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# Stochastically Ordered Distributions and Monotonicity of the OC-Function of Sequential Probability Ratio Tests

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**Summary.** Some characterizations for the stochastic ordering of probability distributions are given. Especially a general sufficient condition for stochastic ordering is proved and the question of existence of upper and lower bounds in the class of all distributions with given marginals is considered. Stochastic ordering is applied to prove a general theorem on monotonicity of the OC-function of sequential probability ratio tests in stochastic processes.

*Key words:* Stochastic ordering, monotone functions, Rosenblatt transformation, OC-function, sequential probability ratio tests.

## 1. Introduction

Let $(E, \mathfrak{A})$ be a polish space supplied with the Borel $\sigma$-Algebra $\mathfrak{A}$ and let $\leqq$ be a closed partial order on E. The set

$$M := \{f: (E, \mathfrak{A}) \to (R^1, \mathfrak{B}^1);\quad f \text{ isotone and bounded}\}$$

defines a partial order $\leqq_M$ on the set of all probability measures on $(E, \mathfrak{A})$ ([14]) by $P_1 \leqq_M P_2$ if and only if

$$\int f\,dP_1 \leqq \int f\,dP_2 \quad \text{for all } f \in M\,. \tag{1}$$

In section 2 of this paper we prove some characterizations of $\leqq_M$. Especially we give a sufficient condition for $\leqq_M$ in the case of $(E, \mathfrak{A}) = (R^n, \mathfrak{B}^n)$ which weakens the known sufficient conditions due to VEINOTT [21], FRANKEN, KIRSTEIN [7], KAMAE, KRENGEL and O'BRIEN [14], KALMYKOV [13], STOYAN [20] and FRANKEN, STOYAN [6]. There exist no lower and upper bounds w.r.t. $\leqq_M$ in the set of probability measures with given marginals. In contrast to this probability measures with ordered marginals are stochastically ordered.

Stochastic ordering is applied in the second part of this paper in order to prove monotonicity of the OC-function of sequential probability ratio tests. This connection was considered the first time by LEHMANN [15]. We get a general result, which includes the results known for this question. The methods are general enough in order to be applied to sequential probability ratio tests in stochastic processes. A connection to GIRSANOV's [11] Theorem on the representation of measures in function spaces continuous w.r.t. a given measure is indicated.

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## 2. Stochastic ordering

Let $P_1$, $P_2$ be probability measures on $(\mathrm{E}, \mathfrak{A})$. An element $A \in \mathfrak{A}$ is called isotone if $1_A \in M$. A characterization of $\leqq_M$ due to STRASSEN (cp. [14]) and KAMAE, KRENGEL and O'BRIEN [14], Th. 1, says:

$$P_1 \leqq_M P_2 \tag{2}$$

$\Leftrightarrow$ There exist E-valued random variables $X_1$, $X_2$ with distributions $P_1$, $P_2$ such that $X_1 \leqq X_2$ a.s.

$\Leftrightarrow$ (1) holds for the indicator functions of all isotone, closed sets $A \in \mathfrak{A}$.

**Remark 1.** a) The first equivalence implies for $(\mathrm{E}, \mathfrak{A}) = (\mathrm{R}^n, \mathfrak{B}^n)$, that $F_1 \geqq F_2$ and $h_{F_1} \leqq h_{F_2}$, whenever $P_1 \leqq_M P_2$, where $F_i$ are the distribution functions of $P_i$ and $h_{F_i}(x) := P_i([x, \infty))$, $x \in \mathrm{R}^n$, are the survival functions of $P_i$, $i = 1, 2$. For $n = 1$ also the converse is true; for $n \geqq 2$ simple examples show, that the converse is false.

b) The counterexample given by LEHMANN [15], Ex. 2.2 for the first equivalence in (2) is false, since for the set $S := (B \cup D \cup A)_+$ from LEHMANNS example (with $X_+ := \{y;\ \exists x \in X, x \leqq y\}$) it holds true that $P_\vartheta(S) = \frac{15}{16}$ while $P_{\vartheta'}(S) = \frac{14}{16}$ and, therefore, it is not true that $P_\vartheta \leqq_M P_{\vartheta'}$ in contrast to LEHMANN's statement. Example 2.2 of LEHMANN [15] really shows that a multivariate monotone likelihood ratio does not imply stochastic ordering.

The following characterization of $\leqq_M$ allows to reduce or to enlarge the class $M$ in order to prove (1). Especially it gives a connection to weak convergence. We assume that for isotone sets $B$ and $x \leqq y$, $d(x, B) \leqq d(y, B)$, where $d$ denotes a metric on E inducing the topology. This is fulfilled if e.g. E is a vector space, $\leqq$ is consistent with addition and $d$ is invariant under addition.

