Reducing model risk via positive and negative dependence assumptions

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Abstract

We give analytical bounds on the Value-at-Risk and on convex risk measures for a portfolio of random variables with fixed marginal distributions under an additional positive dependence structure. We show that assuming positive dependence information leads to reduced dependence uncertainty spreads compared to the case where only marginals information is known. In more detail we show that the assumption of a positive dependence structure improves the best-possible lower estimate of a risk measure, while leaving unchanged its worst-possible upper risk bounds. In a similar way we derive that the assumption of a negative dependence structure leads to improved upper bounds for the risk while it does not help to increase the lower risk bounds in an essential way. As a result we find that additional assumptions on the dependence structure may result in essentially improved risk bounds.

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1 Preliminaries and motivation

The problem of assessing the model risk associated with the risk measurement of a high dimensional portfolio has recently gathered a lot of interest in the actuarial and financial literature. To set a mathematical framework, we assume that a financial institution holds a d-dimensional risk portfolio over a fixed time period. This risk portfolio is represented by a random vector $\boldsymbol{X} = (X_1, \dots, X_d)$ on a standard non-atomic probability space $(\Omega, \mathcal{F}, P)$. The total loss exposure associated with $\boldsymbol{X}$ is given by the sum

$$X_d^+ = X_1 + \cdots + X_d.$$

Using a risk measure ρ, the aggregate random position X_d^+ is mapped into the real value $\rho(X_d^+)$, to be interpreted as the regulatory capital to be reserved in order to be able to safely hold $\boldsymbol{X}$. In this paper, we mainly deal with the case where ρ is a convex risk measure or the case where ρ is the Value-at-Risk (VaR). The evaluation of $\rho(X_d^+)$ is mainly a numerical issue once the joint distribution of $\boldsymbol{X}$ has been chosen or statistically evaluated. Estimating a multivariate distribution is a challenging task which is usually performed in two steps: first, d individual models F_j for the marginal loss exposures X_j are independently developed. Then, the marginal distributions are merged into a joint distribution using a dependence structure.

In fact, banks/insurance companies typically have better methods/more data for estimating a one-dimensional distribution for each risk type X_j than they have to estimate the overall dependence structure of $\boldsymbol{X}$. It is therefore reasonable to assume that the marginal distributions $F_1, \dots, F_d$ are known, while $F_{\boldsymbol{X}}$, the joint distribution of $\boldsymbol{X}$, varies in $\mathcal{F}_d(F_1, \dots, F_d)$, the so-called *Fréchet* class of all possible joint distributions having the fixed marginal models $F_1, \dots, F_d$. The choice of a single distribution in $\mathcal{F}_d(F_1, \dots, F_d)$ can lead to the miscalculation of the reserve $\rho(X_d^+)$. The implied model risk is referred to as *dependence uncertainty*.

A natural way to measure dependence uncertainty and, in more generality, model risk consists in finding the minimum and maximum possible values of the risk measure ρ evaluated over the class of candidate models; this is the approach taken in Cont (2006). In our framework, we define the smallest and biggest capitals to be held coherently with the given marginal distributions as

$$\underline{\rho}(X_d^+) = \inf \left\{ \rho(X_d^+); F_{\mathbf{X}} \in \mathcal{F}_d(F_1, \dots, F_d) \right\}, \tag{1.1}$$

and

$$\overline{\rho}(X_d^+) = \sup \left\{ \rho(X_d^+); F_{\mathbf{X}} \in \mathcal{F}_d(F_1, \dots, F_d) \right\}. \tag{1.2}$$

For any risk portfolio $(X_1, \dots, X_d)$ having marginal distributions $F_1, \dots, F_d$, it obviously holds that

$$\underline{\rho}(X_d^+) \leq \rho(X_d^+) \leq \overline{\rho}(X_d^+).$$

The difference $\overline{\rho}(X_d^+) - \underline{\rho}(X_d^+)$ is called the *Dependence Uncertainty spread* (DU-spread) for ρ and is used to measure model uncertainty on the final capital reserve; see Embrechts et al. (2014b) for this terminology.

Computation of DU-spreads has been treated in the recent literature. The *analytical* computation of best- and worst-possible bounds on Value-at-Risk can be performed only under some specific assumptions on the marginal distributions; see the survey paper Embrechts et al. (2014a) for the state-of-the-art and an history of the problem. The analytical computation of worst-possible bounds on Expected Shortfall (ES) is in general straightforward, while for the best-case ES partial analytical results can be found in Wang and Wang (2011) and Bernard et al. (2014). For several classes of risk measures (including convex and distortion risk measures) Wang et al. (2014) provide a systematic way to compute the worst (and best) possible bounds across any homogeneous portfolio.

The *numerical* computation of DU-spreads of VaR and ES for arbitrary portfolios can be performed using the Rearrangement Algorithm described in Embrechts et al. (2013) (for the case of VaR) and in Puccetti (2013) (for ES) for dimensions d in the several hundreds or possibly thousands. Even if DU-spreads of VaR and ES are numerically available for practically any joint portfolio of risks, their relevance in actuarial practice has been recently questioned since they can be considerably large; see Aas and Puccetti (2014) for a real case study.

Therefore, in the recent literature many techniques to tighten DU-spreads were introduced. One possibility is to add extra (statistical) information on top of the knowledge of the marginal distributions. For instance, in Embrechts et al. (2013, Section 4) it is shown that having higher order (typically two-dimensional) marginals information on the joint portfolio leads to strongly improved bounds. The DU-spread of the VaR can be similarly reduced by specifying the copula on some subset of its domain (see Bernard et al., 2013a) or putting a variance constraint on the total position (see Bernard et al., 2013b).

In this paper, we show that positive dependence restrictions do not help to improve upper risk bounds essentially. It however allows to increase the lower risk bounds and therefore to reduce the model risk faced by an institution. Positive dependence information is introduced in a natural way in Section 2 by the notions of orthant orders and weakly conditional increasing in sequence order. These orders are particularly capable to capture the concept of *stronger dependence* in the comparison of portfolios with fixed marginal distributions. In Section 3, we introduce a class of models for joint portfolios described by several independent groups with given marginals while entailing comonotonic dependence within the groups. We provide analytical upper and lower bounds on the VaR of the joint portfolio which are easily computable, widely applicable and are compared with the corresponding unconstrained bounds obtained without positive dependence assumption. In Section 4, we deal with the case of law-invariant, convex risk measures, where we draw similar conclusions. While assuming a positive dependence structure typically improves the best-possible lower bound of a risk measure, it generally leaves unchanged the worst-possible upper risk bound. Finally, in Section 5 we discuss how *negative* dependence assumptions moderate also worst-case scenarios. We give a variety of applications of interest in quantitative risk management that can be easily adapted and closed formulas to be used in the risk management of real portfolios.

2 Dependence orders between risk vectors

In quantitative risk management, the components of a risk portfolio often have some positive dependence structure. A simple way to describe positive dependence is by using suitable stochastic orders between random vectors. In this section, we recall some natural positive dependence orders needed in the remainder of the paper. For more details on these dependence notions we refer to Chapter 6 in Rüschendorf (2013). For a random vector $\boldsymbol{X} = (X_1, \dots, X_d)$ in $\mathbb{R}^d$ we indicate with $F_{\boldsymbol{X}}$ its joint distribution function and with $\overline{F}_{\boldsymbol{X}}$ its survival function. Formally, for $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, we denote

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = P(X_1 \le x_1, \dots, X_d \le x_d), \; \overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) = P(X_1 > x_1, \dots, X_d > x_d).$$

For two random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathbb{R}^d$, we define

- the *upper orthant order* $\boldsymbol{Y} \le_{\text{uo}} \boldsymbol{X}$, if $\overline{F}_{\boldsymbol{Y}}(\boldsymbol{x}) \le \overline{F}_{\boldsymbol{X}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^d$;
- the *lower orthant order* $\boldsymbol{Y} \le_{\text{lo}} \boldsymbol{X}$, if $F_{\boldsymbol{Y}}(\boldsymbol{x}) \le F_{\boldsymbol{X}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^d$;
- the *concordance order* $\boldsymbol{Y} \le_{\text{co}} \boldsymbol{X}$, if both $\boldsymbol{Y} \le_{\text{uo}} \boldsymbol{X}$ and $\boldsymbol{Y} \le_{\text{lo}} \boldsymbol{X}$ hold.
- the *weakly conditional increasing in sequence order* $\boldsymbol{Y} \le_{\text{wcs}} \boldsymbol{X}$, if, for all $x \in \mathbb{R}$, all i with $1 \le i \le d$, and all component-wise increasing functions f, we have

$$\operatorname{Cov}(\mathbf{1}(Y_i > x), f(Y_{i+1}, \dots, Y_n)) \le \operatorname{Cov}(\mathbf{1}(X_i > x), f(X_{i+1}, \dots, X_n)). \tag{2.1}$$

A random vector $\boldsymbol{Y}$ is smaller than $\boldsymbol{X}$ in the upper (lower) orthant order if the probabilities for upper (lower) orthants are ordered, i.e. the probability that all components jointly assume large (small) values is lower for $\boldsymbol{Y}$ rather than for $\boldsymbol{X}$. A random vector $\boldsymbol{Y}$ is smaller than $\boldsymbol{X}$ in the weakly conditional increasing in sequence order if on any level of values larger than x, the i-th component X_i is more strongly positively correlated to $(X_{i+1}, \dots, X_n)$ than Y_i is to $(Y_{i+1}, \dots, Y_n)$.

