Consistent risk measures for portfolio vectors

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Abstract

The main purpose to study risk measures for portfolio vectors $X = (X_1, \dots, X_d)$ is to measure not only the risk of the marginals X_i separately but to measure the joint risk of X caused by the variation of the components and their possible dependence.

Thus an important property of risk measures for portfolio vectors is consistency with respect to various classes of convex and dependence orderings. From this perspective we introduce and study convex risk measures for portfolio vectors defined axiomatically and further introduce two natural and easy to interprete and calculate classes of examples of risk measures for portfolio vectors and investigate their consistency properties.

Key words: convex risk measure, multivariate portfolio, convex ordering
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1 Introduction

In this paper we consider risk measures defined for portfolio vectors $X = (X_1, \dots, X_d)$ on a probability space $(\Omega, \mathfrak{A}, P)$. The aim is to have a measure of risk not only for the marginals X_i separately but to measure the joint risk of all components caused by their variation and by possible dependence of the X_i.

In the first part of this paper we introduce two classes of examples of risk measures for portfolio vectors which are easy to interprete and typically can be calculated in more or less explicit form. An explicit calculation would be prohibitive for an extension of the idea of worst case conditional expectation to the multivariate case (see [6]). A natural idea to measure the joint risk of X is to consider the one-dimensional risk of the joint portfolio or that of the maximal component

$$\Psi(X) = \Psi_1\Big(\sum_{i=1}^{d} X_i\Big) \tag{1}$$

or

$$\Psi(X) = \Psi_1(\max_{i \le 1} X_i), \tag{2}$$

where Ψ_1 is a suitable one-dimensional risk measure like the expected shortfall, the average value at risk or a distorted risk measure. One might also want to consider a combination of both,

$$\Psi(X) = \alpha\Psi_1\Big(\sum_{i=1}^{d} X_i\Big) + \beta\Psi_1(\max_{i \le d} X_i)$$

An extension of the idea of measuring the risk of the joint portfolio or the maximal portfolio as in (1) and (2) is to introduce some class $\mathcal{F}_0 = \{f_\alpha; \alpha \in A\}$ or real functions on $\mathbb{R}^d$ and to measure the risk of the real 'aspects' $f_\alpha(X)$ of X, $\alpha \in A$, by

$$\Psi_A := \sup_{\alpha \in A} \Psi_\alpha(f_\alpha(X)), \tag{3}$$

and

$$\Psi_M := \sup_{\mu \in M} \int \Psi_\alpha(f_\alpha(X)) d\mu(\alpha). \tag{4}$$

Here M is some class of weighting measures on A and $\{\Psi_\alpha\}$ is a class of one-dimensional risk measures for $\alpha \in A$. Thus we are measuring the maximal risk of the real aspects $f_\alpha(X)$ or the maximal average risk over some weighting class M. If e.g. $A = \Delta := \{\alpha \in \mathbb{R}^d_+; \sum \alpha_i = 1\}$, then

$$\Psi_\Delta := \sup_{\alpha \in \Delta} \Psi_1(\alpha \cdot X) \tag{5}$$

is the maximal risk of X over all positive directions α and

$$\Psi_\mu := \int_\Delta \Psi_1(\alpha \cdot X) d\mu(\alpha) \tag{6}$$

is the risk of X averaged over all positive directions. The class of proposed risk measures Ψ_A,Ψ_M will be investigated in the first part of this paper with respect to its consistency properties concerning several types of multivariate convex stochastic orderings.

In the second part of the paper we introduce axiomatically the class of all convex risk measures i.e. monotone, translation invariant and convex functionals and establish a representation result by scenario measures similar to the one-dimensional case. It is however in the multivariate portfolio case not clear how to single out axiomatically interesting subclasses of risk measures as in the one-dimensional case, where one has as an important tool the Choquet integral and the related distortion risk measures with nice characterizations of their properties (see Wang et al. (1997), Wirch and Hardy (2000), Yaari (1987), and Dhaene et al. (2004)). This is the reason to restrict to risk measures Ψ_A, Ψ_M of the form (3), (4) which by their definition exhibit a simple representation form, which in dimension one can be characterized axiomatically as in the Kusuoka representation result. Further in the multivariate case there is not a natural and simple generating class of convex functions and quantiles as in dimension 1 to compare the risk of the tails. It is therefore of interest to study general ordering properties of multivariate risk measures. In particular it turns out that all convex, law invariant risk measures are consistent with respect to convex ordering for any dimension $d \geq 1$.