**Theorem 1.** *The following statements are equivalent.*

a) $P_1 \leqq_M P_2$.

b) (1) *holds for isotone functions $f$ with $f(x) \in \{0, 1\}$, $\forall x \in \mathrm{E}$.*

c) (1) *holds for all isotone functions $f$, such that the integrals exist.*

d) (1) *holds for all continuous, bounded, isotone functions $f \geqq 0$.*

Proof. The equivalence of a) and b) follows from (2). We show that

b) $\Rightarrow$ c). Let $f$ be integrable w.r.t. $P_1$, $P_2$.

By means of a well known integration formula we get for E-valued random variables $X$, $Y$ on $(\mathrm{X}, \mathfrak{B}, \mathsf{P})$ with $P^X = P_1$, $P^Y = P_2$

$$\mathsf{E}f(X) = \int_0^\infty \mathsf{P}(f(X) \geqq t)\, dt - \int_{-\infty}^0 \mathsf{P}(f(X) \leqq t)\, dt\,.$$

But $\{x; f(x) \geqq t\}$ and $\{x; f(x) > t\}$ are isotone sets since $f$ is isotone. Therefore b) implies

$$\mathsf{P}(f(X) \geqq t) \leqq \mathsf{P}(f(Y) \geqq t)$$

and

$$\mathsf{P}(f(X) > t) \leqq \mathsf{P}(f(Y) > t), \ \forall t \in \mathrm{R}^1 .$$

This implies

$$\int f \mathrm{d}P_1 = \mathsf{E}f(X) \leqq \mathsf{E}f(Y) = \int f \mathrm{d}P_2 .$$

It remains to show:

d) $\Rightarrow$ b). We proof at first, that d) implies (1) for isotone functions $f$, continuous from the right with $f(x) \in \{0, 1\}$, $\forall x$.

Let $A := \{x; f(x) = 1\}$ and let $d(t, A) := \inf \{d(t, y); y \in A\}$, $t \in \mathrm{E}$, where $d$ denotes a metric on E which induces the topology on E. The functions

$$u^{(k)}(t) := \begin{cases} 0 & d(t, A) \geqq \frac{1}{k} \\ 1 - kd(t, A) & d(t, A) < \frac{1}{k} \end{cases}$$

$k \in \mathbb{N}$, are bounded, continuous, isotone (since $A$ is isotone) and $\geqq 0$. Right continuity of $f$ implies, that $u^{(k)}$ converges antitone to $f$. The theorem of monotone convergence implies

$$\int u^{(k)} \mathrm{d}P_i \to \int f \mathrm{d}P_i \quad i = 1, 2$$

and, therefore, $\int f \mathrm{d}P_1 \leqq \int f \mathrm{d}P_2$ for all binary, right continuous functions $f$.

Let now in the second step $f$ be an arbitrary isotone function and define again $A := \{x; f(x) = 1\}$. Then there exists a compact subset $K \subset A$ with

$$P_1(K) + \varepsilon \geqq P_1(A),$$

since by Th. 1.4. of Billingsley [1] each probability measure on a complete separable metric space is tight. Let now $K_+$ be the isotonic closure of $K$, then $I_{K_+}$ is isotone, right continuous and $K \subset K_+ \subset A$. The first step of this proof implies $P_1(K_+) \leqq P_2(K_+)$ and, therefore,

$$P_1(A) \leqq P_1(K) + \varepsilon \leqq P_1(K_+) + \varepsilon \leqq P_2(K_+) + \varepsilon \leqq P_2(A) + \varepsilon ,$$

which implies the conclusion since $\varepsilon$ is arbitrary.

**Remark 2.** Theorem 1 implies Proposition 3 of Kamae, Krengel and O'Brien [14] which states that $\leqq_M$ is a closed partial order on the set of all probability measures with the weak convergence topology. To show this let $(P_n)$, $(Q_n)$ be sequences of probability measures on a polish space with closed order such that

$$P_n \xrightarrow{\mathfrak{D}} P, \quad Q_n \xrightarrow{\mathfrak{D}} Q \quad (\xrightarrow{\mathfrak{D}} \text{ weak convergence})$$

and assume $P_n \leqq_M Q_n$, $\forall n \in \mathbb{N}$. Then by definition of weak convergence $\int f \mathrm{d}P \leqq \int f \mathrm{d}Q$ for all continuous, bounded, isotone functions $f \geqq 0$ which implies by Theorem 1 that $P \leqq_M Q$ holds.

Theorem 1 also implies Proposition 2 of [14]. To show this let $P$, $Q$ be probability measures on $E^\infty$ and let $P^{(i)}$, $Q^{(i)}$ be the marginals of $P$, $Q$ on $E^i$ the $i$-fold product space. Assume that $P^{(i)} \leqq_M Q^{(i)}$, $i \in \mathbb{N}$, and let $z \in E$ and $z_\infty := (z, z, \ldots) \in E^\infty$. Then

$$P_{(i)} := P^{(i)} \otimes \varepsilon_{z_\infty} \leqq_M Q^{(i)} \otimes \varepsilon_{z_\infty} =: Q_{(i)}$$

for all $i \in \mathbb{N}$, where $\varepsilon_{z_\infty}$ is the one point measure in $z_\infty$ and, furthermore, $P_{(i)} \xrightarrow{\mathfrak{D}} P$, $Q_{(i)} \xrightarrow{\mathfrak{D}} Q$. So by the first part of this remark $P \leqq_M Q$.

