closure of A).*

Generally $E(\bar{\mathscr{P}}) \neq \overline{E(\mathscr{P})}$ (consider for example $X = [0, \Omega)$, Ω first uncountable ordinal number, $\mathscr{A}_0$ σ-algebra generated by $[0, \alpha)$, $0 \leqslant \alpha < \Omega$, $\mathscr{A}$ generated by $\mathscr{A}_0 \cup \{\Omega\}$, and $\mathscr{P} = \{\delta_\alpha \mid 0 \leqslant \alpha < \Omega\}$), and for $P \in M_1(\mathscr{X}, \mathscr{A}_0)$ the set $E(P)$ may be empty. (For a discussion of this point cf. Lipecki [8].) A simple condition for $E(P) \neq \phi$ and an interesting extension is given by the following proposition. Let $M(\mathscr{X}, \mathscr{A})$ denote the set of all measures μ on $(\mathscr{X}, \mathscr{A})$ such that $\mu \mid \mathscr{A}_0$ is σ-finite.

Proposition 2. *For* $\mu \in M(\mathscr{X}, \mathscr{A})$ *and* $P \ll \mu \mid \mathscr{A}_0$ *define* $P_\mu(A) = \int_A \frac{dP}{d\mu \mid \mathscr{A}_0} d\mu$, $A \in \mathscr{A}$, *and* $(E(P))_\mu = \{Q \in E(P) \mid Q \ll \mu\}$. *Then it holds:*

(a) $P_\mu \in E(P)$,

(b) $Q \ll P_\mu$ *for* $Q \in (E(P))_\mu$.

Proof.

(a) is immediate by definition.

(b) If $A \in \mathscr{A}$, $P_\mu(A) = 0$, then $1_A \frac{dP}{d\mu \mid \mathscr{A}_0} = 0[\mu]$ and, therefore, $1_A \frac{dP}{d\mu \mid \mathscr{A}_0} = 0[Q]$. Furthermore, $\frac{dP}{d\mu \mid \mathscr{A}_0} > 0[P]$ implies that $\frac{dP}{d\mu \mid \mathscr{A}_0} > 0[Q]$ and, therefore, $1_A = 0[Q]$.

P_μ is the uniquely determined μ-continuous extension which has a $\mathscr{A}_0$-measurable density. The set of all μ-continuous extensions is given by the following proposition. Let $L(\mathscr{A})$ denote the $\mathscr{A}$-measurable real functions.

Proposition 3. $\mu \in M(\mathscr{X}, \mathscr{A})$ *and* $P \ll \mu \mid \mathscr{A}_0$ *implies* $E(P)_\mu = \{h\mu \mid h \in L(\mathscr{A}),\ h \geqslant 0,\ E_\mu(h \mid \mathscr{A}_0) = \frac{dP}{d\mu \mid \mathscr{A}_0}\}$.

Proof. If $Q \in E(P)_\mu$, then by the theorem of Radon–Nikodym $Q = h\mu$, $h \in L(\mathscr{A})$, $h \geqslant 0$, and for $A_0 \in \mathscr{A}_0$, $Q(A_0) = \int_{A_0} h \, \mathrm{d}\mu = P(A_0) = \int_{A_0} \frac{\mathrm{d}P}{\mathrm{d}\mu \mid \mathscr{A}_0} \mathrm{d}\mu$. Therefore, $E_\mu(h \mid \mathscr{A}_0) = \frac{\mathrm{d}P}{\mathrm{d}\mu \mid \mathscr{A}_0} [\mu \mid \mathscr{A}_0]$. If $E_\mu(h \mid \mathscr{A}_0) = \frac{\mathrm{d}P}{\mathrm{d}\mu \mid \mathscr{A}_0}$, then $\int_{A_0} h \, \mathrm{d}\mu = \int_{A_0} E_\mu(h \mid \mathscr{A}_0) \, \mathrm{d}\mu = P_\mu(A_0) = P(A_0)$ and, therefore, $h\mu \in E(P)_\mu$.

Corollary 1. *Let* $Q \in E(P)$, $f \in L_1(\mathscr{A}, Q)$ *with* $f \geqslant 0$ *and assume* $E_Q(f \mid \mathscr{A}_0) > 0$. *Define* $h = \frac{f}{E_Q(f \mid \mathscr{A}_0)}$ *and* $Q^{(f)} = hQ$, *then* $Q^{(f)} \in E(P)$.

A consequence of Corollary 1 is, that if there is any extension of P, then there are many extensions. The following proposition expresses this fact in statistical terms: If a test $\varphi \in \Phi$ is similar w.r.t. $E(P)_Q$ then φ is Q-a.s. $\mathscr{A}_0$-measurable, where Φ denotes the set of all tests.

Proposition 4. *Let* $Q \in E(P)$, $\varphi \in \Phi$ *and* $E_Q \varphi = E_{Q^{(f)}} \varphi$ *for all* $f \in L_1(\mathscr{A}, Q)$ *with* $f \geqslant 0$ *and* $E_Q(f \mid \mathscr{A}_0) > 0$. *Then* $\varphi = E_Q(\varphi \mid \mathscr{A}_0)[Q]$.

Proof. Because of $A_0 := \{E_Q(\varphi \mid \mathscr{A}_0) = 0\} \subset \{\varphi = 0\}[Q]$ one obtains $\varphi = E_Q(\varphi \mid \mathscr{A}_0)[Q]$ on A_0 and

$$(*) \qquad \int \varphi \, \mathrm{d}Q = \int_{A_0^c} E_Q(\varphi \mid \mathscr{A}_0) \, \mathrm{d}Q = \int_{A_0^c} \frac{E_Q(\varphi f \mid \mathscr{A}_0)}{E_Q(f \mid \mathscr{A}_0)} \, \mathrm{d}Q.$$

Choosing $f = \varphi + \delta$, $\delta > 0$ one gets by the lemma of Fatou for $\delta \to 0$

$$\int_{A_0^c} E_Q(\varphi \mid \mathscr{A}_0) \, \mathrm{d}Q \geqslant \int_{A_0^c} \frac{E_Q(\varphi^2 \mid \mathscr{A}_0)}{E_Q(\varphi \mid \mathscr{A}_0)} \, \mathrm{d}Q.$$

By Jensen's inequality $E_Q(\varphi^2 \mid \mathscr{A}_0) \geqslant E_Q^2(\varphi \mid \mathscr{A}_0)[Q]$, which yields $E_Q(\varphi^2 \mid \mathscr{A}_0) = E_Q^2(\varphi \mid \mathscr{A}_0)[Q]$ on A_0^c implying $\varphi = E_Q(\varphi \mid \mathscr{A}_0)[Q]$ on A_0^c.

Together, we have $\varphi = E_Q(\varphi \mid \mathscr{A}_0)[Q]$.