In order to have monotonicity of the risk measures Ψ w.r.t. stochastic ordering we adapt the insurance risk interpretation of X. In order to apply the following results in the financial context where $-X$ is the liability we have to switch to

$$\varrho(X) = \Psi(-X). \tag{7}$$

2 Consistency against various convex orderings

In this section we investigate consistency of the two classes of risk measures in (3) and (4) against various convex orderings. In the multivariate case there are several different relevant convex type stochastic orderings which all serve different purposes but all are relevant for relating two random vectors X, Y with respect to their diffusiveness and thus their risky status. The orderings are defined via function classes $\mathcal{F}$ like the convex functions $\mathcal{F}_{\mathrm{cx}}$, the symmetric convex functions $\mathcal{F}_{\mathrm{symm,cx}}$, the directionally convex functions $\mathcal{F}_{\mathrm{dcx}}$, the supermodular functions $\mathcal{F}_{\mathrm{sm}}$, the $\triangle$-monotone functions $\mathcal{F}_{\triangle}$ and the Schur-convex functions $\mathcal{F}_{\mathrm{m}}$ or with respect to the increasing resp. decreasing elements of these classes like $\mathcal{F}_{\mathrm{icx}}$, $\mathcal{F}_{\mathrm{idcx}}$. The corresponding stochastic orders are defined by

$$X \leq_{\mathcal{F}} Y \text{ if } Ef(X) \leq Ef(Y), \tag{8}$$

for all $f \in \mathcal{F}$ such that the integrals exist. The stochastic orderings are denoted by $X \leq_{\mathrm{cx}} Y, X \leq_{\mathrm{dcx}} Y, X \leq_{\mathrm{sm}} Y, \ldots$ For a survey of these stochastic orderings, for criteria and their relevance for risk measurement see Müller and Stoyan (2002), Rüschendorf (2005), and Bäuerle and Müller (2005).

In order to establish consistency of risk measurs Ψ_A, Ψ_M with respect to convex type stochastic orderings a basic property is that the functions $f_\alpha \in \mathcal{F}_0$ preserve the convex stochastic orderings. We discuss this property for the various convexity classes mentioned above.

a) Convex order $\leq_{\mathrm{cx}}$

In order to obtain an ordering result for the risk measures Ψ_A, Ψ_M we need the assumption

Assumption $\boldsymbol{A}_{\mathbf{icx}}$: *Let $\{\Psi_\alpha\}$ be one-dimensional law invariant risk measures* *such that Ψ_α is monotone and preserves the increasing convex order $\leq_{\mathrm{icx}}$, $\alpha \in A$.*

It is easy to see that for $\mathcal{F} = \mathcal{F}_{\mathrm{cx}}$ the class of convex functions $f : \mathbb{R}^d \to \mathbb{R}^1$, and for $f \in \mathcal{F}_{\mathrm{cx}}$ holds:

$$X \leq_{\mathrm{cx}} Y \Rightarrow f(X) \leq_{\mathrm{icx}} f(Y). \tag{9}$$

As consequence of (9) we obtain the following proposition.

Proposition 2.1 (Consistency with the convex order) *Under assumption A_{icx} on $\{\Psi_\alpha\}$ let $\mathcal{F}_0 = \{f_\alpha, \alpha \in A\} \subset \mathcal{F}_{\mathrm{cx}}$, then Ψ_A and Ψ_M are consistent with the convex order, i.e.*

$$X \leq_{\mathrm{cx}} Y \Rightarrow \Psi_A(X) \leq \Psi_A(Y) \quad \text{and} \quad \Psi_M(X) \leq \Psi_M(Y). \tag{10}$$

If $\mathcal{F}_0 \subset \mathcal{F}_{\mathrm{icx}}$, then Ψ_A, Ψ_M are consistent with $\leq_{\mathrm{icx}}$.

Remark 2.2 *In particular it follows from* (10) *that $\Psi_1\left(\sum_{i=1}^d X_i\right)$, $\Psi_1(\max_i X_i)$ and $\Psi_\triangle(X) = \sup_{\alpha\in\triangle} \Psi_1(\alpha \cdot X)$ are risk measures consistent with the convex order if Ψ_1 satisfies A_{icx}.*

b) Supermodular and directionally convex ordering The supermodular ordering and the directionally convex ordering are of particular interest for multivariate risk comparison (for definition see [9] or [12, Definition 4.1]). Twice differentiable functions f are **supermodular** if

$$\frac{\partial^2}{\partial x_i \partial x_j} f(x) \geq 0 \quad \text{for all } x \text{ and } i < j; \tag{11}$$

f is **directionally convex** if

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \quad \forall\, i \leq j. \tag{12}$$