In this way we get a proof of Propositions 2, 3 of [14] without refering to the involved pointwise equivalence theorem of STRASSEN (cf. (2)) as is done in [14]. A further useful aspect of Theorem 1 is that it allows to apply results of the theory of balayage – which are generally proven for compact cones of continuous functions – to stochastic ordering. So e.g. the pointwise equivalence theorem of STRASSEN (cf. (2)) for probabilities on compact sets can be implied in this way by a general theorem of the theory of balayage.

Some further statements in [14] show, that in order to prove $\leqq_M$ in function spaces in many cases (e.g. SKOROHOD space, space of continuous functions) it is enough to show $\leqq_M$ for the finite dimensional distributions. Therefore, we want to consider in the following the special case $E = [0, 1]^n$ (the case $E = \mathbb{R}^n$ is implied by application of isotone transformations).

The proof of the following lemma follows by approximation on the lattice

$$G_m := \{(i_1/_2m, \ldots, i_n/_2m);\quad 0 \leqq i_j \leqq 2^m,\ 1 \leqq j \leqq n\}.$$

**Lemma 2.** *Let* $f: E \to \mathbb{R}^1$ *be isotone and right continuous, then there exist sequences* $t_{i,m} \in [0, 1]^n$, $\alpha$, $\alpha_{i,m} \in \mathbb{R}^1$, $\alpha_{i,m} \geqq 0$, $m'$, $k_{i,m} \in \mathbb{N}$, $1 \leqq i \leqq m$, $m \in \mathbb{N}$, *such that*

$$f_m(x) = \alpha + \sum_{i=1}^{m'} \alpha_{i,m} I_A(x) \tag{3}$$

*where*

$$A = \bigcup_{j=1}^{k_{i,m}} [t_{i_j,m}, 1]$$

*converges monotonically to* $f$.

Let now $F$, $G$ be the distribution functions (d$f$'s) of $P_1$, $P_2$ and let $X$, $Y$ be E-valued random variables with df's $F$, $G$ then we define $F \leqq_M G$ if $P_1 \leqq_M P_2$. Furthermore, for $\lambda_i \in [0, 1]$, $1 \leqq i \leqq n$, and $x = (x_1, \ldots, x_n) \in E$ we define

$$\begin{aligned} F(x_1, \lambda_1) &:= \mathsf{P}(X_1 < x_1) + \lambda_1 \mathsf{P}(X_1 = x_1) \\ F(x_i, \lambda_i \mid x_1, \ldots, x_{i-1}) &:= \mathsf{P}(X_i < x_i \mid X_j = x_j,\ 1 \leqq j \leqq i-1) \\ &\quad + \lambda_i \mathsf{P}(X_i = x_i \mid X_j = x_j,\ j \leqq i-1),\quad 2 \leqq i \leqq n, \end{aligned}$$

and $\tau_F: [0, 1]^n \times [0, 1]^n \to [0, 1]^n$ by

$$\tau_F(x, \lambda) := (F(x_1, \lambda_1), F(x_2, \lambda_2 \mid x_2), \ldots, F(x_n, \lambda_n \mid x_1, \ldots, x_{n-1}). \tag{4}$$

For $\tau_F(x, 1, \ldots, 1)$ we use the abbreviation $\tau_F(x)$. $\tau_F$ is a modification of a transformation introduced by ROSENBLATT [18]. The inverse of $\tau_F$ is defined inductively by

$$\tau_F^{-1}(x)=(z_1,\ldots,z_n) \quad \text{with}$$
$$z_1:=F^{-1}(x_1)=\inf\{y\in \mathbb{R}^1;\, F(y)\geqq x_1\}$$

(where $F(y)$ denotes the first marginal df. of $F$ applied to $y$.)

$$z_2:=\inf\{y;\, F(y\mid z_1)\geqq x_2\}=F^{-1}(x_2\mid z_1)$$
$$z_n:=F^{-1}(x_n\mid z_1,\ldots,z_{n-1})\,.$$

Similarly we define $\tau_G$, $\tau_G^{-1}$. Let $U_1,\ldots,U_n$ be $\mathrm{R}(0,1)$-distributed random variables such that $\{U_1,\ldots,U_n,X,Y\}$ are independent, then with $U=(U_1,\ldots,U_n)$ we get the following

**Lemma 3.** a) *$\tau_F(X,U)$ is a random variable whose distribution is the product of $n$ independent* $\mathrm{R}(0,1)$*-distributions.*
b) *$\tau_F^{-1}(U)$ is a random variable with* df. *$F$.*

**Remark** (to the proof). In the special case of absolutely continuous df's part a) has been shown by Rosenblatt [18]. The general case can be proved by successive application of the idea of Lemma 1, p. 216 of Ferguson [5] to all conditional df's. b) follows from an integral transformation. In a slight modified form b) is stated in Theorem 1 of O'Brien [17].

