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RATE OF CONVERGENCE FOR SUMS AND MAXIMA AND DOUBLY IDEAL METRICS*

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Abstract. It is well known that the minimal L_p-metrics are ideal with respect to summation and maxima of order $r = \min(p, 1)$. This implies that one can get rate of convergence results in stable limit theorems with $0 < \alpha < 1$ with respect to maxima and sums. It will be shown that one can extend and improve the ideality properties of minimal L_p-metrics to stable limit theorems with $0 < \alpha < 2$. As a consequence one obtains, e.g., an improvement of the classical results on the rate of convergence of sums with values in Banach spaces with respect to the Prokhorov distance. In the second part of the paper it is proved that a problem posed by Zolotarev in 1983 on the existence of doubly ideal metrics of order $r > 1$ has an essential negative answer. In spite of this the minimal L_p-metrics behave like ideal metrics of order $r > 1$ with respect to maxima and sums. This allows to improve results on the stability of queueing models respect to departures from the ideal model.

1. Introduction. Let $(U, \|\cdot\|)$ be a separable Banach space with norm $\|\cdot\|$ and Borel α-algebra $\mathcal{B} = \mathcal{B}(U)$ and let $\mathcal{X} = \mathcal{X}(U)$ be the set of all random variables on a nonatomic probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with values in U. Then the set $\mathcal{P} = \mathcal{P}(U)$ of all distributions $\{\mathbf{P}_X;\ X \in \mathcal{X}\}$ coincides with the set $M^1(U, \mathcal{B})$ of all probability measures on $(U, \mathcal{B})$. A function $\mu\colon \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is called a *probability metric* (cf. [23, p. 374]) if for $X, Y, Z \in \mathcal{X}$

$$\text{(1.1)} \qquad \begin{aligned} &\text{(a)}\ \ \mathbf{P}(X = Y) = 1 \to \mu(X, Y) = 0,\\ &\text{(b)}\ \ \mu(X, Y) = \mu(Y, X),\\ &\text{(c)}\ \ \mu(X, Z) \leqq \mu(X, Y) + \mu(Y, Z). \end{aligned}$$

μ is called a *simple metric* if $X_1 \overset{d}{=} X_2,\ Y_1 \overset{d}{=} Y_2$ implies $\mu(X_1, Y_1) = \mu(X_2, Y_2)$ and *compound* otherwise. A simple metric induces a (usual semi) metric $\mu\colon \mathcal{P}(U) \times \mathcal{P}(U) \to [0, \infty]$ and vice versa.

Considering the rate of convergence problem for the CLT (central limit theorem) Zolotarev [23] introduced the notion of ideal metrics. A probability metric μ is called a *compound* $(r, +)$-*ideal metric* (ideal of order $r > 0$ with respect to summation) if and only if, for all $X, Y, Z \in \mathcal{X},\ c \in \mathbf{R}^1$,

$$\text{(1.2)} \qquad \begin{aligned} &\text{(a)}\ \ \mu(X + Z, Y + Z) \leqq \mu(X, Y),\\ &\text{(b)}\ \ \mu(cX, cY) = |c|^r \mu(X, Y), \end{aligned}$$

hold. μ is called a *simple* $(r, +)$-*ideal metric* if (a) is satisfied for any Z independent of X and Y.

The estimate

$$\text{(1.3)} \qquad \mu\left(\sum_{j=1}^{n} c_j X_j,\ \sum_{j=1}^{n} c_j Y_j\right) \leqq \sum_{j=1}^{n} |c_j|^r \mu(X_j, Y_j)$$

for any $c_j \in \mathbf{R}^1$ and any r.v.'s (random variables) (X_j), (Y_j) is a consequence of the $(r, +)$ ideality of μ. If μ is a simple $(r, +)$-ideal metric, then $\{X_1, \ldots, X_n\}$, as well

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as $\{Y_1, \ldots, Y_n\}$, are supposed to be independent r.v.'s. In particular, if $X_1, X_2, \ldots$ are i.i.d. (identically independent distributed) r.v.'s and $Y_{(\alpha)}$ has a strictly symmetric stable distribution with parameter $\alpha \in (0, 2]$ and μ is a simple $(r, +)$-ideal metric of order $r > \alpha$, then we obtain

$$\mu\left(n^{-1/\alpha}\sum_{i=1}^{n} X_i,\ Y_{(\alpha)}\right) \leqq n^{1-r/\alpha}\mu(X_1,\ Y_{(\alpha)}) \tag{1.4}$$

from (1.3), which gives a precise estimate in the CLT under the only assumption that $\mu(X_1, Y_{(\alpha)}) < \infty$. (Under additional assumptions one can improve the order in (1.4), cf. (2.11) with $\mu = \widehat{L}_p$.)

In several Banach spaces (e.g., in Banach function spaces) one has a natural maximum operation $x \vee y$. With respect to the operation $\vee$ one defines similarly the notion of *compound* and of *simple* $(r, \vee$–ideal metrics assuming condition (b) in (1.2) only for positive c. Especially, if μ is a simple $(r, \vee)$-ideal metric on $\mathbf{R}^1$ and if $Z_{(\alpha)}$ is a α-max-stable distributed r.v. on $\mathbf{R}^1$ (i.e. $F_{Z(\alpha)}(x) = \exp\{-x^{-\alpha}\}$, $x \geqq 0$), then

$$\mu\left(n^{-1/\alpha}\bigvee_{i=1}^{n} X_i,\ Z_{(\alpha)}\right) \leqq n^{1-r/\alpha}\mu(X_1,\ Z_{(\alpha)}) \tag{1.5}$$

for any i.i.d. r.v.'s X_i.

In the following sections we construct some ideal metrics for summation and for maxima and discuss the problem formulated by Zolotarev [26, p. 300] to construct metrics which are ideal with respect to both operations simultaneously. As is immediately clear from (1.3)–(1.5) one gets as a consequence rate of convergence results in the CLT. It will be interesting to compare the new results with classical results in terms of, e.g., the Prokhorov distance and also with respect to the assumptions in these theorems.

We will point out the importance of the L_p-metrics in this kind of problems and especially obtain an improvement of Zolotarev's classical estimate of the rate of convergence with respect to the Prokhorov distance for $1 \leqq \alpha < 2$. We will show that Zolotarev's problem has an essential negative answer but that in the range $0 < \alpha < 2$ the $\widehat{L}_p$-metrics (in spite of being only ideal of order $\min(1, p)$) behave like doubly ideal metrics of order $r = 1 + \alpha - \alpha/p \geqq 1$ for $0 < \alpha \leqq p \leqq 2$.