If random vectors X, Y are comparable with respect to the stronger supermodular ordering $\leq_{\mathrm{sm}}$, then necessarily the marginals are identical, i.e. $X_i \stackrel{d}{=} Y_i$, $1 \leq i \leq d$. The comparison w.r.t. the directionally convex order $X \leq_{\mathrm{dcx}} Y$ is possible if the marginals increase convexly i.e. $X_i \leq_{\mathrm{cx}} Y_i$, $1 \leq i \leq d$. Similarly, comparison w.r.t. the increasing directionally convex order $\leq_{\mathrm{idcx}}$ is possible if $X_i \leq_{\mathrm{icx}} Y_i$, $1 \leq i \leq d$.

Proposition 2.3 *Let $\{\Psi_\alpha\}$ fulfill assumption A_{icx}*

a) If $X \leq_{\mathrm{ism}} Y$, then $f(X) \leq_{\mathrm{icx}} f(Y)$ for all $f \in \mathcal{F}_{\mathrm{ism}}$ and for $\mathcal{F}_0 \subset \mathcal{F}_{\mathrm{ism}}$ holds:

$$\Psi_A(X) \leq \Psi_A(Y), \quad \Psi_M(X) \leq \Psi_M(Y). \tag{13}$$

b) If $X \leq_{\mathrm{idcx}} Y$, then $f(X) \leq_{\mathrm{icx}} f(Y)$ for all $f \in \mathcal{F}_{\mathrm{idcx}}$ and for $\mathcal{F}_0 \subset \mathcal{F}_{\mathrm{idcx}}$ holds:

$$\Psi_A(X) \leq \Psi_A(Y), \quad \Psi_M(X) \leq \Psi_M(Y). \tag{14}$$

Proof: By an approximation argument it is sufficient for the ordering result to consider twice differentiable functions f. Then for any $h \in \mathcal{F}_{\mathrm{icx}} \cap C^2(\mathbb{R}^1)$ holds

$$\frac{\partial^2 h \circ f}{\partial x_i \partial x_j} = h'' \circ f \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} + h' \circ f \frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{15}$$

Since $h'' \circ f \geq 0$, $h' \circ f \geq 0$, we obtain that $\frac{\partial^2 h \circ f}{\partial x_i \partial x_j}$ is positive for $i \neq j$ and $f \in \mathcal{F}_{dcx} \cap C^2(\mathbb{R}^d)$ and, therefore, (13) follows. The proof of (14) is similar. □

Remark 2.4 *Sufficient conditions for $\leq_{\mathrm{sm}}$ and $\leq_{\mathrm{dcx}}$ were established in [11], e.g. for $d = 2$ assume:*

$$\overline{F}_X(u,v) - \overline{F}_{X_1}(u)\overline{F}_{X_2}(v) \leq \overline{F}_Y(u,v) - \overline{F}_{Y_1}(u)\overline{F}_{Y_2}(v). \tag{16}$$

Then

$$X_i \leq_{\mathrm{cx}} Y_i, \ \ i = 1,2 \Rightarrow X \leq_{\mathrm{dcx}} Y$$
$$X_i \leq_{\mathrm{icx}} Y_i, \ \ i = 1,2 \Rightarrow X \leq_{\mathrm{idcx}} Y,$$

(see [11, Corollary 3.2]).

In particular (13) *and* (14) *imply results for the type that more positive dependence of random vectors leads to higher risks. The classical result in this direction going back to Tchen (1980) is that*

$$X \leq_{\mathrm{sm}} X^c := \left(F_1^{-1}(U), \ldots, F_d^{-1}(U)\right), \tag{17}$$

where X^c is the comonotonic vector to X and F_i are the d.f.s of X_i. (17) *yields in particular the consequence that the comonotonic vector is most risky for the joint portfolio*

$$\sum_{i=1}^d X_i \leq_{\mathrm{icx}} \sum_{i=1}^d F_i^{-1}(U), \tag{18}$$

and

$$\Psi_M(X) \leq \Psi_M(X^c), \quad \Psi_A(X) \leq \Psi_A(X^c). \tag{19}$$

The comonotonic risk vector has the highest risk in the Fréchet class of X with respect to all risk measures of the form Ψ_M, Ψ_A. For several related statements on this topic see [2,4,9,12].