Let $P_1$, $P_2$, $F$, $G$ be defined as above, then we get

**Theorem 4.** a) *$P_1\leqq_M P_2$ if and only if*

$$\lambda_n\left\{x\in[0,1]^n;\, \tau_F^{-1}\in\bigcup_{j=1}^{k}\,[t_j,1]\right\}\leqq\lambda_n\left\{x\in[0,1]^n;\, \tau_G^{-1}\in\bigcup_{j=1}^{k}\,[t_j,1]\right\},$$
$$\forall t_j\in[0,1]^n,\ \forall k\in\mathbb{N}\,.$$

b) $$\tau_F^{-1}\leqq\tau_G^{-1}[\lambda_n]\,. \tag{5}$$

*implies $P_1\leqq_M P_2$.*

Proof. a) By Theorem 1 it is sufficient to consider bounded continuous functions $f\geqq 0$ in $M$. These functions can be approximated isotonically by Lemma 2 by functions of the type

$$\alpha+\sum_{i=1}^{m'}\alpha_{i,m}I_A(x)\,,\quad\text{where}\quad A=\bigcup_{j=1}^{k_{i,m}}\,[t_{i_j,m},1]$$

From Lemma 3, b) follows statement a).

b) Let $U_1,\ldots,U_n$ be stochastically independent, $\mathrm{R}(0,1)$-distributed, $U:=(U_1,\ldots,U_n)$. Then $X:=\tau_F^{-1}(U)\leqq Y:=\tau_G^{-1}(U)\,[\lambda_n]$. So b) follows from Lemma 3, b). b) is also immediate from a).

The following lemma gives 2 conditions which imply condition (b).

**Lemma 5.** *Let $F$, $G$ be $n$ dim. df's.*

a) *From* $G(x_1) \leqq F(x_1)$, $\forall x_1$ *and* $G(x_{i+1} \mid x_1, \ldots, x_i) \leqq F(x_{i+1} \mid y_1, \ldots, y_i)$, $\forall (x_1, \ldots, x_i) \geqq (y_1, \ldots, y_i)$ *follows*

$$\tau_F^{-1} \leqq \tau_G^{-1} .$$

b) *If* $\tau_G \leqq \tau_F$ *and if* $G(x_{i+1} \mid x_1, \ldots, x_i)$ *is antitone in* $(x_1, \ldots, x_i)$, $i \leqq n-1$, *then*

$$\tau_F^{-1} \leqq \tau_G^{-1} .$$

Proof. Define $\tau_F^{-1}(x_1, \ldots, x_n) = : (z_1, \ldots, z_n)$ and $\tau_G^{-1}(x_1, \ldots, x_n) = : (w_1, \ldots, w_n)$.

a) The first assumption in a) implies $z_1 \leqq w_1$. If $(z_1, \ldots, z_i) \leqq (w_1, \ldots, w_i)$, then

$$\begin{aligned} G^{-1}(x_{i+1} \mid w_1, \ldots, w_i) &= \inf \{y \in \mathrm{R}^1; G(y \mid w_1, \ldots, w_i) \geqq x_{i+1}\} \\ &\geqq \inf \{y; F(y \mid z_1, \ldots, z_i) \geqq x_{i+1}\} = F^{-1}(x_{i+1} \mid z_1, \ldots, z_i) . \end{aligned}$$

This implies $\tau_F^{-1} \leqq \tau_G^{-1}$.

b) If $G(x_{i+1} \mid x_1, \ldots, x_i)$ is antitone in $(x_1, \ldots, x_i)$ and $\tau_G \leqq \tau_F$, then

$$\begin{aligned} &G(x_{i+1} \mid x_1, \ldots, x_i) \leqq G(x_{i+1} \mid y_1, \ldots, y_i) \leqq F(x_{i+1} \mid y_1, \ldots, y_i) , \\ &\forall (y_1, \ldots, y_i) \leqq (x_1, \ldots, \vartheta_i) . \end{aligned}$$

So b) is implied by a).

**Remark 3.** a) From monotonic transformation it follows that Theorem 4 is true also for $\mathrm{E} = \mathrm{R}^n$.

b) KAMAE, KRENGEL and O'BRIEN [14], Theorem 2, FRANKEN, KIRSTEIN [7], Satz 4.1, and VEINOTT [21], Theorem 4 prove stochastic ordering under the conditions a) resp. b) of Lemma 5. So these results are included in Theorem 4.

By Remark 1, a) stochastic ordering on $\mathrm{R}^n$ implies ordering of the df's. KALMYKOV [13], STOYAN [6], Satz 5 (in the case of Markov processes) and O'BRIEN [17], Theorem 4 (in the non markovian case) prove ordering of the df's under the condition b) of Lemma 5. So their results are also included in Theorem 4.

For some different types of comparison of Markov processes cf. FRANKEN, STOYAN [6], STOYAN [20] and FRANKEN, KIRSTEIN [7].