The extreme points of $E(P)$ are characterized by a theorem due to Douglas [3]. Our special extensions of Proposition 2 allow to prove (a slight modification of) Douglas' result by probabilistic methods without using the Hahn–Banach theorem.

Proposition 5. *Let $Q \in E(P)$. Then Q is an extreme point of $E(P)$ if and only if it holds that $\varphi = E_Q(\varphi \mid \mathscr{A}_0)[Q]$ for all $\varphi \in \Phi$ or, equivalently, if for all $A \in \mathscr{A}$ there is a $B \in \mathscr{A}_0$ with $Q(A \,\Delta\, B) = 0$.*

Proof. If $Q = \alpha Q_1 + (1-\alpha)Q_2$, $Q_i \in E(P)$, $i = 1, 2$, then $Q_1 \ll Q$ and $\frac{dQ_1}{dQ} \leqslant \frac{1}{\alpha}$, $\alpha \in (0, 1)$. From $\frac{dQ_1}{dQ} = E_Q\left(\frac{dQ_1}{dQ} \mid \mathscr{A}_0\right)[Q]$ and $Q \mid \mathscr{A}_0 = Q_1 \mid \mathscr{A}_0$ we obtain $Q = Q_1$.

For $\varphi \in \Phi$ define $h = \frac{1 + \varphi}{E_Q(1 + \varphi \mid \mathscr{A}_0)} - 1$ then $-1 \leqslant h \leqslant 1$ and by Corollary 1 $(1+h)Q, (1-h)Q \in E(P)$ and $Q = \frac{1}{2}[(1+h)Q + (1-h)Q]$. Therefore, $\varphi = E_Q(\varphi \mid \mathscr{A}_0)[Q]$.

Remark 1. In a similar way one could also give a simple proof of the following characterization of the extreme points of the set $E^G(P)$ of all extensions of P which are invariant w.r.t. a semigroup G, namely: Let $Q \in E^G(P)$; then Q is an extreme point of $E^G(P)$ iff for all $\varphi \in \Phi$ which are Q-almost G-invariant $\varphi = E_Q(\varphi \mid \mathscr{A}_0)[Q]$ holds. This result is due to L u s c h g y [9], Theorem 4.4.

An important property of extensions gives the following proposition. For a set A in a vector space let $\operatorname{con}(A)$ be the convex hull of A.

Proposition 6.

(a) *If $P_i \in M_1(\mathscr{X}, \mathscr{A}_0)$, $E(P_i) \neq \phi$, $i = 0, 1$, then for all $\alpha \in [0, 1]$ and*

$$E(\alpha P_0 + (1-\alpha)P_1) = \alpha E(P_0) + (1-\alpha)E(P_1).$$

(b) *If $\mathscr{P} \subset M_1(\mathscr{X}, \mathscr{A}_0)$ with $E(P) \neq \phi$ for all $P \in \mathscr{P}$, then $E(\operatorname{con}\mathscr{P}) = \operatorname{con} E(\mathscr{P})$.*

Proof.

(a) If $Q_i \in E(P_i)$, $i = 0, 1$, then trivially, $\alpha Q_0 + (1-\alpha)Q_1 \in E(\alpha P_0 + (1-\alpha)P_1)$. Let $Q \in E(\alpha P_0 + (1-\alpha)P_1)$ and $\alpha \in (0, 1)$ then $P_0 \ll Q \mid \mathscr{A}_0$ and so we obtain by Proposition 2 $\tilde{P}_0(A) = \int_A \frac{dP_0}{dQ \mid \mathscr{A}_0}\, dQ$

defines an extension of P_0 with $\widetilde{P}_0(A) \leqslant \frac{1}{\alpha} Q(A)$. Define $\widetilde{P}_1(A) = \frac{1}{1-\alpha}(Q(A) - \alpha\widetilde{P}_0(A))$. Then $\widetilde{P}_1 \in E(P_1)$ and $Q = \alpha\widetilde{P}_0 + (1-\alpha)\widetilde{P}_1$.

(b) is immediate from (a).

Proposition 7. *Let $P_i \in M_1(\mathscr{X}, \mathscr{A}_0)$, $i = 0, 1$, and $P_0 = hP_1$ where $h \in L_1(\mathscr{A}_0, P_1)$ with $h \geqslant 0$. If $E(P_1) \neq \phi$, then*

$$E(P_0) = hE(P_1) = \{hQ \mid Q \in E(P_1)\}.$$

Proof. If $Q_1 \in E(P_1)$, then for $A_0 \in \mathscr{A}_0$

$$(hQ_1)(A_0) = \int_{A_0} h \,\mathrm{d}Q_1 = \int_{A_0} h \,\mathrm{d}P_1 = P_0(A_0).$$

If $h > 0[P_1]$, then $P_1 = \frac{1}{h} P_0$ and, therefore, by the first inclusion $E(P_1) \supset \frac{1}{h} E(P_0) \supset E(P_1)$ which implies $E(P_0) = hE(P_1)$. Assume now $0 < a = P_1(\{h = 0\}) < 1$ and define $\mathscr{X}' = \mathscr{X} \setminus \{h = 0\}$, $\mathscr{A}' = \mathscr{A} \cap \mathscr{X}'$, $\mathscr{A}'_0 = \mathscr{A}_0 \cap \mathscr{X}'$, $P'_0 = P_0 \mid \mathscr{A}'_0$, $P'_1 = \frac{1}{1-a} P_1 \mid \mathscr{A}'_0$ and $h' = h \mid \mathscr{X}'$. Then $P'_0 = (1-a)h'P'_1$ and $Q_0 \in E(P_0)$ implies $Q'_0 = Q_0 \mid \mathscr{A}' \in E(P'_0)$. Therefore, there exists $Q'_1 \in E(P'_1)$ such that $Q'_0 = (1-a)h'Q'_1$. Let $\widetilde{Q} \in E(P_1)$ and define $Q_1(A) = (1-a)Q'_1(A \cap \{h > 0\}) + \widetilde{Q}(A \cap \{h = 0\})$. We have for $A_0 \in \mathscr{A}_0$:

$$Q_1(A_0) =$$

$$(1-a)P'_1(A_0 \cap \{h > 0\}) + P_1(A_0 \cap \{h = 0\}) = P_1(A_0)$$

and

$$Q_0 = hQ_1.$$

Proposition 7 allows to describe the relation between extensions of two probability measures completely.

Proposition 8. *Let $P_0, P_1 \in M_1(\mathscr{X}, \mathscr{A}_0)$, let $P_0 = aP'_0 + (1-a)P''_0$ where $P'_0, P''_0 \in M_1(\mathscr{X}, \mathscr{A}_0)$ and $P'_0 = hP_1$, $P''_0 \perp P_1$, $a \in [0, 1]$.*

Assume that $E(P_i) \neq \phi$, $i = 0, 1$. Then

(a) $E(P_0'), E(P_0'') \neq \phi$.

(b) $E(P_0) = ahE(P_1) + (1-a)E(P_0'')$.