2. Ideal metric and rate of convergence for summation. Define for $X, Y \in \mathcal{X}(U) = \mathcal{X}$

$$\begin{aligned} L_p(X,\ Y) &= \left(\mathbf{E}\,\|X - Y\|^p\right)^{\min(1,1/p)}, \qquad 0 < p < \infty, \\ L_\infty(X,\ Y) &= \operatorname{ess\,sup}\|X - Y\| \end{aligned} \tag{2.1}$$

and let $\widehat{L}_p$ denote the corresponding *minimal* metrics, i.e.,

$$\widehat{L}_p(X,\ Y) = \inf\left\{L_p(\widetilde{X}, \widetilde{Y});\ \widetilde{X} \overset{d}{=} X,\ \widetilde{Y} \overset{d}{=} Y\right\}, \qquad 0 < p < \infty, \tag{2.2}$$

where $\widetilde{X} \overset{d}{=} X$ means that $\mathbf{P}_{\widetilde{X}} = \mathbf{P}_X$. Note that $\widehat{L}_p(X, Y) < \infty$ $(0 < p \leqq \infty)$ does not imply the finiteness of pth moments of $\|X\|$ and $\|Y\|$; for example, on $\mathbf{R}^1$ a sufficient condition for $\widehat{L}_p(X, Y) < \infty$ $(1 \leqq p < \infty)$ is $\kappa_p(X, Y) := \int |x|^{p-1}|F_X(x) - F_Y(x)|dx < \infty$, see further (2.15). (The advantage of exploring the difference moment condition $\kappa_p(X, Y) < \infty$ in the Berry–Esséen type estimates was demonstrated by Hall [5]). Since L_p is a compound $(r, +)$-ideal metric with $r = r_p = \min(p, 1)$, $\widehat{L}_p$ is a simple $(r, +)$-ideal metric (see [23] and [16]). Therefore, from (1.4) one obtains for i.i.d. r.v.'s (X_i) and for the α-stable r.v. $Y_{(\alpha)}$ the following estimate:

$$\widehat{L}_p\left(n^{-1/\alpha}\sum_{i=1}^{n} X_i,\ Y_{(\alpha)}\right) \leqq n^{1-r/\alpha}\widehat{L}_p(X_1, Y_{(\alpha)}), \tag{2.3}$$

which is useful only for $0 < \alpha < p \leqq 1$.

In the following remark we will discuss the results obtained by means of Zolotarev's ideal metric ζ_r.

Remark 2.1. a) It is easy to see that there is *no* nontrivial compound $(r, +)$-ideal metric μ, when $r > 1$. Since the compound $(r, +)$ ideality would imply $\mu(X, Y) = \mu((X + \cdots + X)/n,\ (Y + \cdots + Y)/n) \leqq n^{1-r}\mu(X, Y)$, for all $n \in \mathbf{N}$, i.e., $\mu(X, Y) \in \{0, \infty\}$ for all $X, Y \in \mathcal{X}(U)$.

b) Zolotarev [23] detected a simple $(r, +)$-ideal metric of any order $r > 1$, namely, if $r = m + \alpha$, $0 < \alpha \leqq 1$, $m \in \mathbf{N}$, then

$$\zeta_r(X, Y) = \sup\left\{\left|\mathbf{E}\left(f(X) - f(Y)\right)\right|;\qquad \left|f^{(m)}(x) - f^{(m)}(y)\right| \leqq \|x - y\|^{\alpha}\right\}, \tag{2.4}$$

$f^{(m)}(x)$ denoting the Frechét-derivative of order m.

A problem with the application of ζ_r for $r > 1$ in the infinite-dimensional case was pointed out by Bentkus and Rachkauskas [1]. In Banach spaces the convergence with respect to ζ_r, $r > 1$, does *not* imply weak convergence (cf. [1]). In Hilbert spaces by results of Senatov [17] there is no inequality of the type $\zeta_r \geqq c\pi^a$, $a > 0$, where π is the Prokhorov-metric, while by a result of Gine, ζ_r-convergence implies weak convergence. (The reference to Gine was pointed out to us by a referee.) Under some smoothnes conditions on the Banach space, Zolotarev [22, Thm. 5] obtained the estimate

$$\pi^{1+r}(\|X\|,\ \|Y\|) \leqq C\,\zeta_r(X, Y), \tag{2.5}$$

where $C = C(r)$. Therefore, under these conditions it follows from (1.4) that

$$\pi\left(n^{-1/\alpha}\left\|\sum_{i=1}^{n} X_i\right\|, \|Y_{(\alpha)}\|\right) \leqq C\,n^{[1/(r+1)](1-r/\alpha)}\zeta_r^{1/(r+1)}\left(X_1,\ Y_{(\alpha)}\right), \qquad 0 < \alpha \leqq r < \infty \tag{2.6}$$

(see also [1, Thm. 19] for a version of (2.6)). It was proved by Senatov [17] that the order in (2.6) is true for $r = 3$, $\alpha = 2$, namely, $n^{-1/8}$. The only known upper estimate for ζ_r applicable in the stable situation is (cf. [24, Thm. 4)

$$\zeta_r \leqq \frac{\Gamma(1+\alpha)}{\Gamma(1+r)}\,\nu_r, \qquad r = m + \alpha, \quad 0 < \alpha \leqq 1, \quad m \in \mathbf{N}, \tag{2.7}$$

where

$$\nu_r(X, Y) = \int \|x\|^r\, d\left|\mathbf{P}_X - \mathbf{P}_Y\right|(x) \tag{2.8}$$

is the absolute pseudomoment and $\mathbf{P}_X$, $\mathbf{P}_Y$ are assumed to have identical pseudomoments of order $\leqq m$. So $\nu_r(X_1, Y_{(\alpha)}) < \infty$ and the assumption of identical pseudomoments ensures the validity of (2.6).

We next would like to show that $\widehat{L}_p$, in spite of being only a simple $(r_p, +)$-ideal metric, $r_p = \min(1, p)$, acts as an ideal $(r, +)$-metric of order $r = 1 + \alpha - \alpha/p$ for $0 < \alpha \leqq p \leqq 2$. We formulate this result for Banach spaces U of type p. Let $(Y_i)_{i\geqq 1}$ be a sequence of independent random signs.

DEFINITION 2.1 (cf. [21] and [7]). Let $1 \leqq p \leqq 2$. A Banach space $(U, \|\cdot\|)$ is said to be of type p, if there exists a constant C such that for all $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in U$

$$\text{(2.9)} \qquad \mathbf{E}\left\|\sum_{i=1}^{n} Y_i x_i\right\|^p \leqq C \sum_{i=1}^{n} \|x_i\|^p.$$

Inequality (2.9) is equivalent to the following condition (cf. [8]): There exists $A > 0$ such that for all $n \in \mathbf{N}$ and $X_1, \ldots, X_n \in \mathcal{X}(U)$ independent with $\mathbf{E}\, X_i = 0$ and finite pth moment holds

$$\text{(2.10)} \qquad \mathbf{E}\left\|\sum_{i=1}^{n} X_i\right\|^p \leqq A \sum_{i=1}^{n} \mathbf{E}\,\|X_i\|^p.$$

Every separable Banach space is of type 1, every finite-dimensional Banach space and every separable Hilbert space is of type 2, $\mathcal{L}^q = \{X \in \mathcal{X}(\mathbf{R}^1)\colon \mathbf{E}\,|X|^q < \infty\}$ is of type $p = \min(2, q)$ for all $q > 0$. Similarly $l_q = \{x \in \mathbf{R}^\infty, \|x\|_q < \infty\}$ is of type $p = \min(2, q)$.