Let $F_1, \ldots, F_n$ be $n$ one-dimensional df's and let $\mathcal{H}(F_1, \ldots, F_n)$ denote the set of all $n$-dim. df's with $F_i$ as $i$-th marginal df. $\mathcal{H}(F_1, \ldots, F_n)$ is of statistical interest since it describes the influence of dependence on a statistical problem. A characterization of $\mathcal{H}(F_1, \ldots, F_n)$ by means of ordering the df's gives the Theorem on the Fréchet bounds from DALL'AGLIO [2].

A df. $H$ is an element of $\mathcal{H}(F_1, \ldots, F_n)$ if and only if

$$H_1(x) \leqq H(x) \leqq H_2(x), \ \forall x \in \mathrm{R}^n ,$$

where

$$H_2(x) := \min_{1 \leqq i \leqq n} F_i(x_i)$$

and

$$H_1(x) := \left( \sum_{i=1}^{n} F_i(x_i) - (n-1) \right)_+$$

for $x = (x_1, \ldots, x_n)$ with $a_+ = \max\{a, 0\}$, $a \in \mathbb{R}^1$. $H_2$ is an element of $\mathcal{H}(F_1, \ldots, F_n)$ while $H_1 \in \mathcal{H}(F_1, \ldots, F_n)$ only for $n=2$ and in few special cases for $n \geqq 3$. In all other cases there does not exist a lower bound in the set $\mathcal{H}(F_1, \ldots, F_n)$ with respect to the pointwise ordering.

This implies by Remark 1. a) that generally for $n \geqq 3$ there does not exist a largest element in $\mathcal{H}(F_1, \ldots, F_n)$ w.r.t. stochastic ordering. The following example considers the question of stochastic ordering between two elements of $\mathcal{H}(F_1, \ldots, F_n)$.

**Example 1.** a) Let $F_1 = F_2$ be the df. of the 2-point distribution $\frac{1}{2}(\varepsilon_{\{0\}} + \varepsilon_{\{1\}})$. By a simple calculation we get:

$$\mathcal{H}(F_1, F_2) = \Big\{ F^\alpha;\ F^\alpha \text{ is df. of}$$

$$P^\alpha := \left(\frac{1}{2} - \alpha\right)(\varepsilon_{(0,1)} + \varepsilon_{(1,0)}) + \alpha\, (\varepsilon_{(0,0)} + \varepsilon_{(1,1)}),\ 0 \leqq \alpha \leqq \frac{1}{2} \Big\}.$$

For $0 \leqq \alpha_1 \leqq \alpha_2 \leqq \frac{1}{2}$ we have $H^{\alpha_1} \leqq H^{\alpha_2}$ and $h_{H^{\alpha_1}} \leqq h_{H^{\alpha_2}}$ and $H^0 = H_1$, $H^{\frac{1}{2}} = H_2$ ($H_1, H_2$ from DALL'AGLIO's Theorem). Let $f_1$, $f_2$ be isotone functions with $f_1(0,0)=0$, $f_1(1,0)=f_1(0,1)=f_1(1,1)=1$ and $f_2(0,0)=f_2(0,1)=f_2(1,0)=0$, $f_2(1,1)=1$. Then $\int f_1 \mathrm{d}P^\alpha = \frac{1}{2} - \alpha$ and $\int f_2 \mathrm{d}P^\alpha = \alpha$, so that for $0 \leqq \alpha_1 < \alpha_2 \leqq \frac{1}{2}$

$$\text{neither } H^{\alpha_1} \leqq_M H^{\alpha_2} \quad \text{nor} \quad H^{\alpha_2} \leqq_M H^{\alpha_1}.$$

So $\mathcal{H}(F_1, F_2)$ is totally disordered w.r.t. $\leqq_M$.

b) Let $F_1, \ldots, F_n$ be df's and define

$$g(x) := (g_1(x), \ldots, g_n(x)),\ h(x) := (h_1(x), \ldots, h_n(x))$$

by $h_1(x) = g_1(x) = F_1(x_1)$ and

$$h_i(x) := I_{[\min_{j \leqq i-1} F_j(x_j), 1]}(F_i(x_i)) \tag{6}$$

$$g_i(x) := I_{\left[\left(i-1-\sum_{j=1}^{i-1} F_j(x_j)\right) \wedge 1, 1\right]}(F_i(x_i)),\ 2 \leqq i \leqq n.$$

Then $h = \tau_{H_2}$ and $g = \tau_{H_1}$ (resp. solution of the corresponding integral equation if $H_1$ is no df.). $h \leqq g$ or $g \leqq h$ holds true only in the case of one point distributions. This example leads to the following result.