c) Schur convex and symmetric convex ordering

Let $\prec$ denote the Schur order on $\mathbb{R}^d$,

$$x \prec y \Leftrightarrow \sum_{i=1}^{k} x_{(i)} \le \sum_{i=1}^{k} y_{(i)}, \quad 1 \le k \le d-1 \text{ and } \sum_{i=1}^{d} x_i = \sum_{i=1}^{d} y_i \tag{20}$$

and the increasing Schur order

$$x \preceq y \Leftrightarrow \sum_{i=1}^{k} x_{(i)} \le \sum_{i=1}^{k} y_{(i)}, \quad 1 \le k \le d, \tag{21}$$

where $x_{(1)} \ge \cdots \ge x_{(d)}$ is the ordered vector. The corresponding stochastic orders are called majorization orders (see [8]) and denoted by $\prec_{\mathrm{m}}$ and $\preceq_{\mathrm{m}}$. These are relevant diffusion orders. Note that choosing the convex cone K as

$$K = \{x \in \mathbb{R}^d;\ 0 \preceq x\} \tag{22}$$

we have that $\mathbb{R}^{\mathrm{d}}_+ \subset K$. Monotonicity of a risk measure Ψ w.r.t. the cone ordering induced by K means that comparability of random vectors X in the increasing Schur order $\preceq$ implies comparison of the risks

$$X \preceq Y \Rightarrow \Psi(X) \le \Psi(Y). \tag{23}$$

Thus studying consistency w.r.t. Schur ordering is equivalent to studying monotonicity w.r.t. the cone ordering induced by K.

Proposition 2.5 *Let $\{\Psi_\alpha\}$ fulfill assumption A_{icx}.*

a)

$$\begin{aligned} &X \prec_{\mathrm{m}} Y \textit{ implies } X \le_{\mathrm{symm,cx}} Y \\ &X \preceq_{\mathrm{m}} Y \textit{ implies } X \le_{\mathrm{isymm,cx}} Y \end{aligned} \tag{24}$$

b) If $X \preceq_{\mathrm{m}} Y$ and $\mathcal{F}_0 \subset \mathcal{F}_{\mathrm{isymm,cx}}$, then

$$\Psi_A(X) \le \Psi_A(Y) \textit{ and } \Psi_M(X) \le \Psi_M(Y). \tag{25}$$

Proof: If $X \prec_{\mathrm{m}} Y$, then by Strassen's representation result (see [10]) there exist versions $\widetilde{X} \stackrel{d}{=} X$, $\widetilde{Y} \stackrel{d}{=} Y$ such that $\widetilde{X} \prec \widetilde{Y}$ and, therefore, $\widetilde{X} = \sum_{\pi \in S_{\mathrm{d}}} \alpha_\pi \widetilde{Y}_\pi$ for some random $\alpha_\pi \ge 0$ with $\sum_{\pi \in S_{\mathrm{d}}} \alpha_\pi = 1$, where $\widetilde{Y}_\pi$ is the reordered vector. If $f \in \mathcal{F}_{\mathrm{symm,cx}}$ then $f(\widetilde{X}) \le \sum \alpha_\pi f(\widetilde{Y}_\pi) = f(\widetilde{Y})$ and thus $Ef(X) \le Ef(Y)$ i.e. $X \le_{\mathrm{symm,cx}} Y$. If $X \preceq_{\mathrm{m}} Y$ then by a variant of Strassen's representation theorem (see [10]) there exist $\widetilde{X} \stackrel{d}{=} X$, $\widetilde{Y} \stackrel{d}{=} Y$ and Z such that $\widetilde{X} \prec Z \le \widetilde{Y}$. If $f \in \mathcal{F}_{\mathrm{isymm,cx}}$, then

$$\begin{aligned} Ef(X) = Ef(\widetilde{X}) &\le Ef(Z) \\ &\le Ef(\widetilde{Y}) = Ef(Y), \quad \text{i.e. } X \le_{\mathrm{isymm,cx}} Y. \end{aligned}$$

The other conclusions are similar to the corresponding ones in Propositions 2.1 and 2.3. □

Conclusion:

a) The risk measures Ψ_A, Ψ_M define meaningful and interpretable classes of risk measures, consistent with respect to the various classes of convex orderings. Also the simple way of construction allows one easily to establish the further relevant risk properties of Ψ_A, Ψ_M, which are inherited from the basic one-dimensional risk measures $\{\Psi_\alpha\}$ used for their construction. If we take e.g. $\Psi_\triangle(X) = \sup_{\alpha\in\triangle} \Psi_1(\alpha \cdot X)$ from (5) then $\Psi_\triangle$ is a convex risk measure if Ψ_1 is convex and $\Psi_\triangle$ is a coherent risk measure if Ψ_1 is coherent.
b) To measure the risk caused by dependence of the componentes of X it is natural to consider the difference

$$\widehat{\Psi}(X) := \Psi(X) - \Psi(X^*) \tag{26}$$

where X^* is the vector with independent marginals $X_i^* \stackrel{d}{=} X_i$, $1 \le i \le d$.