THEOREM 2.2. *If U is of type p, $1 \leqq p \leqq 2$ and $0 < \alpha < p \leqq 2$, then for any i.i.d. r.v.'s $X_1, \ldots, X_n \in \mathcal{X}(U)$ and for a strictly symmetric stable r.v. $Y_{(\alpha)}$ with* $\mathbf{E}\,(X_i - Y_{(\alpha)}) = 0$

$$\text{(2.11)} \qquad \widehat{L}_p\left(n^{-1/\alpha}\sum_{i=1}^{n} X_i,\, Y_{(\alpha)}\right) \leqq B_p^{1/p} n^{1/p - 1/\alpha}\widehat{L}_p(X_1, Y_{(\alpha)})$$

holds.

Proof. From Proposition 2.1 of Woyczynski [21] one obtains for any $q \geqq 1$ that U is of type p if and only if for some B_p and any $n \in \mathbf{N}$ and for $Z_1, \ldots, Z_n \in \mathcal{X}(U)$ independent with $\mathbf{E}\, Z_i = 0$

$$\text{(2.12)} \qquad \mathbf{E}\left\|\sum_{i=1}^{n} Z_i\right\|^q \leqq B_p^q\, \mathbf{E}\left(\sum_{i=1}^{n} \|Z_i\|^p\right)^{q/p}.$$

Let $Y_1, \ldots, Y_n \in \mathcal{X}(U)$ be independent, $Y_i \stackrel{d}{=} Y_{(\alpha)}$, such that $Z_i = X_i - Y_i$, $1 \leqq i \leqq n$ are independent also. Then with $q = p$ and $Y_{(\alpha)} = n^{-1/\alpha}\sum_{i=1}^n Y_i$ from (2.11) follows $L_p(n^{-1/\alpha}\sum_{i=1}^n X_i, Y_{(\alpha)}) \leqq B_p n^{-1/\alpha + 1/p} L_p(X_i, Y_1)$. Passing to the minimal metrics (2.11) follows.

From Strassen's representation of the Levy–Prokhorov distance one obtains the relation

$$\text{(2.13)} \qquad \pi^{p+1} \leqq (\widehat{L}_p)^p.$$

Equation (2.12) implies the following corollary.

COROLLARY 2.3. *Under the assumptions of Theorem 2.2 for $1 \leqq p \leqq 2$, $0 < \alpha <$* $p \leqq 2$

$$\pi\left(n^{-1/\alpha}\sum_{i=1}^{n} X_i,\, Y_{(\alpha)}\right) \leqq B_p^{1/(p+1)} n^{[1/(p+1)](1 - p/\alpha)}\widehat{L}_p\big(X_1, Y_{(\alpha)}\big)^{p/(p+1)}.$$

holds.

Remark 2.2 a) For $r = p \in [1, 2]$ the rate in (2.13) and in Zolotarev's estimate (2.6) are the same. If $U = \mathcal{L}^p$, then it can be shown that one can choose the constants

$$(2.14)\qquad \begin{aligned} B_p &= 18\, p^{3/2}/(p-1)^{1/2} \quad \text{for} \quad 1 < p \leqq 2 \\ B_1 &= 1. \end{aligned}$$

In contrast to (2.6), which concerns the distance between the norms of r.v.'s only, (2.13) concerns the Prokhorov distance itself which is topologically strictly stronger in Banach spaces and much more informative. Furthermore, from [25, p. 272] follows

$$(2.15)\qquad \widehat{L}_p^p(X,\, Y) \leqq 2^p \kappa_p(X,\, Y) \leqq 2^p \nu_p(X,\, Y),$$

where κ_r, $r > 0$, is the rth *difference pseudomoment*

$$(2.16)\qquad \begin{aligned} \kappa_r(X,\, Y) &= \inf\left\{ \mathbf{E}\, d_r(\widetilde{X},\, \widetilde{Y});\quad \widetilde{X} \overset{d}{=} X,\, \widetilde{Y} \overset{d}{=} Y \right\} \\ &= \sup\left\{ \left| \mathbf{E}\, f(X) - \mathbf{E}\, f(Y) \right| \colon f\colon U \to \mathbf{R}, \quad \text{bounded,} \right. \\ &\qquad \left. \left| f(x) - f(y) \right| \leqq d_r(x,\, y), \quad x,\, y \in U \right\}, \end{aligned}$$

and $d_r(x,y) = \|x\|x\|^{r-1} - y\|y\|^{r-1}\|$. Since the problem, whether $\kappa_r(X,Y) < \infty$, $\mathbf{E}\,(X - Y) = 0$ implies $\zeta_r(X,\, Y) < \infty$ is still open for $1 < r < 2$, the right-hand side of (2.13) seems to contain weaker conditions than the right-hand side of (2.6).

b) Bentkus and Rachkauskas [1, Thm. 19] proved that

$$\pi_{\mathcal{A}}\left(n^{-1/\alpha} \sum_{i=1}^n X_i,\; Y_{(\alpha)} \right) \leqq c(r)\, n^{[1/(r+1)](1-r/\alpha)} \zeta_r^{1/(r+1)}\big(X_1,\; Y_{(\alpha)}\big),$$

$$0 \leqq \alpha \leqq 2, \qquad r > \alpha,$$

where $\pi_{\mathcal{A}}$ is the "restricted" Prokhorov metric

$$\pi_{\mathcal{A}}(X,\, Y) := \inf\left\{ \varepsilon > 0\colon\; \mathbf{P}\,(X \in A) \leqq \mathbf{P}\,(Y \in A^\varepsilon) + \varepsilon \quad \text{for all} \quad A \in \mathcal{A} \right\},$$

and $\mathcal{A}$ is a class of Borel sets obeying some very complicated "uniformity" conditions. By the same arguments as in a) one sees that Theorem 2.2 improves the result of Bentkus and Rachkauskas [1] cited above.

Example 2.1. Let $1 \leqq p \leqq 2$, let $(\mathbf{E},\, \mathcal{E},\, \mu)$ be a measure space and define

$$(2.17)\qquad l_{p,\mu} = \left\{ X\colon\, (\mathbf{E},\, \mathcal{E}) \times (\Omega,\, \mathcal{A}) \to (\mathbf{R}^1,\, \mathbf{B}^1)\colon\, \|X\|_{p,\mu} < \infty \right\},$$

where $\|X\|_{p,\mu} = \mathbf{E}\,(\int |X(t)|^p d\mu(t))^{1/p}$. $(l_{p,\mu},\, \|\cdot\|_{p,\mu})$ is a Banach space (identical to $\mathcal{L}^p$ for one point measures μ) of stochastic processes. Let $X_1, \ldots, X_n \in \mathcal{X}(l_{p,\mu})$ with $\mathbf{E}\, X_i = 0$, then

$$\mathbf{E} \left\| \sum_{i=1}^n X_i \right\|_{p,\mu}^p = \mathbf{E} \int \left| \sum_{i=1}^n X_i(t) \right|^p d\mu(t) \leqq \int B_p^p\, \mathbf{E} \left(\sum_{i=1}^n X_i^2(t) \right)^{p/2} d\mu(t)$$

by the Marcinkiewicz–Zygmund inequality (cf. [18, p. 469]). Since $p \leqq 2$ we obtain from the Minkowski-inequality

$$(2.18)\qquad \mathbf{E} \left\| \sum_{i=1}^n X_i \right\|_{p,\mu}^p \leqq B_p \sum_{i=1}^n \mathbf{E} \int |X_i(t)|^p d\mu(t) = B_p \sum_{i=1}^n \|X_i\|_{p,\mu}^p,$$

i.e.,$l_{p,\mu}$ is of type p and,therefore, one can apply Theorem 2.2 and Corollary 2.3 to stochastic processes in $l_{p,\mu}$.