It is well known that in dimension $d = 2$ and assuming identical marginals for the two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ the four orders defined above are equivalent, i.e.

$$\boldsymbol{Y} \le_{\text{uo}} \boldsymbol{X} \Leftrightarrow \boldsymbol{Y} \le_{\text{lo}} \boldsymbol{X} \Leftrightarrow \boldsymbol{Y} \le_{\text{co}} \boldsymbol{X} \Leftrightarrow \boldsymbol{Y} \le_{\text{wcs}} \boldsymbol{X}.$$

The four orders are however different when $d \ge 3$, where we have that

$$\boldsymbol{Y} \le_{\text{wcs}} \boldsymbol{X} \;\Rightarrow\; \boldsymbol{Y} \le_{\text{co}} \boldsymbol{X} \Rightarrow\; \boldsymbol{Y} \le_{\text{lo}} (\le_{\text{uo}})\, \boldsymbol{X},$$

but not vice versa. The orders $\le_{\text{uo}}, \le_{\text{lo}}, \le_{\text{co}}, \le_{\text{wcs}}$ all imply a stronger positive dependence when vectors with the same marginal distributions are to be compared. Here *stronger positive dependence* means having bigger pairwise correlation or bigger rank correlation, as the following proposition (see Remark 6.3 in Rüschendorf, 2013) shows.

Proposition 2.1. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two random vectors in $\mathbb{R}^d$ having joint distributions $F_{\boldsymbol{X}}, F_{\boldsymbol{Y}} \in \mathcal{F}_d(F_1, \dots, F_d)$ such that*

$$\boldsymbol{Y} \le \boldsymbol{X},$$

where $\le$ is one of the orders $\le_{\text{uo}}, \le_{\text{lo}}, \le_{\text{co}}, \le_{\text{wcs}}$. Then:

$$\operatorname{Cov}(Y_i, Y_j) \le \operatorname{Cov}(X_i, X_j); \quad \rho_S(Y_i, Y_j) \le \rho_S(X_i, X_j); \quad \tau(Y_i, Y_j) \le \tau(X_i, X_j);$$

where ρ_S is Spearman's and τ is Kendall's rank correlation coefficient.

Remark 2.1. In the remainder of this section we use the assumptions $\boldsymbol{Y} \le_{\text{lo}} \boldsymbol{X}$ and $\boldsymbol{Y} \le_{\text{uo}} \boldsymbol{X}$, while in Section 4 we assume the stronger condition $\boldsymbol{Y} \le_{\text{wcs}} \boldsymbol{X}$. Note that the assumptions $\boldsymbol{Y} \le_{\text{lo}} \boldsymbol{X}$ or $\boldsymbol{Y} \le_{\text{uo}} \boldsymbol{X}$ do not imply that $F_{\boldsymbol{X}}$ and $F_{\boldsymbol{Y}}$ have the same marginal distributions. On the contrary, assuming $\boldsymbol{Y} \le_{\text{wcs}} \boldsymbol{X}$ or $\boldsymbol{Y} \le_{\text{co}} \boldsymbol{X}$ implies that $F_{\boldsymbol{X}}$ and $F_{\boldsymbol{Y}}$ belong to the same Fréchet class.

Orthant orders lead to bounds for the distribution function, resp. the survival function of an aggregate position. For a d-dimensional distribution function G, we define the generalized G-supremal convolution as

$$\bigvee G(s) = \sup_{\boldsymbol{x} \in \mathcal{G}(s)} G(\boldsymbol{x}),$$

where $\mathcal{G}(s) = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d; \sum_{i=1}^d x_i = s\right\}$. The following proposition will be used in Section 3 to obtain analytical bounds for the VaR of a joint portfolio under positive dependence.

Proposition 2.2 (Puccetti and Rüschendorf, 2012)**.** *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two random vectors in $\mathbb{R}^d$.*

1. *If $\boldsymbol{Y} \leq_{\mathrm{lo}} \boldsymbol{X}$, then*
$$P\left(X_d^+ \leq s\right) \geq \bigvee F_{\boldsymbol{Y}}(s); \tag{2.2}$$

2. *If $\boldsymbol{Y} \leq_{\mathrm{uo}} \boldsymbol{X}$, then*
$$P\left(X_d^+ < s\right) \leq 1 - \bigvee \overline{F}_{\boldsymbol{Y}}(s). \tag{2.3}$$

The two bounds in (2.2) and (2.3) are sharp when $d \leq 2$.

3 Reducing the dependence spread of Value-at-Risk

In this section we introduce positive dependence conditions on the portfolio vector $\boldsymbol{X}$ which allow to describe analytically the increase of the lower VaR bound as well as the decrease of the upper VaR bound from marginals information only. It will turn out that positive orthant dependence alone does not allow a reduction of the dependence spread of VaR. For this reason, in the following we introduce and study a more specific class of positive dependence models. Recall that the Value-at-Risk (VaR) of a loss random variable X, computed at a probability level $\alpha \in (0,1)$, is defined as

$$\mathrm{VaR}_\alpha(X) := F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\},$$

where $F_X(x) = P(X \leq x)$ is the distribution function of X.

Consider a decomposition $\{1, \dots, d\} = \cup_{j=1}^k I_k$ of $\{1, \dots, d\}$ into k disjoint subsets I_j with cardinality $n_j = |I_j|, 1 \leq j \leq k$, and let $\boldsymbol{Y}$ denote a random vector with distribution function

$$F_{\boldsymbol{Y}}(x_1, \dots, x_d) = \prod_{j=1}^k \min_{i \in I_j}\{F_j\left(x_i\right)\}. \tag{3.1}$$

By definition the components within the subgroups I_j are homogeneous with distribution F_j and comonotonic, while the different subgroups are independent. Our positive dependence restriction on $\boldsymbol{X}$ is formulated by the condition that

$$\boldsymbol{Y} \leq \boldsymbol{X}, \tag{3.2}$$

where $\leq$ is one of the orders $\leq_{\mathrm{uo}}$, resp. $\leq_{\mathrm{lo}}$, or $\leq_{\mathrm{wcs}}$ as described in Section 2.

In case $k = 1$ condition (3.2) implies that $\boldsymbol{Y}$ and $\boldsymbol{X}$ are comonotonic vectors, the strongest possible dependence restriction. The weakest form of dependence restriction is instead obtained in the case $k = d$. Then $\boldsymbol{Y} = \boldsymbol{X}^{\perp\!\!\!\perp}$ is an independent vector with the same marginals as $\boldsymbol{X}$ and condition (3.2) postulates that $\boldsymbol{X}$ is positive upper (resp. lower) dependent, i.e. PUOD (resp. PLOD), or $\boldsymbol{X}$ is weakly associated in sequence, i.e. WAS; see Rüschendorf (2004) on this terminology.

We are now ready to give the main result of this section, where we state lower and upper bounds for the VaR when positive dependence information of the type $\boldsymbol{Y} \leq \boldsymbol{X}$ is assumed.