3 Convex risk measures for portfolio vectors

In this section we introduce the general class of all convex risk measures for portfolio vectors. It seems that further axiomatic classes of risk measures have so far not been considered in the literature except in a paper of Jouini et al. (2004) who introduced vector valued coherent risk measures as set valued functionals $R : L_d^\infty(P) \to \mathcal{P}(\mathbb{R}^n)$, where $n \le d$ represents n aspects of the risk and where $L_d^\infty(P) = \prod_{i=1}^d L^\infty(P)$ is the set of risk portfolio vectors $X = (X_1, \dots, X_d)$, $X_i \in L^\infty(P)$ for all i. They proved a representation result for the risk sets which have monotone, homogeneous, translation invariant and subadditive properties. Monotonicity is defined via a closed convex cone $K \subset \mathbb{R}^d$, $\mathbb{R}_+^d \subset K$ and $K \ne \mathbb{R}^d$ by

$$x \ge 0 \;\; \Leftrightarrow \;\; x \in K, \tag{27}$$

which is extended to $L_d^\infty(P)$ by

$$X \succeq 0 \;\; \Leftrightarrow X \in K\,[P]. \tag{28}$$

Jouini et al. (2004) postulated that any entry in the i-th position, $i \ge n+1$ can be substituted by some entry in the first position, i.e. $\forall i \ge n+1 : \exists \alpha, \beta > 0$ such that

$$\alpha e_1 - e_i \in K \text{ and } e_i - \beta e_1 \in K, \tag{29}$$

where $e_i \in \mathbb{R}^d$ are the unit vectors, which is motivated by considering dynamic exchange processes. This assumption excludes e.g. the interesting case that $K = \mathbb{R}^d_+$. In our paper we do not pose this condition.

We use this ordering framework but restrict to one-dimensional risk measures for risky portfolio vectors $X = (X_1, \dots, X_d) \in L^\infty_d(P)$. In this section we use the notation of financial risk measures $\varrho(X) = \Psi(-X)$ (see (7)).

Definition 3.1 *$\varrho : L^\infty_d(P) \to \mathbb{R}$ is a convex risk measure if for $X, Y \in L^\infty_d(P)$ holds:*

M1) $X \succeq Y \Rightarrow \varrho(X) \leq \varrho(Y)$
M2) $\varrho(X + me_i) = -m + \varrho(X)$ for all $m \in \mathbb{R}$ and $1 \leq i \leq d$.
M3) $\varrho(\alpha X + (1-\alpha)Y) \leq \alpha\varrho(X) + (1-\alpha)\varrho(Y)$ for all $\alpha \in (0,1)$.

Thus convex risk measures are monotone, translation invariant convex functionals on $L^\infty_d(P)$.

As in the real case convex risk measures can be equivalently defined in terms of acceptance sets. The risk of a portfolio X is the smallest amount which has to be added to X, such that the payment $X + me_i$ is acceptable for some i.

Definition 3.2 *A subset $\mathcal{A} \subset L^\infty_d(P)$ is called (convex) acceptance set if*

A1) $\mathcal{A}$ is closed and convex
A2) $X, Y \in L^\infty_d(P)$, $X \succeq Y$ and $Y \in \mathcal{A}$ implies $X \in \mathcal{A}$
A3) $X + me_i \in \mathcal{A} \Leftrightarrow X + me_j \in \mathcal{A}$ for all i, j
A4) $\mathbb{R}^d \not\subset \mathcal{A}$.

For any acceptance set $\mathcal{A}$ we define a risk measure $\varrho_{\mathcal{A}}$ by

$$\varrho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R};\ \ X + me_1 \in \mathcal{A}\} \tag{30}$$

Then as in the one-dimensional case holds

Proposition 3.3 *a) If $\mathcal{A}$ is a convex acceptance set, then $\varrho_{\mathcal{A}}$ is a convex risk measure.*
b) If ϱ is a convex risk measure, then

$$\mathcal{A}_\varrho := \{X \in L^\infty_d(P);\ \ \varrho(X) \leq 0\} \tag{31}$$

is a convex acceptance set.