For $0 < \alpha < 2p \leqq 1$ we have the following analogue of Theorem 2.2.

THEOREM 2.4. *Let $X_1, \dots, X_n \in \mathcal{X}(U)$ be i.i.d.,let $Y_1, \dots, Y_n \in \mathcal{X}(U)$ be i.i.d.,and let $0 < \alpha < 2p \leqq 1$, then*

$$\widehat{L}_p\left(n^{-1/\alpha}\sum_{i=1}^{n} X_i,\ n^{-1/\alpha}\sum_{i=1}^{n} Y_i\right) \leqq B_p n^{1/(2p)-1/\alpha}\widehat{T}_p^2(X_1,\ Y_1), \tag{2.19}$$

where $\widehat{T}_p$ is the minimal metric with respect to the compound metric $T_p(X_1, Y_1) = [\sup_{c>0} c^{2p}\,\mathbf{P}(\|X_1 - Y_1\| > c)]^{1/4p}$.

Proof. We have $\mathbf{E}\,\|n^{-1/\alpha}\sum_{i=1}^n X_i - n^{-1/\alpha}\sum_{i=1}^n Y_i\|^p = n^{-p/\alpha}\mathbf{E}\,\|\sum_{i=1}^n (X_i - Y_i)\|^p \leqq n^{-p/\alpha}\mathbf{E}\,(\sum_{i=1}^n \|X_i - Y_i\|)^p \leqq B_p n^{-p/\alpha}\sqrt{n}(\sup_{c>0} c^2\,\mathbf{P}(\|X_1 - Y_1\|^p > c))^{1/2}$, the last inequality following from Lemma 5.3 of Pisier and Zinn [14] with $q = 1/p \geqq 2$. Passing to the minimal metrics, (2.19) follows.

Remark 2.3. a) From the ideality of order p of $\widehat{L}_p$ (cf. (2.3)) one obtains for $0 < \alpha < 1$ the bound

$$\widehat{L}_p\left(n^{-1/\alpha}\sum_{i=1}^{n} X_i,\ n^{-1/\alpha}\sum_{i=1}^{n} Y_i\right) \leqq n^{1-p/\alpha}\widehat{L}_p(X_1,\ Y_1). \tag{2.20}$$

For $0 < \alpha < 2p < 1$ holds $\frac{1}{2}p - 1/\alpha < 1 - p/\alpha$, i.e., the rate in (2.19) is better but (2.20) concerns convergence with respect to the stronger metric $\widehat{L}_p$.

b) In the case $U = l_{p,\mu}$ our results give rates of convergence in "L^p-invariance" principles in the stable case. For some results on L^p-invariance principles we refer to [13, Thm. 1].

3. Ideal metrics and rate of convergence for maxima.

For the maxima of r.v.'s several simple $(r, \vee)$-ideal metrics are known for any $r > 0$, implying by (1.5) the rate of convergence of order $1 - r/\alpha$ (cf. [27] and [12]). In the following example we construct for any $r > 0$ a compound $(r, \vee)$-ideal metric. This shows an essential difference between summation and maxima of r.v.'s.

Example 3.1. (A compound $(r, \vee)$-ideal metric.) For $U = \mathbf{R}^1$ and any $0 < p \leqq \infty$ define for $X, Y \in \mathcal{X}(\mathbf{R}^1)$

$$\Delta_{r,p}(X,\ Y) = \left(\int_{-\infty}^{\infty} \varphi_{X,Y}^p(x)\,|x|^{rp-1}dx\right)^q, \tag{3.1}$$

$$\Delta_{r,\infty}(X,\ Y) = \sup_{x\in\mathbf{R}^1} |x|^r\varphi_{X,Y}(x),$$

where $q = \min(1, 1/p)$ and $\varphi_{X,Y}(x) = \mathbf{P}(X \leqq x < Y) + \mathbf{P}(Y \leqq x < X)$; $\Delta_{r,p}$ is a probability metric. Obviously, for any $c > 0$,

$$\Delta_{r,p}(cX,\ cY) = \left(\int_{-\infty}^{\infty} \varphi_{X,Y}^p(x/c) \times |x|^{rp-1}dx\right)^q = c^{rpq}\Delta_{r,p}(X, Y)$$

and

$$\Delta_{r,\infty}(cX,\ cY) = c^r\Delta_{r,\infty}(X,\ Y)$$

holds. Furthermore, from the relation

$$\{X \vee Z \leqq x < Y \vee Z\} \subset \{X \leqq x < Y\}, \tag{3.2}$$

which can be established for any r.v.'s X, Y, Z by considering the different possible order relations between X, Y, Z, it follows that

$$\Delta_{r,\,p} \text{ is a compound } (r \min(p,1),\, \vee)\text{-ideal metric for } 0 < p \leqq \infty, \quad 0 < r < \infty. \tag{3.3}$$

The corresponding minimal metric $\widehat{\Delta}_{r,p} = \inf\{\Delta_{r,p}(\widetilde{X}, \widetilde{Y});\ \widetilde{X} \overset{d}{=} X,\ \widetilde{Y} \overset{d}{=} Y\}$ is a simple $(r \min(p,1), \vee)$-ideal metric that one can check by the representation

$$\begin{aligned} \widehat{\Delta}_{r,p}(X,\,Y) &= \left(\int_{-\infty}^{\infty} |x|^{rp-1} |F_X(x) - F_Y(x)|^p dx\right)^q, \quad 0 < p < \infty, \\ \widehat{\Delta}_{r,\infty}(X,\,Y) &= \sup_{x \in \mathbf{R}} |x|^r |F_X(x) - F_Y(x)| \end{aligned} \tag{3.4}$$

(cf. [15]). The ideality of $\widehat{\Delta}_{r,\infty}$ was first established and used by Zolotarev in the CLT for maxima of i.i.d. r.v.'s (see [26, p. 299]).

From (1.5) one obtains for a simple probability metric μ which is simple $(r, \vee)$-ideal that $\mu(X_1,\, Z_{(\alpha)}) < \infty$ implies $\mu(n^{-1/\alpha} \bigvee_{i=1}^n X_i,\ Z_{(\alpha)}) \leqq n^{1-r/\alpha} \mu(X_1,\, Z_{(\alpha)})$. For $\mu = \widehat{\Delta}_{r,\infty}$ it was shown by Omey and Rachev [12] that the converse relation is also correct, i.e., the rate in (1.5) is of right order.