Theorem 3.1. *Assume that the random vector $\boldsymbol{Y}$ has distribution $F_{\boldsymbol{Y}}$ defined as in (3.1).*

1. *(Lower bound) If $\boldsymbol{Y} \leq_{\mathrm{uo}} \boldsymbol{X}$, then for any $\alpha \in (0,1)$ we have*

$$\mathrm{VaR}_\alpha(X_d^+) \geq \sup_{\mathcal{L}(\alpha)} \sum_{j=1}^{k} n_j\, F_j^{-1}(u_j), \tag{3.3}$$

where $\mathcal{L}(\alpha) = \{(u_1, \ldots, u_k) \in [0, \alpha]^k; \prod_{j=1}^k (1-u_j) = 1-\alpha\}$.

2. *(Upper bound) If $\boldsymbol{Y} \leq_{\mathrm{lo}} \boldsymbol{X}$, then for any $\alpha \in (0,1)$ we have*

$$\mathrm{VaR}_\alpha(X_d^+) \leq \inf_{\mathcal{U}(\alpha)} \sum_{j=1}^{k} n_j\, F_j^{-1}(u_j), \tag{3.4}$$

where $\mathcal{U}(\alpha) = \{(u_1, \ldots, u_k) \in [\alpha, 1]^k; \prod_{j=1}^k u_j = \alpha\}$.

Before proving the theorem we need the following lemma.

Lemma 3.2. *We have that*

$$\sup_{x_1+\ldots+x_d=s} \prod_{j=1}^{k} \overline{F}_j \left(\max_{i \in I_j} \{x_i\} \right) = \sup_{y_1+\ldots+y_k=s} \prod_{j=1}^{k} \overline{F}_j \left(\frac{y_j}{n_j} \right).$$

Proof. For any $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with $\sum_{i=1}^d x_i = s$ it is possible to define $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$ as

$$z_i = \sum_{j=1}^{k} \mathbf{1}\{i \in I_j\} \cdot \left(\frac{1}{n_j} \sum_{r \in I_j} x_r \right), \ 1 \leq i \leq d.$$

The vector $\boldsymbol{z}$ has the components z_i's with index i in the same subgroup I_j identical and equal to the average of the $x_i's$ with index belonging to the same I_j. It is straightforward to check that $\sum_{i=1}^d z_i = s$ and $\max_{i\in I_j}\{x_i\} \geq \max_{i \in I_j}\{z_i\}, 1 \leq j \leq k$. Since the product operator is coordinatewise increasing while the $\overline{F}_j$'s are decreasing functions, we obtain that

$$\prod_{j=1}^{k} \overline{F}_j \left(\max_{i \in I_j} \{x_i\} \right) \leq \prod_{j=1}^{k} \overline{F}_j \left(\max_{i \in I_j} \{z_i\} \right) = \prod_{j=1}^{k} \overline{F}_j \left(\frac{1}{n_j} \sum_{r \in I_j} x_r \right) = \prod_{j=1}^{k} \overline{F}_j \left(\frac{y_j}{n_j} \right),$$

where $y_j = \sum_{r \in I_j} x_r$ with $\sum_{j=1}^k y_j = s$. Thus,

$$\sup_{x_1+\ldots+x_d=s} \prod_{j=1}^{k} \overline{F}_j \left(\max_{i \in I_j} \{x_i\} \right) \leq \sup_{y_1+\ldots+y_k=s} \prod_{j=1}^{k} \overline{F}_j \left(\frac{y_j}{n_j} \right).$$

The inverse inequality ($\geq$) is immediate. □

Proof of Theorem 3.1, (1). By Proposition 2.2, it follows that

$$P\left(X_d^+ \geq s\right) \geq \sup_{\boldsymbol{x} \in \mathcal{G}(s)} \overline{F}_{\boldsymbol{Y}}(\boldsymbol{x}) = \sup_{\boldsymbol{x} \in \mathcal{G}(s)} \prod_{j=1}^{k} P(Y_i > x_i, i \in I_j) = \sup_{\boldsymbol{x} \in \mathcal{G}(s)} \prod_{j=1}^{k} \overline{F}_j \left(\max_{i \in I_j} x_i \right),$$

where $\overline{F}_j = 1 - F_j, 1 \leq j \leq d$. Applying Lemma 3.2 leads to

$$P\left(X_d^+ \geq s\right) \geq \sup_{y_1+\ldots+y_k=s} \prod_{j=1}^{k} \overline{F}_j \left(\frac{y_j}{n_j} \right) = \sup_{y_1+\ldots+y_k=s} \prod_{j=1}^{k} \overline{G}_j\left(y_j\right), \tag{3.5}$$

where we set $\overline{G}_j(\cdot) = \overline{F}_j(\cdot/n_j), 1 \le j \le k$. From the duality principle given in Theorem 4.1 in Embrechts et al. (2003), it follows that

$$\mathrm{VaR}_\alpha(X_d^+) \ge \sup_{\mathcal{L}'(\alpha)} \sum_{j=1}^{k} G_j^{-1}(u_j) = \sup_{\mathcal{L}'(\alpha)} \sum_{j=1}^{k} n_j F_j^{-1}(u_j),$$

where $\mathcal{L}'(\alpha) = \{(u_1, \dots, u_k) \in [0,1]^k; \prod_{j=1}^{k}(1-u_j) = 1-\alpha\}$. The upper bound in (3.3) now follows by noting that the constraints $\prod_{j=1}^{k}(1-u_j) = 1-\alpha$ and $u_j \in [0,1]$ imply $u_j \le \alpha, 1 \le j \le d$. $\square$

Proof of Theorem 3.1, (2). Using Proposition 2.2, we have that

$$P(X_d^+ \le s) \ge \sup_{x_1+\ldots+x_d=s} \prod_{j=1}^{k} \min_{i\in I_j}\{F_j(x_i)\} = \sup_{x_1+\ldots+x_d=s} \prod_{j=1}^{k} F_j\left(\min_{i\in I_j}\{x_i\}\right). \tag{3.6}$$

Using the same argument as in the proof of Lemma 3.2, we have that

$$P(X_d^+ \le s) \ge \sup_{x_1+\ldots+x_d=s} \prod_{j=1}^{k} F_j\left(\min_{i\in I_j}\{x_i\}\right) = \sup_{y_1+\ldots+y_k=s} \prod_{j=1}^{k} F_j\left(\frac{y_j}{n_j}\right) = \sup_{y_1+\ldots+y_k=s} \prod_{j=1}^{k} G_j(y_j). \tag{3.7}$$

From Theorem 4.1 in Embrechts et al. (2003), it follows that

$$\mathrm{VaR}_\alpha(X_d^+) \le \inf_{\mathcal{U}'(\alpha)} \sum_{j=1}^{k} G_j^{-1}(u_j) = \inf_{\mathcal{U}'(\alpha)} \sum_{j=1}^{k} n_j F_j^{-1}(u_j),$$

where $\mathcal{U}'(\alpha) = \{(u_1, \dots, u_k) \in [0,1]^k; \prod_{j=1}^{k} u_j = \alpha\}$. The upper bound in (3.4) now follows by noting that the constraints $\prod_{j=1}^{k} u_j = \alpha$ and $u_j \in [0,1]$ imply $u_j \ge \alpha, 1 \le j \le k$. $\square$

3.1 Computation of the lower bound on the VaR

Under additional hypotheses on the marginal models $F_1, \dots, F_k$ it is possible to compute analytically the supremum in (3.3).

Theorem 3.3. *Fix $\alpha \in (0,1)$ and assume that the function $\psi_j : [\ln(1-\alpha), 0] \to \mathbb{R}$ defined as*

$$\psi_j(x) = F_j^{-1}(1-e^x)$$

is continuous and convex for all $1 \le j \le k$. Then, we have that

$$\sup_{\mathcal{L}(\alpha)} \sum_{j=1}^{k} n_j\, F_j^{-1}(u_j) = \max_{1\le j\le k} \left\{ n_j F_j^{-1}(\alpha) + \sum_{i\ne j} n_i F_i^{-1}(0) \right\}.$$

Proof. Applying the transformation $u_j = 1 - e^{v_j}$, we obtain

$$\sup_{\mathcal{L}(\alpha)} \sum_{j=1}^{k} n_j\, F_j^{-1}(u_j) = \sup_{\mathcal{V}(\alpha)} \sum_{j=1}^{k} n_j F_j^{-1}(1-e^{v_j}), \tag{3.8}$$

where $\mathcal{V}(\alpha) = \{(v_1, \dots, v_k) \in [\ln(1-\alpha), 0]^k; \sum_{j=1}^{k} v_j = \ln(1-\alpha)\}$ is a convex and compact set. By assumption, the function $\sum_{j=1}^{k} n_j F_j^{-1}(1-e^{v_j})$ is continuous and convex and by the maximum principle it attains the maximum at an extreme point of $\mathcal{V}(\alpha)$, that is at a point $\boldsymbol{v}$ such that $v_j = \ln(1-\alpha)$ for some j and $v_i = 0$ for every $i \ne j$. Hence, we have that

$$\sup_{\mathcal{V}(\alpha)} \sum_{j=1}^{k} n_j F^{-1}(1-e^{v_j}) = \max_{1\le j\le k} \left\{ n_j F_j^{-1}(1-e^{\ln(1-\alpha)}) + \sum_{i\ne j}^{k} n_i F_i^{-1}(1-e^{0}) \right\},$$

which gives the desired result. $\square$

The assumptions of Theorem 3.3 are easy to check and satisfied for some models of interest in quantitative risk management. We give some relevant examples below.