(c) $E(P_0'') \perp E(P_1)$.

Proof.

(a) Since $P_0', P_0'' \ll P_0$ (a) is immediate from Proposition 2.

(b) By Proposition 6 and 7 $E(P_0) = aE(P_0') + (1-a)E(P_0'') = ahE(P_1) + (1-a)E(P_0'')$.

(c) There exists a $A \in \mathscr{A}_0$ with $P_0''(A^c) = 0$, $P_1(A) = 0$. Therefore, $Q(A^c) = 0$ for $Q \in E(P_0'')$ and $R(A) = 0$ for $R \in E(P_1)$. This implies that $E(P_0'') \perp E(P_1)$.

A wellknown criterion for sufficiency implies the following

Corollary 2. *Let P_i, $i = 0, 1$, be as in Proposition 8. Let $Q_i \in E(P_i)$, $i = 0, 1$. Then $\mathscr{A}_0$ is sufficient for $\{Q_0, Q_1\}$ iff*

$$Q_0 \in ahQ_1 + (1-a)E(P_0'')$$

with $h \in L_1(\mathscr{A}_0, P_1)$ and $h \geqslant 0$.

For $k \geqslant 0$ and $P, Q \in M_1(\mathscr{X}, \mathscr{A})$ define the distances

$$d_k(Q, P) = \|Q - kP\| =$$

$$\sup\{Q(A) - kP(A) - (Q(B) - kP(B)),\ A, B \in \mathscr{A}\},$$

and for $P, Q \subset M_1(\mathscr{X}, \mathscr{A})$

$$d_k(\mathscr{P}, \mathscr{Q}) = \inf\{d_k(P, Q) \mid P \in \mathscr{P},\ Q \in \mathscr{Q}\}.$$

Proposition 9. *Let $\mathscr{P}_i \subset M_1(\mathscr{X}, \mathscr{A}_0)$, $i = 0, 1$, let $E(P) \neq \phi$ for all $P \in \mathscr{P}_0 \cup \mathscr{P}_1$ and let $k \geqslant 0$.*

(a) *If $\mu \in M(\mathscr{X}, \mathscr{A})$, $P_0, P_1 \in M_1(\mathscr{X}, \mathscr{A}_0)$, with $P_i \ll \mu \mid \mathscr{A}_0$, $i = 0, 1$, then*

$$d_k(P_0, P_1) = d_k(P_{0,\mu}, P_{1,\mu}).$$

(b) $d_k(\mathscr{P}_0, \mathscr{P}_1) = d_k(E(\mathscr{P}_0), E(\mathscr{P}_1))$.

Proof.

(a) It is easy to see that

$$d_k(P_0, P_1) = \| P_0 - kP_1 \| = \max\{k - 1 + 2(P_0 - kP_1)_+(\mathscr{X})\},$$

where $(P_0 - kP_1)_+$ is the positive part of the Jordan–Hahn decomposition of $P_0 - kP_1$. Therefore, with

$$A = \left\{ \frac{dP_0}{d\mu \mid \mathscr{A}_0} \geq k \frac{dP_1}{d(\mu \mid \mathscr{A}_0)} \right\}$$

$$(P_0 - kP_1)_+(\mathscr{X}) = (P_0 - kP_1)(A) = (P_{0,\mu} - kP_{1,\mu})(A).$$

(b) For $P_i \in \mathscr{P}_i$ and $Q_i \in E(P_i)$, $i = 0, 1$,

$$\sup\{|P_0(A) - kP_1(A)|;\ A \in \mathscr{A}_0\} =$$

$$\sup\{|Q_0(A) - kQ_1(A)|;\ A \in \mathscr{A}_0\} \leq$$

$$\sup\{|Q_0(A) - kQ_1(A)|;\ A \in \mathscr{A}\}.$$

Thus, $\| P_0 - kP_1 \| \leq \| Q_0 - kQ_1 \|$.

By part (a) equality holds for $Q_i = P_{i,\mu}$, $i = 0, 1$. This implies $d_k(\mathscr{P}_0, \mathscr{P}_1) = d_k(E(\mathscr{P}_0), E(\mathscr{P}_1))$.

3. MAXIMIN-TESTS AND LEAST FAVOURABLE DISTRIBUTIONS

Let $\mathscr{P}_0, \mathscr{P}_1 \in M_1(\mathscr{X}, \mathscr{A}_0)$ and let $\mathscr{P}'_i \subset E(\mathscr{P}_i)$, $i = 0, 1$, such that $\overline{\text{con}}\, \mathscr{P}'_i \cap E(P) \neq \phi$, for all $P \in \mathscr{P}_i$, $i = 0, 1$. For the testproblem $\mathscr{P}'_0, \mathscr{P}'_1$ denote the maximin-risk for $\alpha \in [0, 1]$ by

$$\beta(\alpha, \mathscr{P}'_0, \mathscr{P}'_1) = \sup_{\varphi \in \Phi_\alpha(\mathscr{P}'_0, \mathscr{A})} \ \inf_{Q \in \mathscr{P}'_1} E_Q \varphi,$$

where $\Phi_\alpha(\mathscr{P}'_0, \mathscr{A}) = \{\varphi\colon (\mathscr{X}, \mathscr{A}) \to ([0, 1], [0, 1]\mathscr{B}^1) \mid E_Q \varphi \leq \alpha$ for all $Q \in \mathscr{P}'_0\}$ and $\mathscr{B}^1$ is the Borel σ-algebra.

The general assumption $\overline{\text{con}}\, \mathscr{P}'_i \cap E(P) \neq \phi$ for $P \in \mathscr{P}_i$, $i = 0, 1$, implies that $\beta(\alpha, \mathscr{P}'_0, \mathscr{P}'_1) \geq \beta(\alpha, \mathscr{P}_0, \mathscr{P}_1)$. The following theorem gives a sufficient conditions to imply that a maximin solution for $\mathscr{P}_0, \mathscr{P}_1$ is even a solution for $\mathscr{P}'_0, \mathscr{P}'_1$ i.e. optimality of a test for $\mathscr{P}_0, \mathscr{P}_1$ is inherited to the testproblem $\mathscr{P}'_0, \mathscr{P}'_1$.

Theorem 10. *If* $\mathscr{P}_i \ll \mu$, $i = 0, 1$, *and if* $d_k(\text{con}\,\mathscr{P}'_1, \text{con}\,\mathscr{P}'_0) = d_k(\text{con}\,\mathscr{P}_1, \text{con}\,\mathscr{P}_0)$ *for all* $k \geq 0$ *then* $\beta(\alpha, \mathscr{P}'_0, \mathscr{P}'_1) = \beta(\alpha, \mathscr{P}_0, \mathscr{P}_1)$ *for all* $\alpha \in (0, 1]$.