We next want to investigate the properties of the L_p-metrics (cf. (2.1)) with respect to maxima. As in Example 2.1 for $0 < \lambda \leqq \infty$ we consider the Banach space $U = l_{\lambda,\mu} = \{X\colon (\mathbf{E}, \mathcal{E}) \times (\Omega, \mathcal{A}) \to (\mathbf{R}^1, \mathbf{B}^1);\ \|X\|_{\lambda,\mu} < \infty\}$, where

$$\|X\|_{\lambda,\mu} = \begin{cases} \left(\int |X(t)|^\lambda d\mu(t)\right)^{1/\lambda^*} & \text{for } 0 < \lambda < \infty \\ \operatorname{ess\,sup}_t |X(t)| := \inf\left\{\varepsilon > 0;\ \int I\{|X(t)| > \varepsilon\}\, \mu(dt) = 0\right\} & \text{for } \lambda = \infty \end{cases}$$

with $\lambda^* = \max(\lambda, 1)$, and define for $X, Y \in U$, $X \vee Y$ to be the pointwise maximum, $X \vee Y(t) = X(t) \vee Y(t)$, $t \in E$. For related limit results for α-max stable processes consider de Haan and Rachev [4]. In the case $\lambda = \infty$, $l_{\infty,\mu}$ is not separable but since the ess sup $|X(t)|$ is measurable this does not cause difficulties.

LEMMA 3.1. a) *For $0 < \lambda \leqq \infty$ and $0 < p \leqq \infty$, L_p is a compound $(r, \vee)$-ideal metric of order $r = \min(1, p)$.*

b) *If $X_1, \ldots, X_n \in \mathcal{X}(l_{\lambda,\mu})$ are i.i.d., and if $Z_{(\alpha)}$ is a α-max-stable process, then*

$$\widehat{L}_p\left(n^{-1/\alpha} \bigvee_{i=1}^n X_i,\ Z_{(\alpha)}\right) \leqq n^{1-r/\alpha} \widehat{L}_p(X_1,\, Z_{(\alpha)}), \quad \text{where} \quad r = \min(1,\, p). \tag{3.5}$$

Proof. a) For $X, Y, Z \in l_{\lambda,\mu}$ and $0 < p < \infty$, we have $\mathbf{E}\,\|X \vee Z - Y \vee Z\|_{\lambda,\mu}^p = \mathbf{E}\left(\int |X(t) \vee Z(t) - Y(t) \vee Z(t)|^\lambda \mu(dt)\right)^{p/\lambda^*} \leqq \mathbf{E}\,\|X - Y\|_{\lambda,\mu}^p$ and for $c > 0$ we have $\mathbf{E}\,\|cX - cY\|_{\lambda,\mu}^p = c^r\, \mathbf{E}\,\|X - Y\|_{\lambda,\mu}$. The case $p = \infty$ is also obvious.

b) Using the representation $Z_{(\alpha)} = n^{-1/\alpha} \bigvee_{i=1}^n Y_i$ with $Y_1, \ldots, Y_n$ i.i.d., $Y_i \overset{d}{=} Z_{(\alpha)}$, b) follows from a) and (1.5).

The estimate (3.5) is interesting for $p \leqq 1$ only; for $1 < p \leqq \lambda < \infty$ one can improve it as follows.

THEOREM 3.2. *Let $1 \leqq p \leqq \lambda < \infty$, then for $X_1, \ldots, X_n \in \mathcal{X}(l_{\lambda,\mu})$ i.i.d. we have*

$$\widehat{L}_p\Big(n^{-1/\alpha}\bigvee_{i=1}^n X_i,\ Z_{(\alpha)}\Big) \leqq n^{1/p-1/\alpha}\widehat{L}_p(X_1,\ Z_{(\alpha)}). \tag{3.6}$$

Proof. Let $Z_{(\alpha)} = n^{-1/\alpha}\bigvee_{i=1}^n Y_i$ and $\lambda < \infty$, then

$$\begin{aligned}
L_p&\left(n^{-1/\alpha}\bigvee_{i=1}^n X_i,\ n^{-1/\alpha}\bigvee_{i=1}^n Y_i\right)\\
&\leqq n^{-1/\alpha}\left\{\mathbf{E}\left[\int\left|\bigvee_{i=1}^n X_i(t) - \bigvee_{i=1}^n Y_i(t)\right|^{\lambda} d\mu(t)\right]^{p/\lambda}\right\}^{1/p}\\
&\leqq n^{-1/\alpha}\left\{\mathbf{E}\left[\int\sum_{i=1}^n\left|X_i(t) - Y_i(t)\right|^{\lambda} d\mu(t)\right]^{p/\lambda}\right\}^{1/p}\\
&\leqq n^{-1/\alpha}\left\{\sum_{i=1}^n\mathbf{E}\left[\int\left|X_i(t) - Y_i(t)\right|^{\lambda} d\mu(t)\right]^{p/\lambda}\right\}^{1/p}\\
&= n^{-1/\alpha}\left\{\sum_{i=1}^n L_p^p(X_i,\ Y_i)\right\}^{1/p},
\end{aligned}$$

the last inequality following from the Minkowski-inequality, since $p/\lambda \leqq 1$. Passing to the minimal metrics, we obtain (3.6). The case $\lambda = \infty$ is similar.

Remark 3.1 a) Comparing (3.6) with (3.5) we see that actually $\widehat{L}_p$ "acts" in this important case as a simple $(\alpha + 1 - \alpha/p, \vee)$-ideal metric. For $1 < p$ it holds that $1/p - 1/\alpha < 1 - 1/\alpha$, i.e., (3.6) is an improvement over (3.5).

b) An analogue of Theorem 3.2 holds also for the sequence space $l_\lambda \subset \mathbf{R}^\infty$.

4. Doubly ideal metrics. We now investigate the question of the existence and construction of doubly ideal metrics posed by Zolotarev [26]. As we have shown in §§2 and 3, L_p are ideal metrics of order $\min(1, p) \leqq 1$ for both operations simultaneously. Let U be a Banach space with maximum operation $\vee$.

DEFINITION. 4.1 (Doubly ideal metrics). A probability metric μ on $\mathcal{X}(U)$ is called

a) (r, I)-ideal, if μ is compound $(r, +)$-ideal and compound $(r, \vee)$-ideal;

b) (r, II)-ideal, if μ is compound $(r, \vee)$-ideal and simple $(r, +)$-ideal;

c) (r, III)-ideal, if μ is simple $(r, \vee)$-ideal and simple $(r, +)$-ideal.

Remark 4.1 Note that if μ is a (r, II)-ideal metric, then one obtains for (X_i) i.i.d, (X_i^*) i.i.d., $S_k = \sum_{i=1}^k X_i$, $S_k^* = \sum_{i=1}^k X_i^*$, $Z_n = n^{-1/\alpha}\bigvee_{k=1}^n S_k$, $Z_n^* = n^{-1/\alpha}\bigvee_{k=1}^k S_k^*$ the estimate $\mu(Z_n, Z_n^*) \leqq n^{-r/\alpha}\mu(\bigvee_{k=1}^n S_k, \bigvee_{k=1}^n S_k^*) \leqq n^{-r/\alpha}\sum_{k=1}^n \mu(S_k, S_k^*) \leqq n^{-r/\alpha}\sum_{k=1}^n\sum_{j=1}^k \mu(X_j, X_j^*)$ and hence for the minimal metrics (cf. [23])

$$\widehat{\mu}(Z_n,\ Z_n^*) \leqq \frac{n(n+1)}{2}\, n^{-r/\alpha}\widehat{\mu}(X_1,\ X_1^*) < n^{2-r/\alpha}\widehat{\mu}(X_1,\ X_1^*), \tag{4.1}$$

which gives us a rate of convergence if $0 < \alpha < r/2$. Therefore, from the known ideal metrics of order $r \leqq 1$ one gets rates for $\alpha \in (0, \frac{1}{2})$. It is therefore of interest to study Zolotarev's question for the construction of doubly ideal metrics of order $r > 1$.