Example 3.1 (Pareto marginal distributions)**.** Consider the case where all the F_j's are Pareto distributed with parameter $\theta_j > 0$, that is

$$F_j(x) = 1 - (1+x)^{-\theta_j}, x > 0, 1 \leq j \leq d.$$

In this case the functions $F_j^{-1}(1-e^x) = \exp(-x/\theta_j) - 1$ satisfy the assumptions of Theorem 3.3 and for a vector $\boldsymbol{X}$ with $\boldsymbol{Y} \leq_{\mathrm{uo}} \boldsymbol{X}$ we obtain

$$\mathrm{VaR}_\alpha(X_d^+) \geq \max_{1\leq j\leq d} \left\{ n_j \left((1-\alpha)^{-\frac{1}{\theta_j}} - 1 \right) \right\}. \tag{3.9}$$

In Table 1 we report the lower bound in (3.9) obtained under the assumption $\boldsymbol{Y} \leq_{\mathrm{uo}} \boldsymbol{X}$ for a sum of d homogeneous Pareto(2) random variables. In the same table we also report the *unconstrained* bound $\underline{\mathrm{VaR}}_\alpha(X_d^+)$ defined, consistently with (1.1), as

$$\underline{\mathrm{VaR}}_\alpha(X_d^+) = \inf \left\{ \mathrm{VaR}_\alpha(X_d^+); F_{\boldsymbol{X}} \in \mathcal{F}_d(F_1, \dots, F_d) \right\}, \tag{3.10}$$

where $F_i = F_j$ when $i \in I_j$. The bound $\underline{\mathrm{VaR}}_\alpha(X_d^+)$ can be computed in the Pareto case using Corollary 4.7 in Jakobsons et al. (2014). The lower bound obtained in (3.9) under a positive dependence assumption considerably improves the unconstrained bound for all $k < d$. Interestingly enough, the case $k = d = 8$, where the vector $\boldsymbol{X}$ is assumed to be PUOD, exactly returns the unconstrained bound $\underline{\mathrm{VaR}}_\alpha(X_d^+)$, i.e. the PUOD condition alone does not increase the lower risk bound induced by marginals information only. The improvement implied by the extra dependence assumption decreases with increasing k, which corresponds to weaker positive dependence assumption (recall that $k = 1$ gives comonotonic X_j's while $k = 8$ corresponds to independent Y_j's).

Example 3.2 (Exponential marginal distributions)**.** Consider the case where each F_j is an Exponential distribution with positive parameter θ_j, that is

$$F_j(x) = 1 - e^{-\theta_j x}, x > 0, 1 \leq j \leq d.$$

Since $F_j^{-1}(1-e^x) = -\frac{x}{\theta_j}$ is linear, from Theorem 3.3 we obtain

$$\mathrm{VaR}_\alpha(X_d^+) \geq \max_{1\leq j\leq k} \left\{ -\frac{n_j}{\theta_j} \ln(1-\alpha) \right\}. \tag{3.11}$$

Again the lower bound obtained in (3.11) under a positive dependence assumption considerably improves the unconstrained bound for all $k < d$; see Table 2 (with a slight abuse of notation the case $k = 1$ indicates the inhomogeneous case when all the $X_j's$ are comonotonic). Note that Theorem 3.3 allows to compute bounds for portfolios with different families of marginal distributions; see Table 5 below.

In case the conditions of Theorem 3.3 are not satisfied, the supremum in (3.3) might be attained in an internal point of $\mathcal{L}(\alpha)$. We can rewrite (3.3) as

$$\mathrm{VaR}_\alpha(X_d^+) \geq \sup_{u_1,\dots,u_{k-1}\in[0,\alpha]} \sum_{j=1}^{k-1} n_j\, F_j^{-1}(u_j) + n_k F_k^{-1}\left(1 - \frac{1-\alpha}{\Pi_{j=1}^{k-1}(1-u_j)}\right). \tag{3.12}$$

First order conditions for sup = max in (3.12) are given by

$$\begin{cases} \frac{n_i(1-u_i^*)}{f_i(F_i^{-1}(u_i^*))} = \frac{n_k(1-\alpha)}{(\prod_{j\in C}(1-u_j^*)) f_k\left(F_k^{-1}\left(1-\frac{(1-\alpha)}{\prod_{j\in C}(1-u_j^*)}\right)\right)}, & \text{for } i \in C; \\ u_i^* = 0, & \text{for } i \notin C, \end{cases} \tag{3.13}$$

where we define $C = \{j \in \{1, \dots, k-1\} : u_j^* > 0\}$ and $f_j = F_j'$.

Example 3.3 (Homogeneous marginal distributions)**.** The system of equations in (3.13) generally needs to be solved numerically. To get solutions in closed form, we consider the simplified framework where the marginal distributions F_j are assumed to be all identical and each subgroup to have the same cardinality, i.e. $F_j = F$ and $n_j = \frac{d}{k}, 1 \leq j \leq k$. The set of equations in (3.13) then becomes

$$\begin{cases} \frac{1-u_i^*}{f(F^{-1}(u_i^*))} = \frac{1-\alpha}{(\prod_{j \in C}(1-u_j^*)) f\left(F^{-1}\left(1-\frac{(1-\alpha)}{\prod_{j \in C}(1-u_j^*)}\right)\right)}, & \text{for } i \in C; \\ u_i^* = 0, & \text{for } i \notin C. \end{cases} \tag{3.14}$$

The equations in (3.14) are clearly solved by $u_j^* = 1 - (1-\alpha)^{\frac{1}{|C|}}, j \in C$. It is also straightforward to prove that, if the density of F is monotone in $[F^{-1}(0), F^{-1}(\alpha)]$, this solution is unique. Considering all the possible cardinalities of C in (3.14), we collect a set of k candidate vectors, one of which gives (in case uniqueness holds) sup = max in (3.12). Under this simplified framework, the lower bound for VaR in (3.3) becomes

$$\mathrm{VaR}_\alpha\left(X_d^+\right) \geq \max_{0 \leq r \leq k-1} \left\{ \frac{d(k-r)}{k} F^{-1}\left(1 - (1-\alpha)^{\frac{1}{k-r}}\right) + \frac{d\, r}{k} F^{-1}(0) \right\}. \tag{3.15}$$

It is important to remark that the inequality in (3.15) always holds in an homogeneous framework and does not need any extra assumption on the distribution F. In case F is not monotone on $[F^{-1}(0), F^{-1}(\alpha)]$, the bound given in (3.15) might however be improved by finding other different solutions of (3.14).

Figures for the bound (3.15) are reported in Table 3 for an homogeneous portfolio of $d = 16$ Gamma distributions. As one can see from this table, figures for the lower bound tend to deteriorate with increasing dimensions. For example, for $k = 8$ and $k = 16$ the bound (3.15) becomes smaller than the unconstrained one. In Table 3, the unconstrained bound $\underline{\mathrm{VaR}}_\alpha(X_d^+)$ is obtained using the rearrangement procedure described in Embrechts et al. (2013).