Proof. By Baumann [1], Satz 6.3, for a dominated testproblem H, K on $\mathscr{X}$ one has $\beta(\alpha, H, K) = \min\{\alpha k + (Q - kP)_+(\mathscr{X}) \mid k \geq 0,\ Q \in \overline{\text{con}\,K},\ P \in \overline{\text{con}\,H}\}$ (the closure w.r.t. relative weak *-topology). Using $(Q - kP)_+(\mathscr{X}) = \frac{1-k}{2} + \frac{1}{2} d_k(Q, P)$ and the fact that for $\epsilon > 0$ there is a measure $\tilde{\mu}$ on $(\mathscr{X}, \mathscr{A})$ such that $d_k((\text{con}\,\mathscr{P}'_0)_{\tilde{\mu}}, (\text{con}\,\mathscr{P}'_1)_{\tilde{\mu}}) \leq d_k(\text{con}\,\mathscr{P}'_0, \text{con}\,\mathscr{P}'_1) + \epsilon$ (cf. the proof of Satz 6.3 in [1]), we obtain

$$\begin{aligned}
&\beta(\alpha, \mathscr{P}'_0, \mathscr{P}'_1) \leq \beta(\alpha, (\mathscr{P}'_0)_{\tilde{\mu}}, (\mathscr{P}'_1)_{\tilde{\mu}}) = \\
&\inf\left\{\alpha k + \frac{1-k}{2} + \frac{1}{2} d_k(\text{con}\,(\mathscr{P}'_1)_{\tilde{\mu}}, \text{con}\,(\mathscr{P}'_0)_{\tilde{\mu}})\right\} = \\
&\inf\left\{\alpha k + \frac{1-k}{2} + \frac{1}{2} d_k((\text{con}\,\mathscr{P}'_1)_{\tilde{\mu}}, (\text{con}\,\mathscr{P}'_0)_{\tilde{\mu}})\right\} \leq \\
&\inf\left\{\alpha k + \frac{1-k}{2} + \frac{1}{2} d_k(\text{con}\,\mathscr{P}'_1, \text{con}\,\mathscr{P}'_0)\right\} + \epsilon = \\
&\inf\left\{\alpha k + \frac{1-k}{2} + \frac{1}{2} d_k(\text{con}\,\mathscr{P}_1, \text{con}\,\mathscr{P}_0)\right\} + \epsilon = \\
&\beta(\alpha, \mathscr{P}_0, \mathscr{P}_1) + \epsilon, \quad \forall \epsilon > 0.
\end{aligned}$$

Observe that by Proposition 9 the assumptions of Theorem 10 are fulfilled for $\mathscr{P}'_i = E(\mathscr{P}_i)$. Without assuming that $\mathscr{P}_0 \cup \mathscr{P}_1$ is dominated we obtain:

Theorem 11. *If for each* $P \in \mathscr{P}_i$ *there exists a* $Q_P \in E(P) \cap \overline{\text{con}\,\mathscr{P}'_i}$, $i = 0, 1$, *such that* $\mathscr{A}_0$ *is sufficient for* $M = \{Q_P \mid P \in \mathscr{P}_0 \cup \mathscr{P}_1\}$, *then* $\beta(\alpha, \mathscr{P}'_0, \mathscr{P}'_1) = \beta(\alpha, \mathscr{P}_0, \mathscr{P}_1)$.

Proof. Let $\varphi \in \Phi_\alpha(\mathscr{P}'_0, \mathscr{A})$ and define $\psi = E_\mu(\varphi \mid \mathscr{A}_0)$, $\mu \in M$. Then for $P \in \mathscr{P}_0$ $E_P \psi = E_{Q_P} \psi = E_{Q_P} \varphi \leq \alpha$ which implies $\psi \in \Phi_\alpha(\mathscr{P}'_0, \mathscr{A})$. Furthermore, for $P \in \mathscr{P}_1$, and $Q \in \mathscr{P}'_1 \cap E(P)$ it holds that $E_Q \psi = E_P \psi = E_{Q_P} \psi = E_{Q_P} \varphi$ and, therefore, $\inf_{Q \in \mathscr{P}'_1} E_Q \varphi \leqslant \inf_{Q \in \mathscr{P}'_1} E_Q \psi$. This implies that $\beta(\alpha, \mathscr{P}'_0, \mathscr{P}'_1) = \beta(\alpha, \mathscr{P}_0, \mathscr{P}_1)$.

Remark 2.

(a) The condition of Theorem 11 corresponds to an assumption made by Hajek [5] in the case of estimation in the presence of nuisance parameters. The conclusion of Theorem 11 could be strengthened to

$$\beta(\alpha, \mathscr{P}'_0, E(P) \cap \overline{\operatorname{con} P'_1}) = \beta(\alpha, \mathscr{P}_0, P) \quad \text{for all } P \in \mathscr{P}_1.$$

(b) If $\mathscr{P}_i = \{P_i\}$, $i = 0, 1$, then the condition that there exist $Q_P \in E(P_i) \cap \overline{\operatorname{con} \mathscr{P}'_i}$, $i = 0, 1$, such that $\mathscr{A}_0$ is sufficient for $\{Q_{P_0}, Q_{P_1}\}$ is equivalent to the assumption that there exists a $\mu \in M(\mathscr{X}, \mathscr{A})$ with $P_0, P_1 \ll \mu \mid \mathscr{A}_0$ and $Q_i = P_{i,\mu} \in \overline{\operatorname{con} \mathscr{P}'_i}$, $i = 0, 1$ (cf. also Corollary 2). The determination of a maximin-test is simplified in the presence of least favourable pairs. In the literature there are three different definitions of least favourable pairs for the testproblem $\mathscr{P}'_0, \mathscr{P}'_1$.

Let $P_i \in \overline{\operatorname{con} \mathscr{P}'_i}$, $i = 0, 1$, then

(a) $(P_0, P_1) \in \mathrm{LF}_\alpha(\mathscr{P}'_0, \mathscr{P}'_1)$ iff $\beta(\alpha, P_0, P_1) = \beta(\alpha, \mathscr{P}'_0, \mathscr{P}'_1)$, $(P_0, P_1) \in \mathrm{LF}(\mathscr{P}'_0, \mathscr{P}'_1)$ iff $(P_0, P_1) \in \bigcap_{\alpha \in [0,1]} \mathrm{LF}_\alpha(\mathscr{P}'_0, \mathscr{P}'_1)$ (cf. Baumann [1]).

(b) $(P_0, P_1) \in \widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1)$ iff there exists $\pi \in \frac{\mathrm{d}P_1}{\mathrm{d}P_0}$ with $P^\pi \leqslant_{\mathrm{st}} P^\pi_0$ for $P \in \mathscr{P}'_0$, $Q^\pi \geqslant_{\mathrm{st}} P^\pi_1$ for $Q \in \mathscr{P}'_1$ where $\leqslant_{\mathrm{st}}$ is the stochastic order and $\frac{\mathrm{d}P_1}{\mathrm{d}P_0} = \left\{\frac{f_1}{f_0}, f_i = \frac{\mathrm{d}P_i}{\mathrm{d}P_\mu}, i = 0, 1, \mu \text{ dominates } P_i, i = 0, 1\right\}$ (cf. Huber, Strassen [6], Rieder [11]).