L_p, $0 < p < \infty$, is an example of a $(\min(1, p), \mathrm{I})$-ideal metric. We have seen in §2 that there does not exist a (r, I)-ideal metric for $r > 1$. $\widehat{L}_p$ is a (r, III)-ideal metric of order $r = \min(1, p)$. We now show that Zolotarev's question on the existence of a (r, II) or a (r, III)-ideal metric has essentially a negative answer.

THEOREM 4.1. *Let $r > 1$ and let μ be a (r, III)-ideal metric in $\mathcal{P}(\mathbf{R}^1)$ and assume that μ satisfies the following regularity conditions*:

C1: *If X_n (respectively, Y_n) converges weakly to a constant a (respectively, b), then*

$$\limsup_{n\to\infty} \mu(X_n, Y_n) \geqq \mu(\alpha, b); \tag{4.2}$$

C2: $\mu(a, b) = 0 \iff a = b$. *Then for any integrable $X, Y \in \mathcal{X}(\mathbf{R})$ we have $\mu(X, Y) \in \{0, \infty\}$.*

Proof. If μ is a simple $(r, +)$-ideal metric, then for integrable $X, Y \in \mathcal{X}(\mathbf{R}^1)$ inequality $\mu((1/n)\sum_{i=1}^n X_i, (1/n)\sum_{i=1}^n Y_i) \leqq n^{1-r}\mu(X, Y)$ holds, where (X_i, Y_i) are i.i.d. copies of (X, Y). By the WLLN (weak law of large numbers) and C1 we have $\mu(\mathbf{E}X, \mathbf{E}Y) \leqq \limsup \mu((1/n)\sum X_i, (1/n)\sum Y_i)$. Hence assuming that $\mu(X, Y) < \infty$, we have $\mu(\mathbf{E}X, \mathbf{E}Y) = 0$, i.e., $\mathbf{E}X = \mathbf{E}Y$ by C2. So $\mu(X, Y) < \infty$ implies that $\mathbf{E}X = \mathbf{E}Y$. Therefore, by $\mu(X \vee a, Y \vee a) \leqq \mu(X, Y)$ we have that $\mathbf{E}(X \vee a) = \mathbf{E}(Y \vee a)$, $\forall a \in \mathbf{R}^1$, i.e., $X \stackrel{d}{=} Y$ and, therefore, $\mu(X, Y) = 0$.

Remark 4.2. Condition C1 seems to be quite natural. Let, e.g., $\mathcal{F}$ be a class of non-negative lower semicontinuous functions (l.s.c.) on $\mathbf{R}^2$ and φ: $[0, \infty) \to [0, \infty)$ continuous nondecreasing. Define the minimal functional

$$\mu(X, Y) = \inf\left\{\varphi\big(\sup_{f\in\mathcal{F}} \mathbf{E}\, f(\widetilde{X}, \widetilde{Y})\big); \widetilde{X} \stackrel{d}{=} X,\ \widetilde{Y} \stackrel{d}{=} Y\right\}. \tag{4.3}$$

Then μ is l.s.c. on $\mathcal{X}(\mathbf{R}^1) \times \mathcal{X}(\mathbf{R}^1)$, i.e., $(X_n, Y_n) \xrightarrow{w} (X, Y)$ implies

$$\liminf \mu(X_n, Y_n) \geqq \mu(X, Y); \tag{4.4}$$

so C1 is fullfilled. Actually, for $f \in \mathcal{F}$ the mapping $h_f(X, Y) = \mathbf{E}f(X, Y)$ is l.s.c. Therefore, also $\varphi(\sup_{f\in\mathcal{F}} h_f)$ is l.s.c. and there exists a sequence $(\widetilde{X}_n, \widetilde{Y}_n)$ with $\widetilde{X}_n \stackrel{d}{=} X_n$, $\widetilde{Y}_n \stackrel{d}{=} Y_n$, such that $\mu(X_n, Y_n) = \varphi(\sup_{f\in\mathcal{F}} h_f(\widetilde{X}_n, \widetilde{Y}_n))$.

The sequence of distributions λ_n of (X_n, Y_n) is tight. For any weakly convergent subsequence λ_{n_k} with limit λ, obviously, λ has marginals p^X and p^Y. If $\liminf \mu(X_n, Y_n) < \mu(X, Y)$, then for some subsequence $(m) \subset \mathbf{N}$, $\mu(X_m, Y_m)$ would converge to some $a < \mu(X, Y)$ in contradiction to the l.s.c.-property proved above.

Nevertheless we shall show next that for $0 < \alpha \leqq 2$ the metrics $\widehat{L}_p$ for $1 < p \leqq 2$ "act" as (r, II)-ideal metrics in the rate of convergence problem for $Z_n = n^{-1/\alpha}\bigvee_{k=1}^n S_k$, $Z_n^* = n^{-1/\alpha}\bigvee_{k=1}^n S_k$, where $S_k = \sum_{i=1}^k X_i$, $S_k^* = \sum_{i=1}^k X_i^*$ are sums of i.i.d. r.v.'s. The order of ideality is $r = 2\alpha + 1 - \alpha/p > 2\alpha$ and, therefore, we obtain a rate of convergence $n^{2-r/\alpha}$ (cf. (4.1)).

We consider at first the case that (X_i), (X_i^*) are i.i.d. r.v.'s in $(U, \|\cdot\|) = (l_p, \|\cdot\|)$, where for $x = (x^{(j)}) \in l_p$, $\|x\|_p = (\sum_{j=1}^\infty |x^{(j)}|^p)^{1/p}$. For $x, y \in l_p$ we define $x \vee y = (x^{(j)} \vee y^{(j)})$.