$d=8$	$k=1$		$k=2$		$k=4$		$k=8$	
	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$
$\alpha=0.990$	9.00	72.00	9.00	36.00	9.00	18.00	9.00	9.00
$\alpha=0.995$	13.14	105.14	13.14	52.57	13.14	26.28	13.14	13.14
$\alpha=0.999$	30.62	244.98	30.62	122.49	30.62	61.25	30.62	30.62

Table 1: Values (rounded) for the VaR bound in (3.9) (denoted by $\mathrm{VaR}_\alpha^{\mathrm{lb}}$) for an homogeneous portfolio with d Pareto(2) risks, k subgroups, d/k variables in each subgroup and dependence assumption $\boldsymbol{Y} \leq_{\mathrm{uo}} \boldsymbol{X}$. The unconstrained bound $\underline{\mathrm{VaR}}_\alpha(X_d^+)$ in (3.10) is also reported. In this and the forthcoming tables, the indication of the random variable X_d^+ is omitted.

$d=8$	$k=1$		$k=2$		$k=4$		$k=8$	
	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$
$\alpha=0.990$	2.30	13.82	2.30	9.21	2.30	4.61	2.30	2.30
$\alpha=0.995$	2.65	15.89	2.65	10.60	2.65	5.30	2.65	2.65
$\alpha=0.999$	3.45	20.72	3.45	13.82	3.45	6.91	3.45	3.45

Table 2: Values (rounded) for the VaR bound in (3.11) (denoted by $\mathrm{VaR}_\alpha^{\mathrm{lb}}$) for an inhomogeneous portfolio with $d/2$ Exp(2) risks and $d/2$ Exp(4) risks, k subgroups, d/k variables in each subgroup and dependence assumption $\boldsymbol{Y} \leq_{\mathrm{uo}} \boldsymbol{X}$. The unconstrained bound $\underline{\mathrm{VaR}}_\alpha$ is also reported.

$d=16$	$k=1$		$k=2$		$k=4$		$k=8$		$k=16$	
	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$
$\alpha=0.990$	23.47	67.25	23.47	42.58	23.47	28.20	23.47	19.43	23.47	13.82
$\alpha=0.995$	23.70	74.19	23.70	46.53	23.70	30.55	23.70	20.89	23.70	14.77
$\alpha=0.999$	23.93	89.83	23.93	55.31	23.93	35.70	23.93	24.05	23.93	16.81

Table 3: Values (rounded) for the VaR bound in (3.15) (denoted by $\mathrm{VaR}_\alpha^{\mathrm{lb}}$) for an homogeneous portfolio of d Gamma(3,1/2) risks and dependence assumption $\boldsymbol{Y} \leq_{\mathrm{uo}} \boldsymbol{X}$. The unconstrained bound $\underline{\mathrm{VaR}}_\alpha$ is also reported. The parameterization of the Gamma distribution used can be found in Example 4.1.

3.2 Computation of the upper bound on the VaR

We can rewrite (3.4) as

$$\mathrm{VaR}_\alpha(X_d^+) \le \inf_{u_1,\dots,u_{k-1}\in[\alpha,1]} \sum_{j=1}^{k-1} n_j\, F_j^{-1}(u_j) + n_k F_k^{-1}\left(\frac{\alpha}{\Pi_{j=1}^{k-1} u_j}\right). \tag{3.16}$$

First order conditions for inf = min in (3.16) are given by

$$\begin{cases} \frac{n_i u_i^*}{f_i(F_i^{-1}(u_i^*))} = \frac{n_k \alpha}{(\prod_{j\in B} u_j^*) f_k\left(F_k^{-1}\left(\frac{\alpha}{\prod_{j\in B} u_j^*}\right)\right)}, & \text{for } i \in B; \\ u_i^* = 1, & \text{for } i \notin B, \end{cases} \tag{3.17}$$

where we denote $B = \{j \in \{1,\dots,k-1\} : u_j^* < 1\}$ and $f_j = F_j'$.

Example 3.4 (Homogeneous marginal distributions)**.** In general, the set of equations in (3.17) needs to be solved numerically. To get solutions in closed form, we consider a simplified framework, where we assume that all the F_j are identical, and each subgroup has the same cardinality. In practice, we set $F_j = F$ and $n_j = d/k$ for $1 \le j \le d$. The equations in (3.17) then become

$$\begin{cases} \frac{u_i^*}{f(F^{-1}(u_i^*))} = \frac{\alpha}{(\prod_{j\in B} u_j^*) f\left(F^{-1}\left(\frac{\alpha}{\prod_{j\in B} u_j^*}\right)\right)}, & \text{for } i \in B; \\ u_i^* = 1, & \text{for } i \notin B. \end{cases} \tag{3.18}$$

The equations in (3.18) are clearly solved by $u_j^* = \alpha^{\frac{1}{|B|}}, j \in B$. It is also straightforward to prove that, if the density of F is monotone in $[F^{-1}(\alpha), F^{-1}(1)]$, this solution is unique. Considering all the possible cardinalities of B in (3.18), we collect a set of k candidate vectors, one of which gives (in case uniqueness holds) inf = min in (3.16). Under this simplified framework, the upper bound for VaR in (3.4) becomes

$$\mathrm{VaR}_\alpha(X_d^+) \le \min_{1\le r\le k} \left\{ \frac{d\,r}{k} F^{-1}\left(\alpha^{\frac{1}{r}}\right) + \frac{d(k-r)}{k} F^{-1}(1) \right\}. \tag{3.19}$$

It is important to remark that the inequality in (3.19) always holds in our homogeneous framework and do not need any extra assumption on the distribution F. In case F is not monotone on $[F^{-1}(\alpha), F^{-1}(1)]$, the bound given in (3.19) might however be improved by finding other different solutions of (3.18).

In Table 4 we report the upper bound in (3.19) obtained under the assumption $\boldsymbol{Y} \le_{\mathrm{lo}} \boldsymbol{X}$ as well the unconstrained bound $\overline{\mathrm{VaR}}_\alpha(X_d^+)$ for a sum of d identically distributed Pareto(2) random variables. The unconstrained bound $\overline{\mathrm{VaR}}_\alpha(X_d^+)$ is defined, consistently with (1.2), as

$$\overline{\mathrm{VaR}}_\alpha(X_d^+) = \sup\left\{ \mathrm{VaR}_\alpha(X_d^+); F_{\boldsymbol{X}} \in \mathcal{F}_d(F_1,\dots,F_d) \right\}, \tag{3.20}$$

and is obtained in the homogeneous case from Proposition 4 in Embrechts et al. (2013). Since the Pareto distribution is unbounded from above (i.e. $F^{-1}(1) = \infty$), the min in (3.19) is attained when $r = k$, therefore (3.19) simplifies to

$$\mathrm{VaR}_\alpha(X_d^+) \le d F^{-1}\left(\alpha^{\frac{1}{k}}\right).$$

Unfortunately, the bound given in (3.19) with positive dependence information improves the unconstrained bound only for $k = 1$ (obviously, since in this the case the X_j's are assumed to be comonotonic) and $k = 2$ (when the bound is known to be sharp).

$d=8$	$k=1$		$k=2$		$k=4$		$k=8$	
	$\overline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{ub}}$	$\overline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{ub}}$	$\overline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{ub}}$	$\overline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{ub}}$
$\alpha=0.990$	141.67	72.00	141.67	104.99	141.67	151.70	141.67	217.78
$\alpha=0.995$	203.66	105.14	203.66	151.90	203.66	218.06	203.66	311.65
$\alpha=0.999$	465.29	244.98	465.29	349.73	465.29	497.87	465.29	707.39

Table 4: Values (rounded) for the VaR bound in (3.19) ($\mathrm{VaR}_\alpha^{\mathrm{ub}}$ in the table) for an homogeneous portfolio with d Pareto(2) risks, k subgroups and dependence assumption $\boldsymbol{Y} \leq_{\mathrm{lo}} \boldsymbol{X}$. The unconstrained bound $\overline{\mathrm{VaR}}_\alpha$ defined in (3.20) is also reported.

3.3 General lower and upper bounds

If $\boldsymbol{Y}$ has a set of marginal distributions for which the assumptions of Theorem 3.3 are not satisfied, it is however important to notice that from (3.3) the inequality

$$\mathrm{VaR}_\alpha(X_d^+) \geq \max_{1\leq j\leq k}\{n_j F_j^{-1}(\alpha) + \sum_{i\neq j} n_i F_i^{-1}(0)\}. \tag{3.21}$$

always holds in case $\boldsymbol{Y} \leq_{\mathrm{uo}} \boldsymbol{X}$. Analogously, since the vector $\boldsymbol{u} = (\alpha^{1/k}, \ldots, \alpha^{1/k})$ is admissible in (3.4), we have

$$\mathrm{VaR}_\alpha(X_d^+) \leq \sum_{j=1}^{k} n_j F_j^{-1}(\alpha^{1/k}), \tag{3.22}$$

when $\boldsymbol{Y} \leq_{\mathrm{lo}} \boldsymbol{X}$. The inequalities in (3.21) and in (3.22) might yield improved bounds for general portfolios and are straightforward to compute. Both bounds hold if $\boldsymbol{Y} \leq_{\mathrm{co}} \boldsymbol{X}$ is assumed; see Table 6.