**Proposition 6.** *There do not exist two elements in $\mathcal{H}(F_1, \ldots, F_n)$ which are comparable w.r.t. $\leqq_M$.*

Proof. Let $F, H \in \mathcal{H}(F_1, \ldots, F_n)$ and assume $F \leqq_M H$. Then by the pointwise equivalence theorem (2) there exist random variables $X$, $Y$ on a common probability space with df's $F$, $H$ such that $X \leqq Y$ a.s. Therefore, $X_i \leqq Y_i$ a.s. where $X_i$, $Y_i$ are the $i$-th components of $X$, $Y$. Assume that $\mathsf{P}(X_i < Y_i) > 0$. Then since $\{X_i < Y_i\} = \bigcup_{a \in Q} \{X_i < a < Y_i\}$ where $Q$ is the set of all rational numbers there exists an element $a \in Q$ with $\mathsf{P}(X_i < a < Y_i) > 0$ and, therefore,

$$\mathsf{P}(X_i < a) = \mathsf{P}(X_i < a, X_i \leqq Y_i) = \mathsf{P}(X_i \leqq Y_i < a) + \mathsf{P}(X_i < a \leqq Y_i)$$
$$> \mathsf{P}(X_i \leqq Y_i < a) = \mathsf{P}(Y_i < a)$$

contradicting the assumption that both $X_i$ and $Y_i$ have the same df. $F_i$. Therefore, $X_i = Y_i$ a.s. This implies $X = Y$ a.s. and, therefore, $F = H$.

Distributions whose marginals are ordered are comparable w.r.t. $\leqq_M$ in the following sense:

**Proposition 7.** Let $F_i$, $G_i$, $1 \leqq i \leqq n$ *be one-dim. df's with* $F_i \leqq G_i$. *Then it holds:*

a) *To* $F \in \mathcal{H}(F_1, \ldots, F_n)$ *exists a* $G \in \mathcal{H}(G_1, \ldots, G_n)$ *with* $G \leqq_M F$.

b) *To* $G \in \mathcal{H}(G_1, \ldots, G_n)$ *there exists a* $F \in \mathcal{H}(F_1, \ldots, F_n)$ *with* $G \leqq_M F$.

Proof. a) Let $X = (X_1, \ldots, X_n)$ be a random variable with df. $F$ and let $U_1, \ldots, U_n$ be stochastically independent $\mathrm{R}(0, 1)$-distributed random variables independent of $X$. Define

$$Y := (G_1^{-1}(F_1(X_1, U_1)), \ldots, G_n^{-1}(F_n(X_n, U_n))) . \tag{7}$$

Lemma 3 (for $n=1$) implies $F_Y \in \mathcal{H}(G_1, \ldots, G_n)$. $F_i(X_i, U_i) \leqq F_i(X_i)$ implies

$$G_i^{-1} \circ F_i(X_i, U_i) \leqq G_i^{-1} \circ F_i(X_i) \leqq F_i^{-1} \circ F_i(X_i) \leqq X_i$$

and, therefore, $Y \leqq X$. So with $G := F_Y$ we have $G \leqq_M F$.

b) Let $X$ be a random variable with df. $G \in \mathcal{H}(G_1, \ldots, G_n)$, then $-X = (-X_1, \ldots, -X_n)$ is a random variable with df. in $\mathcal{H}(1 - G_1((-\cdot)-), \ldots, 1 - G_n((-\cdot)-))$ and $1 - G_i((-\cdot)-) \leqq 1 - F_i((-\cdot)-)$. Therefore, by a) there is a random variable $Y = (Y_1, \ldots, Y_n)$ with df. in $\mathcal{H}(1 - F_1((-\cdot)-), \ldots, 1 - F_n((-\cdot)-))$ such that $Y \leqq -X$ so that $-Y \geqq X$. The df. $F$ of $-Y$ is an element of $\mathcal{H}(F_1, \ldots, F_n)$ and it holds that $G \leqq_M F$.

## 3. Monotonicity of the OC-functions of SPRT's

Let $(X_i)_{i \in \mathrm{N}}$ be a sequence of real random variables, let $P_\vartheta, \vartheta \in \Theta \subset \mathrm{R}^k$ be probability measures such that $X_{(n)} := (X_1, \ldots, X_n)$ have under $P_\vartheta$ densities $f_\vartheta^n(x)$ w.r.t. a sequence $\mu_n$ of $\sigma$-finite measures on $(\mathrm{R}^n, \mathfrak{B}^n)$. Let further $\vartheta_0, \vartheta_1 \in \Theta$, $\vartheta_0 < \vartheta_1$ and define the likelihood ratio

$$Z_n := f_{\vartheta_1}^n(X_{(n)}) / f_{\vartheta_0}^n(X_{(n)}) . \tag{8}$$

For boundary sequences $a_i$, $b_i \in \mathrm{R}^1$, $a_i < b_i$ we define the stopping time

$$N(w) := \begin{cases} \inf \{n \in \mathbb{N};\, Z_n(w) \notin (a_n, b_n)\} \\ \infty \text{ if } \{n \in \mathbb{N};\, Z_n(w) \notin (a_n, b_n)\} = \Phi \end{cases}$$