Let

$$L^\infty_d(K) = L^\infty_d(K, P) = \{X \in L^\infty_d(P);\ \ X \in K\} \tag{32}$$

and let $\mathrm{ba}_d(P)$ denote the finitely additive measures on $L_d^\infty(P)$ absolutely continuous w.r.t. P, which are the positive part of the dual space of $L_d^\infty(P)$. We use the notation

$$Q(X) = E_Q(X) = \sum_{i=1}^d E_{Q_i} X_i \tag{33}$$

for $Q \in \mathrm{ba}_d(P)$ and define the elements of $\mathrm{ba}_d(P)$ which are positive on K by

$$\mathrm{ba}_d(K) = \mathrm{ba}_d(K, P) = \{Q \in \mathrm{ba}_d(P);\ \ E_Q X \geq 0, \forall X \in L_d^\infty(K)\}. \tag{34}$$

Then we obtain the following representation of convex risk measures.

Theorem 3.4 *A functional $\varrho : L_d^\infty(P) \to \mathbb{R}^1$ is a convex risk measure if and only if there exists a function $\alpha : \mathrm{ba}_d(K) \to (-\infty, \infty]$ such that*

$$\varrho(X) = \sup_{Q \in \mathrm{ba}_d(K)} \{E_Q(-X) - \alpha(Q)\}. \tag{35}$$

α can be chosen as Legendre–Fenchel inverse of ϱ

$$\begin{aligned} \alpha(Q) &= \sup_{X \in L_d^\infty(K)} (E_Q(-X) - \varrho(X)) \\ &= \sup_{X \in \mathcal{A}_\varrho} E_Q(-X). \end{aligned} \tag{36}$$

Proof: The proof uses similar ideas as in the one-dimensional case in [5]. Obviously any ϱ as in (35) satisfies M1)–M3). Conversely, let ϱ be a convex risk measure and define

$$S(X) := \Big\{ m \in \mathbb{R}^1;\ \ \sup_{Q \in \mathrm{ba}_d(K)} E_Q(-(X + me_1)) - \alpha(Q) \leq 0 \Big\}.$$

Then by definition of α in (36) holds

$$R(X) := \{m \in \mathbb{R}^1;\ \ \varrho(X + me_1) \leq 0\} \subset S(X), \tag{37}$$

as $m \in R(X)$ implies $X + me_1 \in \mathcal{A}_\varrho$. Therefore, for any $Q \in \mathrm{ba}_d(K)$ holds $E_Q(-(X + me_1)) \leq \alpha(Q)$. Thus

$$\sup_{Q \in \mathrm{ba}_d(K)} \{E_Q(-(X + me_1)) - \alpha(Q)\} \leq 0$$

i.e. $m \in S(X)$.

To prove the converse inclusion $S(X) \subset R(X)$, assume that there exist some $m_0 \in S(X)$ with $m_0 \notin R(X)$. Then $\sup_{Q \in \mathrm{ba}_d(K)} \{E_Q(-(X + m_0 e_1)) - \alpha(Q)\} \leq 0$ and, as $m_0 \notin R(X)$ it holds that $X + m_0 e_1 \notin \mathcal{A}_\varrho$. By the separation theorem for convex sets there exists a continuous linear functional $\ell \in (L_d^\infty(P))^*$ with $\inf_{Y\in\mathcal{A}_\varrho} \ell(Y) > \ell(X + m_0 e_1)$.

We next prove that ℓ is positive, i.e. $\ell(Y) \geq 0$ for $Y \in L_d^\infty(K)$. As K is a convex cone $\lambda Y \in K, \forall\lambda > 0$. By M2) holds $\varrho(me_1) \leq \varrho(\lambda Y + me_1)$ and thus $R(0) \subset R(\lambda Y)$. This implies that $\lambda Y + me_1 \in \mathcal{A}_\varrho$ for all $m \in R(0)$. As consequence we get

$$\begin{aligned} -\infty &< \ell(X + m_0 e_1) < \ell(\lambda Y + me_1) \\ &= \lambda\ell(Y) + \ell(me_1) \text{ for all } \lambda > 0, \end{aligned}$$

and, therefore, $\ell(Y) \geq 0$. Thus ℓ induces a finitely additive measure $Q \in \mathrm{ba}_d(K)$ with

$$\inf_{Y\in\mathcal{A}_\varrho} E_Q Y > E_Q(X + m_0 e_1). \tag{38}$$

On the other hand we have

$$E_Q(X + m_0 e_1) \geq -\alpha(Q) = \inf_{Y\in\mathcal{A}_\varrho} E_Q Y, \tag{39}$$

a contradiction.