(c) $(P_0, P_1) \in \mathrm{LF}'_\alpha(\mathscr{P}'_0, \mathscr{P}'_1)$ iff $\beta_\alpha(P_0, P_1) \leqslant \beta_\alpha(Q_0, Q_1)$ for all $Q_i \in \overline{\operatorname{con} \mathscr{P}'_i}$, $i = 0, 1$, $(P_0, P_1) \in \mathrm{LF}'(\mathscr{P}'_0, \mathscr{P}'_1)$ iff $(P_0, P_1) \in \bigcap_{\alpha \in [0,1]} \mathrm{LF}'_\alpha(\mathscr{P}'_0, \mathscr{P}'_1)$ (cf. Lehmann [7], pg. 325).

The following remark is concerned with connections of these notions of least favourable pairs and with methods to find least favourable pairs.

Remark 3.

(1a) $(P_0, P_1) \in \mathrm{LF}_\alpha(\mathscr{P}'_0, \mathscr{P}'_1)$ iff there exists a most powerful level α test for P_0, P_1 which is maximin-test for $\mathscr{P}'_0, \mathscr{P}'_1$ at level α.

(1b) $(P_0, P_1) \in \widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1)$ iff there exists a $\pi \in \frac{\mathrm{d}P_1}{\mathrm{d}P_0}$ such that $\varphi_{\pi,\alpha}$ (the LQ-test at level α which is constant on the randomized region) is maximin-test for $\mathscr{P}'_0, \mathscr{P}'_1$ for each $\alpha \in [0, 1]$.

(1c) From (1b) follows $\widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1) \subset \mathrm{LF}(\mathscr{P}'_0, \mathscr{P}'_1)$ (there is no equality in general). Equality holds if for instance the distribution of $\pi \in \frac{\mathrm{d}P_1}{\mathrm{d}P_0}$ is nonatomic under $\mathscr{P}'_i$, $i = 0, 1$, for all $(P_0, P_1) \in \mathrm{LF}(\mathscr{P}'_0, \mathscr{P}'_1)$.

(1d) If $\widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1) \neq \phi$, then $\mathrm{LF}(\mathscr{P}'_0, \mathscr{P}'_1) = \mathrm{LF}'(\mathscr{P}'_0, \mathscr{P}'_1)$ (cf. Rieder [11], Proposition 2.2).

(2) $\widetilde{\mathrm{LF}}((\mathscr{P}'_0)^{(n)}, (\mathscr{P}'_1)^{(n)}) = (\widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1))^{(n)}$, where $(\mathscr{P}'_i)^{(n)} = \{P^{(n)}: P \in \mathscr{P}'_i\}$ (cf. Huber, Strassen [6], Corollary 4).

(3a) $(P_0, P_1) \in \widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1) \Rightarrow d_k(P_0, P_1) = d_k(\mathrm{con}\,\mathscr{P}'_0, \mathrm{con}\,\mathscr{P}'_1)$ for all $k \geq 0$.

(3b) If $\widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1) \neq \phi$, $P_i \in \mathrm{con}\,\mathscr{P}'_i$, $i = 0, 1$, with $d_k(P_0, P_1) = d_k(\mathrm{con}\,\mathscr{P}'_0, \mathrm{con}\,\mathscr{P}'_1)$ for all $k \geq 0$, then $(P_0, P_1) \in \widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1)$. For similar facts concerning $\mathrm{LF}(\mathscr{P}'_0, \mathscr{P}'_1)$, cf. Reinhardt [10].

(4) Elements of $\widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1)$ can also be determined by minimization of certain different distance measures containing for example the measure of divergence of Csiszár [2]. Let $\varphi: [0, 1] \to \mathbf{R}^1$ be twice continuously differentiable with $\varphi'' > 0$ and define:

$$H(P, Q) = \int \varphi\left(\frac{\mathrm{d}P}{\mathrm{d}(P + Q)}\right) \mathrm{d}(P + Q)$$

for probability measures P, Q on $(\mathscr{X}, \mathscr{A})$. Then by a slight modification of Theorem 6.1 of Huber, Strassen [6]:

$$H(\mathscr{P}, \mathscr{Q}) = \inf\{H(P,Q),\ P \in \mathscr{P},\ Q \in \mathscr{Q}\}.$$

(a) $(P_0, P_1) \in \widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1) \Rightarrow H(P_0, P_1) = H(\mathrm{con}\,\mathscr{P}'_0, \mathrm{con}\,\mathscr{P}'_1)$.

(b) If $P_i \in \overline{\mathrm{con}}\,\mathscr{P}'_i$, $i = 0, 1$, $\widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1) \neq \phi$ and $H(P_0, P_1) = H(\mathrm{con}\,\mathscr{P}'_0, \mathrm{con}\,\mathscr{P}'_1)$, then $(P_0, P_1) \in \widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1)$.

Returning to our testproblem $\mathscr{P}'_0 \subset E(\mathscr{P}_0)$, $\mathscr{P}'_1 \subset E(\mathscr{P}_1)$ we have

Theorem 12. *If* $(P_0, P_1) \in \mathrm{LF}_\alpha(\mathscr{P}_0, \mathscr{P}_1)$ $(\widetilde{\mathrm{LF}}(\mathscr{P}_0, \mathscr{P}_1))$ *and if there exist* $Q_i \in \overline{\mathrm{con}}\,\mathscr{P}'_i \cap E(P_i)$, $i = 0, 1$, *such that* $\mathscr{A}_0$ *is sufficient for* $\{Q_0, Q_1\}$, *then*

(a) $(Q_0, Q_1) \in \mathrm{LF}_\alpha(\mathscr{P}'_0, \mathscr{P}'_1)$ $(\widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1))$.

(b) *There is a most powerful level* α *test* φ_α *for* P_0, P_1 *which is maximin-test at level* α *for* $\mathscr{P}'_0, \mathscr{P}'_1$. ($\varphi_{\pi,\alpha}$ *is maximin-test at level* α *for* $\mathscr{P}'_0, \mathscr{P}'_1$ *where* $\pi \in \frac{\mathrm{d}P_1}{\mathrm{d}P_0}$).

Proof.

I. Let $(P_0, P_1) \in \mathrm{LF}_\alpha(\mathscr{P}_0, \mathscr{P}_1)$.

(1) By Remark 3 (1a) there is a most powerful level α test φ_α for P_0, P_1 which is maximin-test for $\mathscr{P}_0, \mathscr{P}_1$. Therefore, $\varphi_\alpha \in \Phi_\alpha(\mathscr{P}'_0, \mathscr{A})$.