In Tables 5–7 ($k=2$) we report the bounds (3.21) and (3.22) for some inhomogeneous portfolios of Pareto, Exponential and LogNormal marginal distributions. In these tables, the lower bound in (3.21) is shown to improve the corresponding unconstrained one for $k=2,4$, while the upper bound in (3.22) is useful to reduce model uncertainty only in the case $k\leq 2$. In Tables 5–7 , the unconstrained bounds $\underline{\mathrm{VaR}}_\alpha$ and $\overline{\mathrm{VaR}}_\alpha$ are computed according to Corollary 4.7 in Jakobsons et al. (2014) (when possible) or via the RA technique described in Embrechts et al. (2013).

$d=8, k=4$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\overline{\mathrm{VaR}}_\alpha$	DU-S	DU-S'	ΔDU-S
$\alpha=0.990$	9.21	18.42	77.41	68.20	58.99	-13.5%
$\alpha=0.995$	13.14	26.28	98.45	85.31	72.17	-15.4%
$\alpha=0.999$	30.61	61.25	175.46	144.85	114.21	-21.2%

Table 5: Values (rounded) for the VaR bounds in (3.21) (denoted by $\mathrm{VaR}_\alpha^{\mathrm{lb}}$) for a inhomogeneous portfolio with $k=4$ subgroups of two Pareto(2), Pareto(3), Exp(1), Exp(2) distributed risks and dependence assumption $\boldsymbol{Y} \leq_{\mathrm{uo}} \boldsymbol{X}$. Values for $\mathrm{VaR}_\alpha^{\mathrm{ub}}$ are not reported since they do not improve the corresponding unconstrained bounds. Dependency uncertainty spreads prior (DU-S=$\overline{\mathrm{VaR}}_\alpha - \underline{\mathrm{VaR}}_\alpha$) and after (DU-S'=$\overline{\mathrm{VaR}}_\alpha - \mathrm{VaR}_\alpha^{\mathrm{lb}}$) the introduction of dependence information are also reported.

$d=8, k=2$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\mathrm{VaR}_\alpha^{\mathrm{ub}}$	$\overline{\mathrm{VaR}}_\alpha$	DU-S	DU-S'	ΔDU-S
$\alpha=0.990$	9.00	36.00	73.68	89.05	80.05	37.68	-52.9%
$\alpha=0.995$	13.14	52.57	99.91	120.58	107.44	47.34	-55.9%
$\alpha=0.999$	30.62	122.49	205.27	248.24	217.62	82.78	-62.0%

Table 6: Values (rounded) for the VaR bounds in (3.21) (denoted by $\mathrm{VaR}_\alpha^{\mathrm{lb}}$) and in (3.22) (denoted by $\mathrm{VaR}_\alpha^{\mathrm{ub}}$) for a inhomogeneous portfolio with $k=2$ subgroups of four Pareto(2), Exp(1) distributed risks and dependence assumption $\boldsymbol{Y} \leq_{\mathrm{co}} \boldsymbol{X}$.

$d=8, k=4$	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\mathrm{lb}}$	$\overline{\mathrm{VaR}}_\alpha$	DU-S	DU-S'	ΔDU-S
$\alpha=0.990$	285.1	570.1	1975.9	1690.8	1405.8	-16.9%
$\alpha=0.995$	469.5	939.0	3338.2	2868.7	2399.2	-16.4%
$\alpha=0.999$	1313.5	2627.0	11119.2	9805.7	8492.2	-13.4%

Table 7: Values (rounded) for the VaR bounds in (3.21) (denoted by $\mathrm{VaR}_\alpha^{\mathrm{lb}}$) for a inhomogeneous portfolio with $k=4$ subgroups of two LogNormal(0,1), LogNormal(1,2), Pareto(2), Pareto(3) distributed risks and dependence assumption $\boldsymbol{Y} \leq_{\mathrm{uo}} \boldsymbol{X}$. Values for $\mathrm{VaR}_\alpha^{\mathrm{ub}}$ are not reported since they do not improve the corresponding unconstrained bounds.

4 Reducing dependency spreads of convex risk measures

To find reduced dependence uncertainty spreads on convex risk measures, throughout this section we assume

$$\boldsymbol{Y} \leq_{\text{wcs}} \boldsymbol{X}, \tag{4.1}$$

where the vector $\boldsymbol{Y}$ is a random vector having the same univariate marginal distributions as $\boldsymbol{X}$. We also denote by $\boldsymbol{X}^*$ a comonotonic vector having the same univariate marginal distributions as $\boldsymbol{X}$, that is having joint distribution

$$F_{\boldsymbol{X}^*}(x_1, \dots, x_d) = \min\{F_1(x_1), \dots, F_d(x_d)\}, \text{ where } F_i = F_j, \text{ if } i \in I_j, 1 \leq i \leq d.$$

We shall make use of the *convex order* $X \leq_{\text{cx}} Y$ between random *variables* X, Y defined by $X \leq_{\text{cx}} Y$ iff $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all convex functions $f : \mathbb{R} \to \mathbb{R}$ such that the expectations exist.

Let $Y_d^+ = \sum_{j=i}^d Y_j$, $X_d^+ = \sum_{j=i}^d X_j$ and $S_d^* = \sum_{j=1}^d X_j^*$. The dependence ordering result given in Theorem 2.1 in Rüschendorf (2004) and a classical result in Meilijson and Nádas (1979) imply that

$$Y_d^+ \leq_{\text{cx}} X_d^+ \leq_{\text{cx}} S_d^*. \tag{4.2}$$

According to Theorem 4.3 in Bäuerle and Müller (2006), law-invariant, convex risk measures satisfying the so-called Fatou property are consistent with respect to the convex order. Therefore (4.2) implies the following result.

Theorem 4.1. *Let ρ be a law-invariant, convex risk measure satisfying the Fatou property and assume that $\boldsymbol{Y} \leq_{\text{wcs}} \boldsymbol{X}$. Then*

$$\rho(Y_d^+) \leq \rho(X_d^+) \leq \rho(S_d^*). \tag{4.3}$$

The Fatou property is a technical condition which is satisfied by the risk measures used in the following.

Remark 4.1. Theorem 4.1 holds under the weaker assumption $\boldsymbol{Y} \leq_{\text{sm}} \boldsymbol{X}$, resp. $\boldsymbol{Y} \leq_{\text{dcx}} \boldsymbol{X}$, where $\leq_{\text{sm}}$ is the *supermodular order* and $\leq_{\text{dcx}}$ is the *directionally convex order*; see Remark 6.27 (b) in Rüschendorf (2013). Since $\leq_{\text{wcs}}$ has a more direct positive dependence interpretation and can be checked in several functional models (see Rüschendorf, 2004) we state Theorem 4.1 under this ordering condition. For the non convex risk measure VaR the conclusion in (4.3) is not true; see the several examples given in Embrechts et al. (2014a).

The upper bound $\rho(S_d^*)$ in (4.3) corresponds to the case of maximally correlated risks and can be easily computed in the examples to follow. Furthermore, once fixed the marginal distributions of $\boldsymbol{X}$, the upper bound $\rho(S_d^*)$ in (4.3) holds without any dependence assumption, therefore we have

$$\rho(S_d^*) = \overline{\rho}(X_d^+).$$

Again, the dependence assumption $\boldsymbol{Y} \leq_{\text{wcs}} \boldsymbol{X}$ has only influence on the lower bound $\rho(Y_d^+)$ which is generally strictly bigger than the unconstrained bound $\underline{\rho}(X_d^+)$ obtained without any positive dependence information.

To keep the connection with Section 3, in the examples below we assume that $\boldsymbol{Y}$ has the joint distribution defined in (3.1) and we study portfolios of Gamma distributed risks since the Gamma distribution allows to obtain all the quantities in (4.3) in closed form. For general portfolios: the unconstrained lower bounds $\underline{\rho}(X_d^+)$ can be computed using the numerical procedures described in Puccetti and Rüschendorf (2015) (for entropic risk measures) and in Puccetti (2013) (for the Expected Shortfall); the constrained lower bound $\rho(Y_d^+)$ can be generally computed numerically or by Monte Carlo simulation; the upper bound $\rho(S_d^*) = \overline{\rho}(X_d^+)$ is easily available numerically considering that $\boldsymbol{X}^*$ is comonotonic.