$(Z_n, n \in \mathbb{N}, N)$ is a SPRT. The OC-function is determined by

$$Q_n(\vartheta) := P_\vartheta\,(N \leqq n, Z_N \leqq a_N)$$

and

$$P_n(\vartheta) := P_\vartheta\,(N \leqq n, Z_N \geqq b_N)\,.$$

We are interested in conditions on $P_\vartheta$, $\vartheta \in \Theta$ such that for $\vartheta \leqq \vartheta'$:

$$Q_n(\vartheta) \geqq Q_n(\vartheta') \quad \text{and} \quad P_n(\vartheta) \leqq P_n(\vartheta')\,, \quad \forall n \in \mathbb{N}\,. \tag{9}$$

**Theorem 8.** *The condition* $P_\vartheta^{(Z_1,\ldots,Z_n)} \leqq_M P_{\vartheta'}^{(Z_1,\ldots,Z_n)}$ *for* $\vartheta \leqq \vartheta'$ *and all* $n \in \mathbb{N}$ *implies* (9).

Proof. The set

$$M_n := \{(z_1, \ldots, z_n) \in \mathrm{R}^n; \quad \exists i \leqq n,\, z_i \notin (a_i, b_i),$$
$$\text{and } z_{i_0} \geqq b_{i_0} \quad \text{for} \quad i_0 := \min \{i;\, z_i \notin (a_i, b_i)\}\}$$

is monotone. So the condition on $(Z_1, \ldots, Z_n)$ implies

$$P_n(\vartheta) = P_\vartheta^{(Z_1,\ldots,Z_n)}(M_n) \leqq P_{\vartheta'}^{(Z_1,\ldots,Z_n)}(M_n) = P_n(\vartheta')\,.$$

The proof of $Q_n(\vartheta) \geqq Q_n(\vartheta')$ is analogous.

Theorem 8 together with Theorem 4 and Lemma 5 weaken a condition given by Hoel [12]. Furthermore, our proof seems to be much simpler than the proof given by Hoel [12]. As a corollary we get the following theorem.

**Theorem 9.** *If* $f_{\vartheta_1}^n, f_{\vartheta_2}^n$ *have a monotone likelihood-ratio in* $x_{(n)} \in \mathrm{R}^n$ *and if* $P_\vartheta^{X_{(n)}} \leqq_M P_{\vartheta'}^{X_{(n)}}$ *for* $\vartheta \leqq \vartheta'$ *then condition* (9) *holds.*

Proof. By assumption $\dfrac{f_{\vartheta_1}^n(x_{(n)})}{f_{\vartheta_0}^n(x_{(n)})} = h_{\vartheta_1,\vartheta_0}(x_{(n)})$ where $h_{\vartheta_1,\vartheta_0}$ is isotone. This implies that $(Z_1, \ldots, Z_n)$ is an isotonic function of $(X_1, \ldots, X_n)$ and, therefore, $P_\vartheta^{(Z_1,\ldots,Z_n)} \leqq_M P_{\vartheta'}^{(Z_1,\ldots,Z_n)}$. So Theorem 9 follows from Theorem 8.

**Remark 4.** Lehmann [16], Lemma 4, p. 101 showed that Theorem 9 holds for independent random variables with a monotone likelihood ratio. Ghosh [9] proved that (9) holds if the densities $\{f_\vartheta^n, \vartheta \in \Theta\}$ have a monotone likelihood ratio in a real valued statistic $T(x_{(n)})$. This assumption implies $P_\vartheta^{(Z_1,\ldots,Z_n)} \leqq P_{\vartheta'}^{(Z_1,\ldots,Z_n)}$ for $\vartheta \leqq \vartheta'$; so Ghosh's result is contained in Theorem 8.

Theorem 9 was formulated by Lehmann [15], Theorem 2. But Lehmann's proof is not correct. Lehmann concludes from the assumption of the monotone likelihood ratio that $P^{Z_n} \leqq_M P^{Z_n}$ holds for $\vartheta \leqq \vartheta'$. By (2) there exist mappings $f_n(z) \geqq z$ such that $P_{\vartheta'}^{Z_n} = P_{\vartheta}^{f_n(Z_n)}$. LEHMANN uses this relation to compare the paths $(i, Z_i), i \geqq 1$ with the paths $(i, f_i(Z_i)), i \geqq 1$ concluding

$$P_{\vartheta'}\{((1, Z_1), \ldots, (n, Z_n)) \in B\} = P_{\vartheta}\{((1, f_1(Z_1)), \ldots, (n, f_n(Z_n))) \in B\},$$

which generally is not true.

By means of Theorem 4 of KAMAE, KRENGEL and O'BRIEN [14] similar results can be given for SPRT's in stochastic processes with continuous time parameter.