THEOREM 4.2. *Let $0 \leqq \alpha < p \leqq 2$, $1 \leqq p \leqq 2$ and $\mathbf{E}(X_1 - X_1^*)$, then under the conditions formulated above we have*

$$(4.5) \qquad \widehat{L}_p(Z_n, Z_n^*) \leqq \left(\frac{p}{p-1}\right)^{1/p} B_p^{1/p} n^{1/p-1/\alpha} \widehat{L}_p(X_1, X_1^*).$$

In the Hilbert space $(l_2, \|\cdot\|_2)$

$$(4.5') \qquad \widehat{L}_2(Z_n, Z_n^*) \leqq 2n^{1/2-1/\alpha} \widehat{L}_2(X_1, X_1^*).$$

holds. In particular, for the Prokhorov metric π we have

$$(4.6) \qquad \pi(Z_n, Z_n^*) \leqq \left(\frac{p}{p-1}\right)^{1/(p+1)} B_p^{1/(p+1)} n^{(1/(p+1))(1-p/\alpha)} \widehat{L}_p^{p/(p+1)}(X_1, X_1^*).$$

Proof. Let $(\widetilde{X}_i, \widetilde{X}_i^*)$ be independent pairs of random variables in $\mathcal{X}(l_p)$. Then for $\widetilde{S}_k = \sum_{i=1}^k \widetilde{X}_i$, $\widetilde{S}_k^* = \sum_{i=1}^k \widetilde{X}_i^*$ we have

$$(4.7) \qquad \begin{aligned} &L_p^p\left(n^{-1/\alpha} \bigvee_{k=1}^n \widetilde{S}_k, n^{-1/\alpha} \bigvee_{k=1}^n \widetilde{S}_k^*\right) \\ &= n^{-p/\alpha} L_p^p\left(\bigvee_{k=1}^n \widetilde{S}_k, \bigvee_{k=1}^n \widetilde{S}_k^*\right) = n^{-p/\alpha}\, \mathbf{E}\left[\sum_{j=1}^{\infty}\left|\bigvee_{k=1}^n \widetilde{S}_k^{(j)} - \bigvee_{k=1}^n \widetilde{S}_k^{*(j)}\right|^p\right]^{1/p-p} \\ &\leqq n^{-p/\alpha}\, \mathbf{E} \sum_{j=1}^{\infty} \bigvee_{k=1}^n \left|\widetilde{S}_k^{(j)} - \widetilde{S}_k^{*(j)}\right|^p = n^{-p/\alpha} \sum_{j=1}^{\infty} \mathbf{E} \bigvee_{k=1}^n \left|\widetilde{S}_k^{(j)} - \widetilde{S}_k^{*(j)}\right|^p \\ &\leqq n^{-p/\alpha} \sum_{j=1}^{\infty} \frac{p}{p-1}\, \mathbf{E}\left|\widetilde{S}_n^{(j)} - \widetilde{S}_n^{*(j)}\right|^p, \end{aligned}$$

the last inequality following from Doob's inequality. Therefore, we can continue applying the Marcinkiewicz–Zygmund inequality with

$$\begin{aligned} &\leqq n^{-p/\alpha} \sum_{j=1}^{\infty} \frac{p}{p-1} B_p\, \mathbf{E}\left(\sum_{i=1}^n \left(\widetilde{X}_i^{(j)} - \widetilde{X}_i^{*(j)}\right)^2\right)^{p/2} \\ &\leqq \frac{p}{p-1} B_p n^{-p/\alpha} \sum_{j=1}^{\infty} \sum_{i=1}^n \mathbf{E}\left|\widetilde{X}_i^{*(j)} - \widetilde{X}_i^{(j)}\right|^p = \frac{p}{p-1} B_p n^{1-p/\alpha} L_p^p(\widetilde{X}_1, \widetilde{X}_1^*), \end{aligned}$$

the last inequality following from the assumption that $2/p \leqq 1$. Passing to the minimal metrics we obtain (4.5), (4.5′). Finally by means of $\pi^{p+1} \leqq \widehat{L}_p^p$ we have (4.6).

The same proof also applies to the Banach space $l_{p,\mu}$ (cf. Theorem 3.2 and Example 2.1).

THEOREM 4.3. *If $0 \leqq \alpha < p \leqq 2$, $1 \leqq p \leqq 2$ and $X_1, \ldots, X_n \in \mathcal{X}(l_{p,\mu})$ are i.i.d. and $X_1^*, \ldots, X_n^* \in \mathcal{X}(l_{p,\mu})$ are i.i.d. such that $\mathbf{E}(X_1 - X_1^*) = 0$, then*

$$(4.8) \qquad \widehat{L}_p\left(n^{-1/\alpha} \bigvee_{k=1}^n S_k, n^{-1/\alpha} \bigvee_{k=1}^n S_k^*\right) \leqq \frac{p}{p-1} B_p n^{1/p-1/\alpha} \widehat{L}_p(X_1, X_1^*)$$

and

$$\pi(Z_n,\ Z_n^*) \leqq \left(\frac{p}{p-1}\right)^{p/(1+p)} B_p n^{(1/(p+1))(1-p/\alpha)} \widehat{L}_p^{p/(p+1)}(X_1,\ X_1^*). \tag{4.9}$$

5. Stability of queueing systems. The stability problem in queueing theory concerns the "domain" within which the "ideal" queueing model may be applied as a good approximation of the real queueing system under consideration. We consider here the $G|G|1|\infty$ model. For this system the random variables $\zeta_n = s_n - e_n$ are i.i.d., $\mathbf{E}\,\zeta_1 < 0$, s_n (respectively, e_n) denoting the waiting (respectively, interarrival times). Then the one-dimensional stationary distribution of the waiting time coincides with the distribution of the following maximum

$$W = \sup_{k\geqq 0} S_k, \qquad S_k = \sum_{j=-k}^{-1} \zeta_j, \qquad Y_0 = 0, \quad \zeta_{-j} \overset{d}{=} \zeta_j. \tag{5.1}$$

The superscript $(*)$ will denote the characteristics of the disturbed model (i.e., e_k^*, s_k^*, S_k^*) which we assume to be also of type $G|G|1|\infty$. For reference to these problems we refer to [6], [11], [9], [19], [20], [2, Chap. IV], and [10]. As Borovkov [2, p. 239], noted, one of the aims of the stability theorems is to estimate the closeness of $\mathbf{E}\,f^*(W^*)$ and $\mathbf{E}\,f(W)$ for various kind of functions f, f^*. Borovkov [2, pp. 239–240] proposed to consider the case

$$f^*(x) - f(y) \leqq A|x-y| \qquad \text{for all} \quad x,\ y \in \mathbf{R}^1. \tag{5.2}$$

He proved in [2, p. 270] that

$$\sup\left\{\left|\mathbf{E}\ f(W^*) - \mathbf{E}\ f(W)\right| : \left|f(x) - f(y)\right| \leqq A|x-y|, \quad x,\ y \in \mathbf{R}^1\right\} \leqq c\varepsilon, \tag{5.3}$$

assuming that $|\zeta_1^* - \zeta_1| \leqq \varepsilon$ a.s. Here and in what follows c stands for an absolute constant that may be different in different places.

By the Kantorovich duality theorem we have

$$A\widehat{L}_1(W^*,\ W) = \sup\left\{\mathbf{E}\,f^*(W^*) - \mathbf{E}\,f(W); \quad (f^*,\ f) \ \text{ satisfy } \ (5.2)\right\} \tag{5.4}$$

provided that $\mathbf{E}\,|W^*| + \mathbf{E}\,|W| < \infty$. So the estimate in (5.3) essentially says that

$$\widehat{L}_1(W^*,\ W) \leqq c\,\widehat{L}_\infty(\zeta_1^*,\ \zeta_1). \tag{5.5}$$

The estimate in (5.5) needs strong assumptions on the disturbances to conclude stability. In this section, we shall precise and extend the estimate (5.5) considering estimates of

$$\begin{aligned} A\,\widehat{L}_p(W^*,\ W) = \sup\{\,&\mathbf{E}\ f^*(W^*) - \mathbf{E}\ f(W); \quad f^*(x) - f(y) \leqq A|x-y|^p \\ &\text{for all} \quad x,\ y \in \mathbf{R}^1\}, \qquad 0 < p < \infty, \end{aligned} \tag{5.6}$$

assuming that $\mathbf{E}\,|W^*|^p + \mathbf{E}\,|W|^p < \infty$. The following lemma considers the closeness of the prestationary distributions of $W_n = \max(0,\ W_{n-1} + \zeta_{n-1})$, $W_0 = 0$, and of W_n^* (defined like W_n).