(2) Let $\varphi \in \Phi_\alpha(\mathscr{P}'_0, \mathscr{A})$ and define $\psi = E_{\{Q_0, Q_1\}}(\varphi \mid \mathscr{A}_0)$. Then $\inf_{Q \in \mathscr{P}'_1} E_Q \varphi \leq E_{Q_1}\varphi$, since $Q_1 \in \overline{\mathrm{con}}\,\mathscr{P}'_1$, and $E_{Q_1}(\varphi) = E_{Q_1}\psi = E_{P_1}\psi$.

Clearly $\psi \in \Phi_\alpha(\mathscr{P}_0, \mathscr{A}_0)$ and, therefore,

$$E_{P_1}\psi \leq E_{P_1}\varphi_\alpha = \inf_{P \in \mathscr{P}_1} E_P \varphi_\alpha = \inf_{Q \in \mathscr{P}'_1} E_Q \varphi_\alpha.$$

This implies $\beta(\alpha, Q_0, Q_1) = \beta(\alpha, P_0, P_1) = \beta(\alpha, \mathscr{P}'_0, \mathscr{P}'_1)$, i.e. $(Q_0, Q_1) \in \mathrm{LF}_\alpha(\mathscr{P}'_0, \mathscr{P}'_1)$ and clearly φ_α is a maximin-test for $\mathscr{P}'_0, \mathscr{P}'_1$.

II. The case $(P_0, P_1) \in \widetilde{\mathrm{LF}}(\mathscr{P}'_0, \mathscr{P}'_1)$ is similar.

Corollary 3. *If* $(P_0, P_1) \in \mathrm{LF}(\mathscr{P}_0, \mathscr{P}_1)$ *and if there exist* $Q_i \in \overline{\mathrm{con}}\,\mathscr{P}'_i \cap E(P_i)$, $i = 0, 1$, *such that* $\mathscr{A}_0$ *is sufficient for* $\{Q_0, Q_1\}$, *then*

$$(Q_0^{(n)}, Q_1^{(n)}) \in \widetilde{\mathrm{LF}}(({\mathscr P}'_0)^{(n)}, ({\mathscr P}'_1)^{(n)}).$$

Proof. Observe that

$$\overline{\mathrm{con}}\,(E(P) \times E(Q)) \subset E(P \times Q)$$

for $P, Q \in M_1(\mathscr{X}, \mathscr{A})$ and, therefore, $Q_i^{(n)} \in E(P_i^{(n)})$, $i = 0, 1$. By Remark 3.2 $(P_0^{(n)}, P_1^{(n)}) \in \widetilde{\mathrm{LF}}(\mathscr{P}_0^{(n)}, \mathscr{P}_1^{(n)})$ and, furthermore, $\mathscr{A}_0^{(n)}$ (the n-fold product of $\mathscr{A}_0$) is sufficient for $\{Q_0^{(n)}, Q_1^{(n)}\}$. Therefore, Corollary 3 follows from Theorem 12.

Corollary 4. *Let* $(P_0, P_1) \in \mathrm{LF}_\alpha(\mathscr{P}_0, \mathscr{P}_1)$ *and let* φ_α *be a most powerful level* α *test for* P_0, P_1, *which is maximin for* $\mathscr{P}_0, \mathscr{P}_1$. *Then* φ_α *is maximin-test at level* α *for* $E(\mathscr{P}_0), E(\mathscr{P}_1)$.

4. UNIFORMLY MOST POWERFUL TESTS

Again let $\mathscr{P}_i \subset M_1(\mathscr{X}, \mathscr{A}_0)$ and $\mathscr{P}'_i \subset E(\mathscr{P}_i)$, $i = 0, 1$, and let $E(P) \neq \phi$, $\forall P \in \mathscr{P}_0 \cup \mathscr{P}_1$. For $P, Q \in M_1(\mathscr{X}, \mathscr{A}_0)$ let $\Phi_\alpha^*(P, Q)$ denote the set of most powerful level α tests for P, Q.

Theorem 13.

(a) *Let* φ_0 *be a* UMP-*test for* $\mathscr{P}_0, \mathscr{P}_1$ *at level* α,

(b) *Let there exist* $P_0 \in \mathscr{P}_0$ *such that for all* $P_1 \in \mathscr{P}_1$, $\varphi_0 \in \Phi_\alpha^*(P_0, P_1)$,

(c) *For all* $Q_1 \in \mathscr{P}'_1$ *let there exist a* $Q_0 \in \overline{\mathrm{con}}\,\mathscr{P}'_0 \cap E(P_0)$ *such that* $\mathscr{A}_0$ *is sufficient for* $\{Q_0, Q_1\}$.

Then φ_0 *is a* UMP-*test for* $\mathscr{P}'_0, \mathscr{P}'_1$ *at level* α.

Proof. Clearly $\varphi_0 \in \Phi_\alpha(\mathscr{P}'_0, \mathscr{A})$. Let $\varphi \in \Phi_\alpha(\mathscr{P}'_0, \mathscr{A})$ and let $Q_1 \in \mathscr{P}'_1$. Then there exists $P_1 \in \mathscr{P}_1$ such that $Q_1 \in E(P_1)$ and so by (c) there is a $Q_0 \in \overline{\mathrm{con}}\,\mathscr{P}'_0 \cap E(P_0)$ such that $\mathscr{A}_0$ is sufficient for $\{Q_0, Q_1\}$. Define $\psi = E_{\{Q_0, Q_1\}}(\varphi \mid \mathscr{A}_0)$, then $E_{P_0}\psi = E_{Q_0}\psi = E_{Q_0}\varphi \leqslant \alpha$ and, therefore, (b) implies

$$E_{Q_1}\varphi = E_{Q_1}\psi = E_{P_1}\psi \leqslant E_{P_1}\varphi_0 = E_{Q_1}\varphi_0.$$

This yields that φ_0 is UMP at level α for $\mathscr{P}'_0, \mathscr{P}'_1$.

Corollary 5. *Let* $P_i \in M_1(\mathscr{X}, \mathscr{A}_0)$ *with* $E(P_i) \neq \phi$, $i = 0, 1$, *and* $\varphi_0 \in \Phi^*_\alpha(P_0, P_1)$. *Then*

(a) φ_0 *is* UMP-*test at level* α *for* $E(P_0), E(P_1)$,

(b) *If* φ^* *is a* UMP-*test at level* α *for* $E(P_0), E(P_1)$ *and if* $E_{Q_1}\varphi_0 < 1$ *for all* $Q_1 \in E(P_1)$, *then* $\varphi^* = E_Q(\varphi^* \mid \mathscr{A}_0)[Q]$ *for all* $Q = \frac{1}{2}(Q_0 + Q_1)$, $Q_i \in E(P_i)$, $i = 0, 1$.

Proof.