In consequence for all $X \in L_d^\infty(P)$ holds $S(X) = R(X)$, which implies the representation $\varrho(X) = \sup_{Q\in\mathrm{ba}_d(K)}\{E_Q(-X) - \alpha(Q)\}$ as in (35). The equivalence in (36) can be shown as in [5]. □

To obtain a representation of ϱ by P-continuous σ-additive measures positive on K we need the Fatou property of ϱ.

Definition 3.5 *A functional $\varrho : L_d^\infty(P) \to \mathbb{R}$ has the Fatou property if for any uniformly bounded sequence $(X_n) \subset L_d^\infty(P)$ with $X_n \xrightarrow{P} X$ for some $X \in L_d^\infty(P)$ holds*

$$\varrho(X) \leq \varliminf_{n\to\infty} \varrho(X_n).$$

The class of σ-additive P-continuous measures, positive on K, can be represented by the corresponding class $L_d^1(K)$ of densities $f = (f_1, \dots, f_d)$.

Theorem 3.6 *Let $\varrho : L_d^\infty$ be a convex risk measure. Then the following are equivalent:*

1) *The class $\mathrm{ba}_d(K)$ in the representation (35) of ϱ can be replaced by the class $L_d^1(K)$ of σ-additive measures.*
2) *The acceptance set $\mathcal{A}_\varrho = \{X \in L_d^\infty(P); \varrho(X) \leq 0\}$ is w^*-closed in $L_d^\infty(P)$.*
3) *ϱ has the Fatou property.*

Proof: The proof is analogously to that of Theorem 3.4. If $L_d^\infty(P)$ is supplied with the w^*-topology, then the dual space $(L_d^\infty(P))^*$ is given by $L_d^1(P)$. Therefore, the linear functional ℓ in the proof can be identified with a σ-additive measure. The proof of the equivalence to 2) and 3) is as in the one-dimensional case. $\square$

Remark 3.7 *a) A risk measure $\varrho : L_d^\infty(P) \to \mathbb{R}$ is called* ***coherent*** *if M1), M2) hold and the convexity condition M3) is replaced by the homogeneity and subadditivity conditions M4) and M5):*
M4) $\varrho(tX) = t\varrho(X)$ for all $t > 0$
M5) $\varrho(X+Y) \le \varrho(X) + \varrho(Y)$ for all $X, Y \in L_d^\infty(P)$.
For coherent risk measures Theorems 3.4 and 3.6 imply the representation

$$\varrho(X) = \sup_{Q \in \mathcal{P}} E_Q(-X) \tag{40}$$

with some subset $\mathcal{P} \subset \mathrm{ba}(K,P)$ resp. $\mathcal{P} \subset M^1(K,P)$ under the Fatou-continuity assumption. This representation corresponds in the result of Jouini et al. (2004) for $n=1$.

b) The representation results (35) *and* (40) *generalize the one-dimensional representation results in Artzner et al. (1998), Delbaen (2002) and Föllmer and Schied (2004).*

An important property of risk measures is consistency with respect to stochastic orderings related to risk measurement. A necessary condition for this consistency is the law invariance of the risk measures.

Definition 3.8 *A risk measure $\varrho : L_d^\infty(P) \to \mathbb{R}$ is law invariant if for $X, Y \in L_d^\infty(P)$ with identical law w.r.t. P i.e. $P^X = P^Y$ holds $\varrho(X) = \varrho(Y)$.*

In the following we generally assume that the underying probability space $(\Omega, \mathfrak{A}, P)$ is rich enough in the usual sense to allow the construction of enough r.v.s on $(\Omega, \mathfrak{A}, P)$. Let $\mathcal{F}_{\mathrm{i}}$ denote the class of increasing functions on $\mathbb{R}^d$, i.e. $x \preceq y$ in the sence of (27) implies that $f(x) \le f(y)$. Similarly $\mathcal{F}_{\mathrm{de}}$ denotes the class of decreasing functions. The stochastic ordering is defined for random vectors X, Y on $(\Omega, \mathfrak{A}, P)$ by

$$X \le_{\mathrm{st}} Y \text{ if } Ef(X) \le Ef(Y) \tag{41}$$

for all $f \in \mathcal{F}_{\mathrm{i}} \cap L^1(\{P^X, P^Y\})$ or equivalently for all $f \in \mathcal{F}_{\mathrm{i}} \cap L^\infty(P)$, where the expectation is with respect to P.