LEMMA 5.1. *For any* $0 < p < \infty$ *and* $\mathbf{E}\,\zeta_1 = \mathbf{E}\,\zeta_1^*$

$$\widehat{L}_p(W_n^*,\ W_n) \leqq A_p, \tag{5.7}$$

where

$$A_p = \min\left(\tfrac{n(n+1)}{2}\,\varepsilon_p,\quad c \min_{1/p-1<\delta\le 2/p-1} n^{1/(1+\delta)}\varepsilon^p_{p(1+\delta)}\right) \quad \text{for} \quad p\in(0,\,1],$$
$$A_p = c\,n^{1/p}\varepsilon_p \quad \text{for} \quad 1<p\le 2,$$
$$A_p = c\,n^{1/2}\varepsilon_p \quad \text{for} \quad p>2,$$

and $\varepsilon_p = \widehat{L}_p(\zeta_1,\,\zeta_1^*)$.

Proof. We have that

$$W_n = \max(0,\,\zeta_{n-1},\,\zeta_{n-1}+\zeta_{n-2},\,\dots,\,\zeta_{n-1}+\dots+\zeta_1) \overset{d}{=} \max_{0\le j\le k} S_j,$$

$$W_n^* = \max(0,\,\zeta_{n-1}^*,\,\zeta_{n-1}^*+\zeta_{n-2}^*,\,\dots,\,\zeta_{n-1}^*+\dots+\zeta_1^*) \overset{d}{=} \max_{0\le j\le k} S_j^*.$$

If $0<p\le 1$ then by the (p,II) ideality of L_p follows (see Remark 4.1)

$$\widehat{L}_p(W_n^*,\,W_n) \le \frac{n(n+1)}{2}\,\widehat{L}_p(\zeta_1,\,\zeta_1^*). \tag{5.8}$$

If $1<p\le 2$ then from Theorem 4.3. follows

$$\widehat{L}_p(W_n^*,\,W_n) \le \left(\frac{p}{p-1}\right)^{1/p} B_p n^{1/p}\varepsilon_p. \tag{5.9}$$

From (5.9) and $\widehat{L}_p \le \widehat{L}_{p(1+\delta)}$ for any $0<p<1$ and $1/p-1<\delta\le 2/p-1$ we have $1\le p(1+\delta)\le 2$ and

$$\widehat{L}_p(W_n^*,\,W_n) \le c\,n^{1/(1+\delta)}\varepsilon^p_{p(1+\delta)}. \tag{5.10}$$

For $p\ge 2$ we have

$$L_p^p(W_n,\,W_n^*) = \mathbf{E}\left|\bigvee_{k=1}^n S_k - \bigvee_{k=1}^n S_k^*\right|^p \le \frac{p}{p-1}\,\mathbf{E}\,|S_n-S_n^*|^p \le c\,n^{p/2}L_p(\zeta_1,\,\zeta_1^*)^p.$$

This last inequality is a consequence of the Marcinkiewicz–Zygmund inequality (cf. [3, p. 357]).

Remark 5.1. a) The estimates in (5.7) are of the right order as can be seen by examples.

If, e.g., $p\ge 2$ consider $\zeta_i \overset{d}{=} N(0,\,1)$ and $\zeta_i^*=0$, then $\widehat{L}_p(W_n^*,\,W_n) = c\,n^{1/2}$.

b) If $p=\infty$, then $\widehat{L}_\infty(W_n^*,\,W_n)\le n\,\varepsilon_\infty$.

Define now the stopping times

$$\theta = \inf\left\{k\colon\, W_k = \max_{0\le j\le k} S_j = W = \sup_{j\ge 0} S_j\right\}, \tag{5.11}$$
$$\theta^* = \inf\{k\colon\, W_k^* = W^*\}.$$

From Lemma 5.1 we now obtain estimates for $\widehat{L}_p(W^*,\,W)$ in terms of the distributions of θ, θ^*. Define $G(n) := \mathbf{P}(\max(\theta^*,\,\theta)=n) \le \mathbf{P}(\theta^*=n) + \mathbf{P}(\theta=n)$.

THEOREM 5.2. *If* $1<p\le 2$, $\lambda,\,\mu\ge 1$ *with* $1/\lambda+1/\mu=1$ *and* $\mathbf{E}\,\zeta_1 = \mathbf{E}\,\zeta_1^* < 0$, *then*

$$\widehat{L}_p^p(W^*,\,W) \le c\,\varepsilon_{p\lambda}\sum_{n=1}^\infty n^{1/\lambda}\big(G(n)\big)^{1/\mu}. \tag{5.12}$$

Proof.

$$\begin{aligned} L_p^p(W^*, W) &= \mathbf{E}\,|W^* - W|^p = \sum_{n=0}^{\infty} \mathbf{E}\,|W^* - W|^p\, \mathbf{1}\{\max(\theta^*, \theta) = n\} \\ &= \sum_{n=0}^{\infty} \mathbf{E}\,|W_n^* - W_n|^p\, \mathbf{1}\{\max(\theta^*, \theta) = n\} \\ &\leqq \sum_{n=0}^{\infty} \left(\mathbf{E}\,|W_n^* - W_n|^{p\lambda}\right)^{1/\lambda} G(n)^{1/\mu} \leqq \sum_{n=0}^{\infty} A_{p\lambda}^p G(n)^{1/\mu} \\ &= \sum_{n=0}^{\infty} c\, n^{1/\lambda} \varepsilon_{p\lambda} G(n)^{1/\mu} \quad \text{(by (5.7))}. \end{aligned}$$

Remark 5.2. a) If

$$G(n) \leqq C\, n^{-\mu(1/\lambda + 1 + \varepsilon)} \tag{5.13}$$

for some $\varepsilon > 0$, then $\sum_{n=1}^{\infty} n^{1/\lambda} G(n)^{1/\mu} \leqq C \sum_{n=1}^{\infty} 1/n^{1+\varepsilon} < \infty$. For conditions on ζ_i, ζ_i^* ensuring (5.13) cf. [2, pp. 229, 230, 240].

b) For $0 < p \leqq 1$ and $p > 2$ we similarly get from Lemma 5.1 corresponding estimates for $\widehat{L}_p(W^*, W)$.

c) Note that $\widehat{L}_1(W^*, W) \leqq \widehat{L}_p(W^*, W)$ (i.e. $\widehat{L}_p$ considers more functions w the deviation) but on the other hand $\varepsilon_{p\lambda} = \widehat{L}_{p\lambda}(\zeta_1^*, \zeta_1) \leqq \widehat{L}_\infty(\zeta_1^*, \zeta_1)$. Therefore, the estimates in Theorem 5.2 improve that of Borovkov.

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