Let $(X_t)_{t \geqq 0}$ be a stochastic process on $(\Omega, \mathfrak{A}, P_\vartheta)$, $\vartheta \in \Theta \subset \mathrm{E}$, E partially ordered, and let for $\vartheta_0 < \vartheta_1$ fixed, $P_{\vartheta_1}$ be continuous w.r.t. $P_{\vartheta_0}$ restricted on $\mathfrak{A}_t := \mathfrak{A}(X_s;\, s \leqq t)$, $t \geqq 0$. Let $a, b : \mathrm{R}_+ \to \mathrm{R}^1$ be boundary functions $a \leqq b$ and define

$$z_t := \mathsf{E}\left(\frac{\mathrm{d}P_{\vartheta_1}}{\mathrm{d}P_{\vartheta_0}} \,\middle|\, \mathfrak{A}_t\right),$$

$$N := \inf\{t \in \mathrm{R}_+;\, Z_t \notin (a_t, b_t)\} \quad (\inf\{t;\, t \in \emptyset\} := \infty)$$

$$Q_t(\vartheta) := P_\vartheta\,(N \leqq t, Z_N \leqq a_N)$$

and

$$P_t(\vartheta) := P_\vartheta\,(N \leqq t, Z_N \leqq b_N)\,.$$

We assume that $(Z_t)_{t \geqq 0}$ defines a stochastic process on the Skorohod space $\mathrm{D}[0, \infty)$ and that the sets defined above are measurable.

$(Z_t, t \geqq 0, N)$ can be looked upon as SPRT. In the literature (cf. GHOSH [10], ZACKS [22], DWORETZKY, KIEFER, WOLFOWITZ [4], FRANZ, WINKLER [8]) usually the case of exponential classes for which $X_t$ is a sufficient statistic and $Z_t = f(t, X_t)$ is considered. Examples are the Wiener process, the Poisson process and the Gammaprocess. We have the following result.

**Theorem 10.** $P_\vartheta^{(Z_{t_1}, \ldots, Z_{t_n})} \leqq P_{\vartheta'}^{(Z_{t_1}, \ldots, Z_{t_n})}$ *for all* $t_i \in \mathrm{R}_+$, $1 \leqq i \leqq n$, $\forall n \in \mathbb{N}$ *implies* $P_t(\vartheta) \leqq P_t(\vartheta')$ *and* $Q_t(\vartheta) \geqq Q_t(\vartheta')$, $\forall t \geqq 0$.

**Remark 5.** Stochastic ordering of stochastic processes can be managed in some cases by means of GIRSANOV's [11] theorem or its generalizations. Let $X = (X_t)_{t \geqq 0}$ be a stochastic process such that $(X, P)$ is a Wiener-process and let $Q$ be a probability measure continuous w.r.t. $P$ on $\mathfrak{A} = \bigvee_{t \geqq 0} \mathfrak{A}_t$. By GIRSANOV's theorem $X$ has w.r.t. $Q$ the representation

$$X_t = \int_0^t \Phi_s \mathrm{d}s + w_t\,,$$

where $\Phi_t$ is the predictable projection on $\mathfrak{A}_t$, $\Phi_s \in \mathrm{L}^2(\langle x \rangle, P)$, where $\langle x \rangle_t = t$ is the associated increasing process and where $(w_t, t \geqq 0, Q)$ is a Wiener process. Furthermore, the exponentiation formula gives

$$\mathsf{E}\left(\frac{\mathrm{d}Q}{\mathrm{d}P} \,\middle|\, \mathfrak{A}_t\right) = \exp\left\{\int_0^t \Phi_s \mathrm{d}X_s - \frac{1}{2}\int_0^t \Phi_s^2 \mathrm{d}s\right\}$$

where the first integral is a stochastic integral. So $Q$ is determined by $\Phi$ and $P$ on $\mathfrak{A}$ and $\Phi$ can be determined from the density. If $\int_0^t \Phi_s ds \geqq 0$ a.s. then $X$ is under $Q$ stochastically larger than under $P$. This implies immediately statements on the monotonicity of SPRT's for measures which are continuous w.r.t. Wiener measure.

By means of the generalization of GIRSANOV's Theorem given by VAN SCHUPPEN, WONG [19] to general local martingales and by means of the generalized exponentiation formula given by DOLÉANS-DADE [3] similar results are possible also for jump processes.

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**Résumé**

On donne de characterisations de l'ordre stochastique de probabilités. Un des buts de cet recherche est de donner une condition suffisante pour l'ordre stochastique. La question d'existence de bornes supérieures et inférieures dans la classe de toutes les probabilités avec des marges préscrites est considerée. À l'aide de l'ordre stochastique on prouve un théorème général sur la monotonie de l'OC-fonction des SPRT's en processus stochastiques.

**Zusammenfassung**

Es werden einige Charakterisierungen der stochastischen Ordnung von Wahrscheinlichkeitsmaßen bewiesen. Insbesondere wird eine allgemeine hinreichende Bedingung für die stochastische Ordnung angegeben und die Frage nach der Existenz von oberen und unteren Schranken in der Klasse aller Wahrscheinlichkeitsmaße mit vorgegebenen Randverteilungen untersucht. Mit Hilfe der stochastischen Ordnung wird ein allgemeiner Satz über die Monotonie der OC-funktion von sequentiellen Dichtequotiententests in stochastischen Prozessen bewiesen.

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