(a) Let $P_0 = aP'_0 + (1-a)P''_0$ be a decomposition as in Proposition 8 and let $Q_1 \in E(P_1)$. Then $Q_0 := ahQ_1 + (1-a)\widetilde{Q}_0$ (with $\widetilde{Q}_0 \in E(P''_0)$ and h a version of $\frac{dP'_0}{dP_1}$) is an element of $E(P_0)$ such that $\mathscr{A}_0$ is sufficient for $\{Q_0, Q_1\}$. So (a) is implied by Theorem 13.

(b) For all $Q_0 \in E(P_0)$ we can find a $Q_1 \in E(P_1)$ (as in (a)) such that $\mathscr{A}_0$ is sufficient for $\{Q_0, Q_1\}$. Therefore, $\varphi_0 \in \Phi^*_\alpha(Q_0, Q_1)$ and, therefore, also $\varphi^* \in \Phi^*_\alpha(Q_0, Q_1)$. Since $\beta = E_{Q_1}\varphi_0 < 1$ we have $E_{Q_0}\varphi^* = \alpha$. This implies that φ^* is a UMP-test at level α for $E(P_0)$ against $\frac{1}{2}E(P_0) + \frac{1}{2}E(P_1) = E\left(\frac{1}{2}(P_0 + P_1)\right)$ by Proposition 6, and, therefore, $E_Q\varphi^* = \frac{\alpha+\beta}{2}$ for all $Q \in E\left(\frac{P_0 + P_1}{2}\right)$. Proposition 5 implies that $\varphi^* = E_Q(\varphi^* \mid \mathscr{A}_0)[Q]$ for all $Q \in E\left(\frac{1}{2}(P_0 + P_1)\right)$.

Remark 4. Corollary 5 generalizes a result of Fraser [4], Theorem 2, which is concerned with the case of nuisance parameters.

5. EXAMPLES

(1) Let $a \leqslant b$ and $0 \leqslant \alpha \leqslant \beta$, $\alpha + \beta \leqslant 1$ and consider the test-problem

$$\mathscr{P}'_0 = \{P^{(n)} \mid P \in M_1(\mathbf{R}^1, \mathscr{B}^1),\ P(-\infty, a] \leqslant \alpha,\ P(a, b] \leqslant \alpha\},$$

$$\mathscr{P}'_1 = \{P^{(n)} \mid P \in M_1(\mathbf{R}^1, \mathscr{B}^1),\ P(-\infty, a] \geqslant \alpha,\ P(a, b] \geqslant \beta\}.$$

Let $T: \mathbf{R}^n \to \mathbf{R}^2$. $T(x) = (s_1(x), s_2(x))$, where $s_1(x) = \sum_{i=1}^{n} 1_{(-\infty, a]}(x_i)$, $s_2(x) = \sum_{i=1}^{n} 1_{(a,b]}(x_i)$. Let $\mathscr{A}_0 = \mathscr{A}(T)$ the σ-algebra induced by T and let $\mathscr{P}_i$ denote the restriction of $\mathscr{P}'_i$ on $\mathscr{A}_0$, $i = 0, 1$, so that we have the situation considered in Sections 3, 4.

To determine $\mathscr{P}_i$ let $A_0 = (-\infty, a]$, $A_1 = (a, b]$, $A_2 = (b, \infty)$ and use the representation

$$\mathscr{P}'_0 = \Big\{\Big(\sum_{i=0}^{2} \alpha_i P_i\Big)^{(n)},\ P_i \in M_1(A_i, A_i \mathscr{L}_1),\quad i = 0, 1, 2,$$
$$0 \leqslant \alpha_i \leqslant 1,\ \sum \alpha_i = 1 \text{ and } \alpha_0 \leqslant \alpha,\ \alpha_1 \leqslant \alpha\Big\},$$

$$\mathscr{P}'_1 = \Big\{\Big(\sum \alpha_i P_i\Big)^{(n)}\ P_i \in M_1(A_i, A_i \mathscr{L}_1),\quad i = 0, 1, 2,$$
$$0 \leqslant \alpha_i \leqslant 1,\ \sum \alpha_i = 1,\ \alpha_0 \geqslant \alpha,\ \alpha_1 \geqslant \beta\Big\}.$$

If $Q = \big(\sum \alpha_i P_i\big)^{(n)} \in \mathscr{P}'_0 \cup \mathscr{P}'_1$, then

$$Q(s_1 = k,\ s_2 = m) = \binom{n}{k, m} \alpha_0^k \alpha_1^m \alpha_2^{n-(k+m)};$$

so $Q/\mathscr{A}_0 = Q(\alpha_0, \alpha_1)$ and

$$\mathscr{P}_0 = \{Q(\alpha_0, \beta_0) \mid \alpha_0 \leqslant \alpha,\ \beta_0 \leqslant \alpha\},$$
$$\mathscr{P}_1 = \{Q(\alpha_1, \beta_1) \mid \alpha_1 \geqslant \alpha,\ \beta_1 \geqslant \beta\}.$$

Let $Q_i = Q_i(\alpha_i, \beta_i) \in \mathscr{P}_i$, $i = 0, 1$, then

$$\frac{Q_1(s_1 = k,\ s_2 = m)}{Q_0(s_1 = k,\ s_2 = m)} =$$

$$\Big(\frac{\alpha_1(1-(\alpha_0+\beta_0))}{\alpha_0(1-(\alpha_1+\beta_1))}\Big)^k \Big(\frac{\beta_1(1-(\alpha_0+\alpha_0))}{\beta_0(1-(\alpha_1+\beta_1))}\Big)^m.$$

From this we easily obtain that

$$(Q(\alpha, \alpha), Q(\alpha, \beta)) \in \widetilde{\mathrm{LF}}(\mathscr{P}_0, \mathscr{P}_1)$$

with most powerful level α test of the type

$$\varphi_0(x) = \begin{cases} 1 & \text{if } \left(\frac{\beta}{\alpha}\right)^{s_1(x)} \left(\frac{1-2\alpha}{1-(\alpha+\beta)}\right)^{s_1(x)+s_2(x)} > k_\alpha, \\ 0 & \qquad\qquad\qquad\qquad \leqslant k_\alpha. \end{cases}$$

Clearly

$$Q_0 = (\alpha P_0 + \alpha P_1 + (1-2\alpha)P_2)^{(n)},$$

$$Q_1 = (\alpha P_0 + \beta P_1 + (1-(\alpha+\beta))P_2)^{(n)}$$

define extensions of $Q(\alpha,\alpha), Q(\alpha,\beta)$ in $\mathscr{P}'_i$, such that $\mathscr{A}_0$ is sufficient for $\{Q_0, Q_1\}$. (P_i are any elements of $M_1(A_i, A_i\mathscr{L}_1)$).

So by Theorem 12 φ_0 is a maximin-test at level α for $\mathscr{P}'_0, \mathscr{P}'_1$. Clearly no UMP-test exists in this situation.