Proposition 3.9 *Let ϱ be a law invariant risk measure satisfying M1), then ϱ is consistent with stochastic ordering, i.e.,*

$$X \le_{\mathrm{st}} Y \textit{ implies } \varrho(Y) \le \varrho(X).$$

Proof: Strassen's theorem which is valid for closed orderings $X \leq_{\mathrm{st}} Y$ implies the existence of versions $\widetilde{X} \stackrel{d}{=} X$ and $\widetilde{Y} \stackrel{d}{=} Y$ such that $\widetilde{X} \preceq \widetilde{Y}$. Therefore, by the monotonicity conditon M1) and the law invariance

$$\varrho(Y) = \varrho(\widetilde{Y}) \leq \varrho(\widehat{X}) = \varrho(X). \qquad \square$$

The class $\mathcal{F}_{\mathrm{cx}}$ of convex functions on $\mathbb{R}^d$ is suitable to measure diffusiveness as in $d = 1$. Let $\mathcal{F}_{\mathrm{icx}}$ and $\mathcal{F}_{\mathrm{decx}}$ denote the class of increasing resp. decreasing convex functions on $\mathbb{R}^d$. The induced stochastic orderings are defined for random vectors X, Y by

$$X \leq_{\mathrm{cx}} Y (\text{resp. } X \leq_{\mathrm{icx}} Y \text{ resp. } X \leq_{\mathrm{decx}} Y) \tag{42}$$

if $Ef(X) \leq Ef(Y)$ for all $f \in \mathcal{F}_{\mathrm{cx}}$ resp. $\mathcal{F}_{\mathrm{icx}}$ resp. $\mathcal{F}_{\mathrm{decx}}$ such that $f(X)$ and $f(Y)$ are in $L^1(P)$.

The ordering $-X \leq_{\mathrm{icx}} -Y$ is equivalent to $X \leq_{\mathrm{decx}} Y$. In the one-dimensional case $d = 1$ this ordering is also called second order stochastic dominance. In contrast to $d = 1$ there is in $d \geq 2$ no simple and natural generating class of $\mathcal{F}_{\mathrm{cx}}$ resp. $\mathcal{F}_{\mathrm{decx}}$ like $f_t(x) = (t - x)_+$, which relates the one-dimensional case uniquely to the tail probabilities and quantiles.

Convex law invariant risk measures are consistent with the convex ordering.

Theorem 3.10 *Let ϱ be a convex law invariant risk measure on $L_d^\infty(P)$. Then ϱ is consistent with the convex orderings $\leq_{\mathrm{cx}}$ and $\leq_{\mathrm{decx}}$, i.e.,*

$$X \leq_{\mathrm{cx}} Y \textit{ implies } \varrho(X) \leq \varrho(Y) \tag{43}$$

and

$$X \leq_{\mathrm{decx}} Y \textit{ implies } \varrho(X) \leq \varrho(Y). \tag{44}$$

Proof: By a recent result of Jouini et al. (2005) law invariant risk measures have the Fatou property. Further by an extension of Lemma 2.2 in Schied (2004) to the multivariate case we obtain that for any convex law invariant risk measure ϱ on $L_d^\infty(P)$ and any $X, Y \in L_d^\infty(P)$ holds:

$$\varrho(X) \geq \varrho(E(X \mid Y)). \tag{45}$$

1) By Strassens's a.s. representation result (see e.g. [10]) there exist versions $\widetilde{X} \stackrel{d}{=} X$, $\widetilde{Y} \stackrel{d}{=} Y$ on Ω such that $E(\widetilde{Y} \mid \widetilde{X}) = \widetilde{X}\,[P]$. Therefore, by (45)

$$\varrho(Y) = \varrho(\widetilde{Y}) \geq \varrho(E(\widetilde{Y} \mid \widetilde{X})) = \varrho(\widetilde{X}) = \varrho(X).$$

2) If $X \leq_{\mathrm{decx}} Y$, then there are versions $\widetilde{X} \stackrel{d}{=} X$, $\widetilde{Y} \stackrel{d}{=} Y$ on Ω such that $E(\widetilde{Y} \mid \widetilde{X}) \leq \widetilde{X}\,[P]$. Therefore, $\varrho(\widetilde{Y}) \geq \varrho(E(\widetilde{Y} \mid \widetilde{X})) \geq \varrho(\widetilde{X})$. $\square$

Remark 3.11 *In the one-dimensional case the convex ordering result of Theorem 3.10 has been proved as consequence of the Kusuoka representation result for convex risk measures in Föllmer and Schied (2004, Corollary 4.59) and in a direct way in a recent paper in Bäuerle and Müller (2005).*

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