Example 4.1 (Expected Shortfall). We now compute the bounds in (4.3) when $\rho = \mathrm{ES}_\alpha$ for a portfolio of Gamma-distributed random variables. For a random variable X with $\mathbb{E}[|X|] < \infty$, the *Expected Shortfall* (ES) at confidence level $\alpha \in [0,1)$ is defined as

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_q(X)\,dq.$$

If a random variable X follows a $F_{a,s} = \mathrm{Gamma}(a,s)$ distribution with positive parameters a, s and density function

$$f_{a,s}(x) = \frac{1}{s^a\Gamma(a)}x^{a-1}e^{-\frac{x}{s}}, 1 \le j \le d,$$

it is elementary to show that

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\frac{\Gamma(a+1)s}{\Gamma(a)}\left(\overline{F}_{a+1,s}\left(F_{a,s}^{-1}(\alpha)\right)\right). \tag{4.4}$$

Assuming that $F_j = \mathrm{Gamma}(a_j, s), 1 \le j \le d$ in (3.1), it follows that $\sum_{i \in I_j} Y_i$ is again a Gamma random variable with distribution $F_{a_j, n_j s}, 1 \le j \le k$. Setting $n_j = d/k, 1 \le j \le k$ and taking the convolution of Gamma with identical scale parameter, we obtain that Y_d^+ has Gamma distribution $F_{\sum_{j=1}^k a_j, ds/k}$. From (4.4) we finally find that the constrained lower bound $\mathrm{ES}_\alpha(Y_d^+)$ in (4.3) is given by

$$\mathrm{ES}_\alpha(Y_d^+) = \frac{d\ s}{k(1-\alpha)}\frac{\Gamma\left(1+\sum_{j=1}^k a_j\right)}{\Gamma(\sum_{j=1}^k a_j)}\left(\overline{F}_{1+\sum_{j=1}^k a_j, ds/k}\left(F^{-1}_{\sum_{j=1}^k a_j, ds/k}(\alpha)\right)\right).$$

The unconstrained lower bound $\underline{\mathrm{ES}}_\alpha(X_d^+)$ defined in (1.2) with $\rho = \mathrm{ES}_\alpha$ is very well approximated by the sum of the first moment $\mu = 12$ of the Gamma marginals under study, that is

$$\underline{\mathrm{ES}}_\alpha(X_d^+) \simeq \mathrm{ES}_\alpha(\mu) = \mu.$$

In fact, using the mixability detection procedure in Puccetti and Wang (2014), it is possible to show that the set of Gamma distributions F_j used in Table 8, conditional to the intervals $[0, F_j^{-1}(1-10^{-7})]$, are very close to being *jointly mixable*, that is there exist a random vector $(W_1, \dots, W_d)$ having the prescribed conditional marginal distributions under which

$$P(W_1 + \cdots + W_d \in [\mu - 10^{-3}, \mu + 10^{-3}]) = 1.$$

Note that this also directly implies that $\underline{\rho}(X_d^+) \simeq \rho(\mu)$ in the forthcoming examples where other convex risk measures ρ are considered. For more details on joint mixability, we refer to Puccetti and Wang (2014) and references therein.

Since ES_α is a *comonotonic additive* convex risk measure, the upper bound $\overline{\mathrm{ES}}_\alpha(X_d^+) = \mathrm{ES}_\alpha(S_d^*)$ is equal to the sum of marginal expected shortfall, i.e. we have

$$\overline{\mathrm{ES}}_\alpha(X_d^+) = \mathrm{ES}_\alpha(S_d^*) = \sum_{j=1}^d \mathrm{ES}_\alpha(Y_j) = \frac{d\ s}{k(1-\alpha)}\sum_{j=1}^k \frac{\Gamma(a_j+1)}{\Gamma(a_j)}\left(\overline{F}_{a_j+1,s}\left(F_{a_j,s}^{-1}(\alpha)\right)\right).$$

In Table 8, we report the bounds $\underline{\mathrm{ES}}_\alpha(X_d^+), \mathrm{ES}_\alpha(Y_d^+)$ and $\overline{\mathrm{ES}}_\alpha(X_d^+) = \mathrm{ES}_\alpha(S_d^*)$ for a portfolio of inhomogeneous Gamma distributions as well as the implied DU-spreads. With a slight abuse of notation the case $k = 1$ indicates in the forthcoming tables the case when all the $X_j's$ are comonotonic.

$d=8$	unconstrained			$k=1$		$k=2$		$k=4$		$k=8$	
	$\underline{\mathrm{ES}}_\alpha$	$\overline{\mathrm{ES}}_\alpha$	DU-S	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	ΔDU-S	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	ΔDU-S	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	ΔDU-S	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	ΔDU-S
$\alpha=0.990$	12.00	38.27	26.27	38.27	-100%	29.15	-65.3%	23.29	-43.0%	19.56	-28.8%
$\alpha=0.995$	12.00	41.64	29.64	41.64	-100%	31.15	-64.6%	24.52	-42.2%	20.33	-28.1%
$\alpha=0.999$	12.00	49.27	37.27	49.27	-100%	35.63	-63.4%	27.21	-40.8%	22.02	-26.9%

Table 8: Values (rounded) for the bounds $\mathrm{ES}_\alpha(Y_d^+)$ (denoted by $\mathrm{ES}_\alpha^{\mathrm{lb}}$), $\underline{\mathrm{ES}}_\alpha(X_d^+)$ and $\overline{\mathrm{ES}}_\alpha(X_d^+)$ for an inhomogeneous portfolio with $d/2$ Gamma(2,1/2) risks and $d/2$ Gamma(4,1/2) risks, k subgroups, $n=d/k$ variables in each subgroup and dependence assumption $\boldsymbol{Y} \leq_{\mathrm{wcs}} \boldsymbol{X}$. In this and in the forthcoming tables, the reduction of DU-spreads implied by additional dependence assumption is also reported.

Example 4.2 (Entropic risk measure)**.** For a random variable X with $\mathbb{E}[|X|]<\infty$, the *Entropic Risk Measure* (ERM) with (risk aversion) parameter $\beta>0$ is defined as

$$\mathrm{ERM}_\beta(X)=\frac{1}{\beta}\log\mathbb{E}\left[e^{\beta X}\right],$$

provided the expectation is finite. The entropic risk measure is well known to be a convex risk measure (see for instance Föllmer and Schied, 2004). For a random variable X with Gamma distribution $F_{a,s}$, it follows immediately by the definition of the moment generating function ($M_X(t)=\mathbb{E}[e^{tX}]$) that:

$$\mathrm{ERM}_\beta(X)=\frac{1}{\beta}\log\left(1-s\beta\right)^{-a}\qquad\text{for } 0<\beta<\frac{1}{s}.$$

We already know from Example 4.1 that Y_d^+ has Gamma distribution $F_{\sum_{j=1}^k a_j, ds/k}$, hence we obtain

$$\mathrm{ERM}_\beta(Y_d^+)=\frac{1}{\beta}\log\left(1-\frac{ds}{k}\beta\right)^{-\sum_{j=1}^k a_j}\quad\text{for } 0<\beta<\frac{k}{ds}.$$

From Theorem 4.1 the upper bound in (4.3) for the entropic risk measure is obtained when all the X_j's are comonotonic, that is

$$\overline{\mathrm{ERM}}_\beta(X_d^+)=\mathrm{ERM}_\beta(S_d^*)=\frac{1}{\beta}\log\mathbb{E}\left[e^{\beta\left(n_1F^{-1}_{a_1,s}(U)+\cdots+n_kF^{-1}_{a_k,s}(U)\right)}\right],$$

where $F^{-1}_{a_j,s}$ is the Gamma inverse cumulative distribution function with parameters a_j, s and U is a random variable with uniformly distributed on $(0,1)$. Similarly to what discussed in Example 4.1, we obtain $\underline{\mathrm{ERM}}_\beta(X_d^+)\simeq 12$. In Table 9, we report the bounds $\underline{\mathrm{ERM}}_\beta(X_d^+)$, $\mathrm{ERM}_\beta(Y_d^+)$ and $\overline{\mathrm{ERM}}_\beta(X_d^+)$ for a portfolio of inhomogeneous Gamma distributions as well as the corresponding dependence uncertainty spreads.