(2) Let $P_{(\mu,\sigma^2)} = \bigotimes_{i=1}^{n} N(\mu,\sigma^2)$, $\mu \in \mathbf{R}^1$, $\sigma^2 > 0$ and consider the testproblem:

(a) $\mathscr{P}'_0 = \{P_{\mu,\sigma^2} \mid \sigma^2 \leqslant \sigma_0^2\}$, $\mathscr{P}'_1 = \{P_{\mu,\sigma^2} \mid \sigma^2 \geqslant \sigma_1^2\}$, where $\sigma_0^2 < \sigma_1^2$. If $s^2(x) = \sum_{i=1}^{n} (x_i - \bar{x})^2$, $\mathscr{A}_0 = \mathscr{A}(s^2)$, then $P^{s^2}_{\mu,\sigma^2} = h_\sigma \lambda^n$ has monotone likelihood ratio in σ^2 so that

$$\varphi_0 = \begin{cases} 1, & s^2(x) > k_\alpha, \\ 0, & s^2(x) \leqslant k_\alpha \end{cases}$$

is a UMP-test at level α for

$$\mathscr{P}_0 = \{h_\sigma \lambda^n \mid \sigma^2 \leqslant \sigma_0^2\},\ \mathscr{P}_1 = \{h_\sigma \lambda^n \mid \sigma^2 \geqslant \sigma_1^2\}.\ P_0 = h_{\sigma_0^2} \lambda^n$$

satisfies condition (b) of Theorem 13. For condition (c) let $\mu \in R^1$, $\sigma^2 \geqslant \sigma_1^2$ and $Q_1 = P_{(\mu,\sigma^2)}$. We are looking for

$$Q_0 \in \overline{\text{con}}\,\{P_{\mu',\sigma_0^2} \mid \mu' \in \mathbf{R}^1\},$$

such that s^2 is sufficient for $\{Q_0, Q_1\}$. Let P_{μ',σ_0^2} be the density of P_{μ',σ_0^2} with respect to λ^n, then

$$P_{\mu',\sigma_0^2}(x) = A(\sigma_0^2)\exp\left(-\frac{s^2}{2\sigma_0^2}\right)\exp\left(-\frac{n(\bar{x}-\mu')^2}{2\sigma_0^2}\right).$$

Using

$$N\left(\bar{x}, \frac{\sigma_0^2}{n}\right) * N\left(\mu - \bar{x}, \frac{\sigma^2}{n} - \frac{\sigma_0^2}{n}\right) = N\left(\mu, \frac{\sigma^2}{n}\right)$$

($*$ denoting convolution), we obtain with $\lambda_0 = N\left(\mu - \bar{x}, \frac{\sigma^2}{n} - \frac{\sigma_0^2}{n}\right)$, $Q_0 := \int P_{\mu',\sigma_0^2}\, d\lambda_0(\mu')$ has λ^n-density

$$A(\sigma_0^2, \sigma^2)\exp\left(-\frac{s^2}{2\sigma_0^2}\right)e^{-\frac{n(\bar{x}-\mu)^2}{2\sigma^2}},$$

which shows that s^2 is sufficient for $\{Q_0, Q_1\}$.

Theorem 13 implies that φ_0 is a UMP-test at level α for $\mathscr{P}'_0, \mathscr{P}'_1$. So in this well-known case we obtain an explanation why to choose the mixing measure λ_0.

(b) Similarly for the testing problem $\mathscr{P}'_0 = \{P_{\mu,\sigma^2} \mid \sigma_0^2 \leqslant \sigma^2 \leqslant K\}$ against $\mathscr{P}'_1 = \{P_{\mu,\sigma^2}\, \sigma^2 < \sigma_0^2\}$ we obtain that s^2 is sufficient for $\{Q_{\sigma^2} \mid 0 < \sigma^2 \leqslant K\}$, where $Q_{\sigma^2} = \int P_{\mu,\sigma^2}\, d\lambda_{\sigma^2}(\mu)$ and where $\lambda_{\sigma^2} = N\left(0, \frac{K-\sigma^2}{n}\right)$. So by Theorem 11

$$\varphi_0 = \begin{cases} 1, & s^2(x) > k_\alpha, \\ 0, & s^2(x) \leqslant k_\alpha \end{cases}$$

yields a maximin-test for $\mathscr{P}'_0, \mathscr{P}'_1$ (which is independent of K).

(3) Let $P_{\alpha,\beta} = f_{\alpha,\beta}\mu$ with $(\alpha, \beta) \in \Theta$ and let A be the projection of Θ onto the first component. Assume

(a) $f_{\alpha,\beta}(x) = f_\alpha(T(x))g_{\alpha,\beta}(x)$, $\forall x \in \mathscr{X}$, $\alpha \in A$, $\beta \in \Theta_\alpha$.

(b) For each $\alpha \in A$ there is a probability measure λ_α on Θ_α, such that $\int_B g_{\alpha,\beta}(x)\, d\lambda_\alpha(\beta) = Q(B)$, $B \in \mathscr{A}$, is independent of $\alpha \in A$, then by Theorem 11 testproblems with respect to α can be reduced to the testproblems for the distributions of T, when considering the maximin-risk. Examples are: $g_{\alpha,\beta}(x) = g_\beta(x)$ with $\lambda_\alpha = \epsilon_{\{\beta_0\}}$, $g_{\alpha,\beta}(x) = h_{\alpha-\beta}(x)$ with $\lambda_\alpha = \epsilon_{\{-\alpha\}}$ and $g_{\alpha,\beta}(x) = h_{\frac{\alpha}{\beta}}(x)$ with $\lambda_\alpha = \epsilon_{\{\alpha\}}$.

(4) Let G denote a finite group of order γ consisting of $(\mathscr{A}, \mathscr{A})$-measurable transformations $g: \mathscr{X} \to \mathscr{X}$ and introduce $\mathscr{A}_0$ as the sub-σ-algebra $\mathscr{A}_0$ of $\mathscr{A}$ consisting of all G-invariant sets belonging to $\mathscr{A}$. If P_i are probability measures on $\mathscr{A}$, $\mathscr{P}_i$, $i = 0, 1$, is defined to be the family $\{P_i^g \mid g \in G\}$, $i = 0, 1$. A UMP-test for $P_0 \mid \mathscr{A}_0, P_1 \mid \mathscr{A}_0$ at level α is in this case according to Theorem 11 a maximin-test for $\mathscr{P}_0, \mathscr{P}_1$ at level α, since $Q_i = \frac{1}{\gamma} \sum_{g \in G} P_i^g \in \operatorname{con} \mathscr{P}_i$, $i = 0, 1$, and $E_{Q_i}(I_A \mid \mathscr{A}_0) = \frac{1}{\gamma} \sum_{g \in G} I_A \circ g$, $A \in \mathscr{A}$, $i = 0, 1$. Especially the version of the sign in Lehmann's book, p. 219–220, is a maximin-test at level α.

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