$d=8$	unconstrained			$k=1$		$k=2$		$k=4$		$k=8$	
	$\underline{\mathrm{ERM}}_\beta$	$\overline{\mathrm{ERM}}_\beta$	DU-S	$\mathrm{ERM}_\beta{}^{\mathrm{lb}}$	ΔDU-S	$\mathrm{ERM}_\beta{}^{\mathrm{lb}}$	ΔDU-S	$\mathrm{ERM}_\beta{}^{\mathrm{lb}}$	ΔDU-S	$\mathrm{ERM}_\beta{}^{\mathrm{lb}}$	ΔDU-S
$\beta=0.10$	12.00	15.22	3.22	15.22	-100%	13.38	-42.9%	12.64	-19.9%	12.31	-9.6%
$\beta=0.15$	12.00	18.14	6.14	18.14	-100%	14.27	-37.0%	13.00	-16.3%	12.47	-7.7%
$\beta=0.20$	12.00	23.80	11.80	23.80	-100%	15.33	-28.2%	13.39	-11.8%	12.64	-5.4%

Table 9: Values (rounded) for the bound $\mathrm{ERM}_\beta(Y_d^+)$ (denoted by $\mathrm{ERM}_\beta{}^{\mathrm{lb}}$), $\underline{\mathrm{ERM}}_\beta(X_d^+)$ and $\overline{\mathrm{ERM}}_\beta(X_d^+)$ for an inhomogeneous portfolio with $d/2$ Gamma(2,1/2) risks and $d/2$ Gamma(4,1/2) risks, k subgroups, $n=d/k$ variables in each subgroup and dependence assumption $\boldsymbol{Y} \leq_{\mathrm{wcs}} \boldsymbol{X}$.

Example 4.3 (Expectiles)**.** We now compute the bounds in (4.3) using expectiles as risk measure, $\rho = e_p$, for a portfolio of Gamma-distributed random variables. Expectiles are attracting increasing interest in the literature on risk measures since they are the only coherent risk measure with the additional property of being elicitable; see for instance Ziegel (2014) and Bellini and Bignozzi (2014) for more details on this. For a random variable X with $\mathbb{E}[|X|] < \infty$, the *expectile* e_p, computed at confidence level $p \in (0, 1)$, is defined as the unique solution to

$$p\mathbb{E}[(X - \mathrm{e}_p(X))_+] = (1 - p)\mathbb{E}[(X - \mathrm{e}_p(X))_-],$$

where $x_+ = \max(0, x)$ and $x_- = \max(0, -x)$. When $p \geq 1/2$, expectiles are coherent risk measures. Expectiles are generally not available in closed form and need to be computed numerically.

Again from Theorem 4.1 the upper bound in (4.3) for the expectiles is obtained when all the X_j's are comonotonic. Hence we have to find numerically the unique solution to

$$\begin{aligned} &p\mathbb{E}[(n_1F^{-1}_{a_1,s}(U) + \cdots + n_kF^{-1}_{a_k,s}(U) - \bar{\mathrm{e}}_p\left(n_1F^{-1}_{a_1,s}(U) + \cdots + n_kF^{-1}_{a_k,s}(U)\right))_+] \\ &\quad = (1 - p)\mathbb{E}[(n_1F^{-1}_{a_1,s}(U) + \cdots + n_kF^{-1}_{a_k,s}(U) - \bar{\mathrm{e}}_p\left(n_1F^{-1}_{a_1,s}(U) + \cdots + n_kF^{-1}_{a_k,s}(U)\right))_-]. \end{aligned}$$

The computation of $\mathrm{e}_p(Y^+_d)$ is analogous while, similarly to what discussed in Example 4.1, we obtain $\underline{\mathrm{e}}_p(X^+_d) \simeq 12$. In Table 10, we report the bounds (4.3) for a portfolio of inhomogeneous Gamma distributions as well as the unconstrained bounds $\underline{\mathrm{e}}_p(X^+_d)$ and $\bar{\mathrm{e}}_p(X^+_d)$.

$d = 8$	unconstrained			$k = 1$		$k = 2$		$k = 4$		$k = 8$	
	$\underline{\mathrm{e}}_p$	$\bar{\mathrm{e}}_p$	DU-S	$\mathrm{e}^{\mathrm{lb}}_p$	ΔDU-S	$\mathrm{e}^{\mathrm{lb}}_p$	ΔDU-S	$\mathrm{e}^{\mathrm{lb}}_p$	ΔDU-S	$\mathrm{e}^{\mathrm{lb}}_p$	ΔDU-S
$p = 0.90$	12.00	18.71	6.71	18.71	−100%	16.67	−69.6%	15.22	−47.8%	14.23	−33.2%
$p = 0.95$	12.00	21.34	9.34	21.34	−100%	18.39	−68.4%	16.36	−46.7%	15.00	−32.1%
$p = 0.99$	12.00	27.52	15.52	27.52	−100%	22.35	−66.7%	18.92	−44.6%	16.70	−30.3%

Table 10: Values (rounded) for the bound $\mathrm{e}_p(Y^+_d)$ (denoted by $\mathrm{e}^{\mathrm{lb}}_p$), $\underline{\mathrm{e}}_p(X^+_d)$ and $\bar{\mathrm{e}}_p(X^+_d)$ for an inhomogeneous portfolio with $\frac{d}{2}$ Gamma(2,1/2) risks and $\frac{d}{2}$ Gamma(4,1/2) risks, k subgroups, $n = \frac{d}{k}$ variables in each subgroup and dependence assumption $\boldsymbol{Y} \leq_{\mathrm{wcs}} \boldsymbol{X}$.

5 Using negative dependence assumptions

We have introduced a class of positive dependence restrictions allowing to determine analytically improved lower risk bounds for the risk of a joint portfolio compared to the case where only marginals information is available. In a variety of examples of interest in quantitative risk management, positive dependence assumptions added on top of marginals information allow to increase the best-possible lower risk $\underline{\rho}(X^+_d)$ and therefore reduce the implied dependence uncertainty spread $\overline{\rho}(X^+_d) - \underline{\rho}(X^+_d)$.

However, positive dependence assumptions seem to be ineffective in reducing the worst-possible estimate $\overline{\rho}(X^+_d)$ apart from some trivial cases of limited applicability. Some kind of negative dependence allows to construct worst-case VaR distributions with VaR value bigger than in the comonotonic case (see Wang and Wang, 2011; Embrechts et al., 2013) and approaching asymptotically the worst ES bound (see Puccetti et al., 2013; Puccetti and Rüschendorf, 2014).

It seems intuitively clear that one has to assume some negative dependence constraints in order to reduce the upper bound on a risk measure. A negative dependence condition on $\boldsymbol{X}$ is obtained assuming that

$$\boldsymbol{X} \leq_{\mathrm{wcs}} \boldsymbol{Y}. \tag{5.1}$$

If $\boldsymbol{Y}$ has the distribution specified in (3.1), then (5.1) does not pose essential restriction on the dependence structure within the groups I_j but affects the joint dependence of the groups which is assumed to be more negatively dependent compared to independent groups. Similarly to Theorem 4.1, the assumption $\boldsymbol{X} \leq_{\mathrm{wcs}} \boldsymbol{Y}$ implies the following result.

Theorem 5.1. *Let ρ be a law-invariant, convex risk measure and assume that $\boldsymbol{X} \leq_{\mathrm{wcs}} \boldsymbol{Y}$. Then*

$$\rho(X_d^+) \leq \rho(Y_d^+) \leq \rho(S_d^*). \tag{5.2}$$

The upper bound $\rho(Y_d^+)$ in (5.2) is in general a strong improvement of the comonotonic upper bound $\rho(S_d^*)$. The assumption $\boldsymbol{X} \leq_{\mathrm{wcs}} \boldsymbol{Y}$ may be realistic in hierarchical insurance models where some branches (groups) of insurance companies are approximately independent or possibly negatively dependent. For examples bad weather type insurance (like hail, heavy rain, storm, etc...) could be supposed realistically to be negatively dependent to hot weather type insurances (like draught, fire, thunderstorm, etc...). Some more detailed study of this type of applications is planned for a future study.

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