On the Monge-Kantorovich Duality Theorem

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The Monge-Kantorovich duality theorem has a variety of applications in probability theory, statistics, and mathematical economics. There has been extensive work to establish the duality theorem under general conditions. In this paper, by imposing a natural stability requirement on the Monge-Kantorovich functional, we characterize the probability spaces (called strong duality spaces) which ensure the validity of the duality theorem. We prove that strong duality is equivalent to each one of (i) extension property, (ii) projection property, (iii) the charge extension property and (iv) perfectness. The resulting characterization enables us to derive many useful properties that such spaces inherit from being perfect.

1 Introduction

Continuous versions of the classical transportation problem dating back to Monge (1781) and developing into infinite-dimensional linear programming in the work of Kantorovich (1940, 1942) concern the validity of a duality theorem for the transportation problem (formally defined in the next section). General treatment of Monge-Kantorovich duality theorems can be found in Rüschendorf (1981), Kellerer (1984), Levin (1984), Rachev (1991) and Ramachandran and Rüschendorf (1995). They arise in the study of (i) probabilities with given marginals and given support, stochastic ordering (Strassen (1965), Sudakov (1975), Hoffmann-Jϕrgensen(1987), Dall'Aglio, Kotz and Salinetti (1991), Rüschendorf, Schweizer and Taylor (1996) and Benes and Stepan (1997)), (ii) probability metrics, central limit theorems and asymptotic analysis of algorithms (Rachev(1991), Rachev and Rüschendorf(1998),

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(iii) equlibria in assignment models in economics (Gretsky, Ostroy and Zame (1992) and Ramachandran and Rüschendorf (1997)), (iv) operator algebras (Arveson (1974), Haydon and Shulman (1996)) and many others.

Ramachandran and Rüschendorf (1995) established a general duality theorem when one of the underlying spaces is perfect. We say that a probability space is a *duality space* if the duality theorem holds for all bounded measurable functions on its product with any other probability space. A duality space is called a *strong duality space* if, further, for each measurable cost function the optimal value is stable with respect to only 'technically' different formulations of the duality problem(in a sense made precise in the next section). The projection property for a probability space is the measure theoretic analogue of the analyticity of the projection of Borel sets in the product of standard Borel spaces. The charge extension property (extension property) concerns the extension of charges (probabilities) with given marginals on a product space when a marginal space is enlarged. We show that strong duality, perfectness, projection property, charge extension property and extension property are all equivalent. This new characterization enables us to obtain several useful properties such spaces inherit from perfectness which are not easily established by direct arguments.

2 Notation and Preliminaries

We use customary measure theoretic terminology and notation (as, for instance, in Neveu (1965)). All measures that we consider are probabilities. A *charge* is a finitely additive probability. For a measurable space $(X, \mathcal{A})$ the notation $f \in \mathcal{A}$ indicates that f is a real-valued, bounded $\mathcal{A}$-measurable function on X. We denote by $\mathcal{B}$ the σ-algebra of Borel subsets of $[0,1]$. If P is a probability on $(X, \mathcal{A})$ then P_* and P^* denote respectively the inner and the outer measures induced by P. A σ-algebra $\mathcal{A}_0$ is said to be countably generated (or c.g. for short) if $\mathcal{A}_0 = \sigma(\{A_n, n \geq 1\})$ in which case $\varphi : (X, \mathcal{A}_0) \to ([0,1], \mathcal{B})$ defined by $\varphi(x) = \sum_{n=1}^{\infty} (2/3^n) 1_{A_n}(x)$ is called the Marczewski function. φ is measurable with $\varphi(x_1) \neq \varphi(x_2)$ if x_1 and x_2 belong to different atoms of $\mathcal{A}_0$ and so we can identify $(X, \mathcal{A}_0)$ with $(\varphi(X), \mathcal{B} \cap \varphi(X))$. We say that $(X, \mathcal{A}, P)$ is a thick subspace of $(X_1, \mathcal{A}_1, P_1)$ and write $(X, \mathcal{A}, P) \subset (X_1, \mathcal{A}_1, P_1)$ whenever $X \subset X_1$, $\mathcal{A} = \mathcal{A}_1 \cap X$ = the trace of $\mathcal{A}_1$ on X, $P_1^*(X) = 1$ and $P = P_1^* | \mathcal{A}$. If $(X, \mathcal{A}, P)$ is a probability space then $\overline{\mathcal{A}}^P$ denotes the completion of $\mathcal{A}$ with respect to P.

Let $(X, \mathcal{A}, P)$ be a probability space. P is called *perfect* (equivalently, the space $(X, \mathcal{A}, P)$ is called *perfect*) if, for every $\mathcal{A}$-measurable, real-valued function f on X we can find a Borel subset B_f of the real line such that $B_f \subset f(X)$ with $P(f^{-1}(B_f)) = 1$. For properties of perfect measures we refer the reader to Ramachandran (1979).

Let $(X_i, \mathcal{A}_i, P_i)$, $i = 1, 2$ be two probability spaces. A probability μ on $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ is said to have marginals P_1 and P_2 if

$$\mu(A_1 \times X_2) = P_1(A_1) \text{ for all } A_1 \in \mathcal{A}_1, \text{ and}$$

$$\mu(X_1 \times A_2) = P_2(A_2) \text{ for all } A_2 \in \mathcal{A}_2.$$

Let

$$\mathcal{M}(P_1, P_2) = \{\mu \text{ on } \mathcal{A}_1 \otimes \mathcal{A}_2 : \mu \text{ has marginals } P_1 \text{ and } P_2\} = \mathcal{M}_{\mathcal{A}_1 \otimes \mathcal{A}_2}(P_1, P_2).$$

$\tilde{\mathcal{M}}(P_1, P_2)$ is used to denote the collection of charges μ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ with the marginals P_1 and P_2. $\pi_i : X_1 \times X_2 \to X_i$ denote the canonical projections for $i = 1, 2$. The abbreviation $\oplus_i g_i$ is used for $\sum_{i=1}^{2} g_i \circ \pi_i$.

For $h \in \mathcal{A}_1 \otimes \mathcal{A}_2$ the *transportation problem* is concerned with

$$S(h) = \sup \left\{ \int_{X_1 \times X_2} h \, d\mu : \mu \in \mathcal{M}(P_1, P_2) \right\} = S_{\mathcal{A}_1 \otimes \mathcal{A}_2}(h)$$

while the *dual problem* deals with

$$I(h) = \inf \left\{ \sum_{i=1}^{2} \int_{X_i} h_i \, dP_i : h_i \in \mathcal{L}^1(P_i) \;\; and \;\; h \leq \oplus_i h_i \right\} \; = \; I_{\mathcal{A}_1 \otimes \mathcal{A}_2}(h).$$

The measure theoretic version of the transportation problem due to Kantorovich(1942) is the validity of the duality

$$S(h) = I(h) \, . \qquad (D)$$

The main duality theorem of Kellerer(1984) deals essentially with second countable or metrizable spaces $X_i, i = 1, 2$ with tight(or Radon) probabilities defined on the Borel sets in which case (D) is shown to hold for a suitably large class containing all the bounded, measurable functions. The following result of Ramachandran and Rüschendorf(1995) is the most general duality theorem of this type.

Theorem 1 *If at least one of the underlying probability spaces is perfect then* (D) *holds for all* $h \in \mathcal{A}_1 \otimes \mathcal{A}_2$.

3 Strong Duality Spaces, Perfectness and Related Properties

In order to study probability spaces which ensure the validity of the general duality theorem we introduce

Definition 1 *A probability space* $(X_1, \mathcal{A}_1, P_1)$ *is called a duality space if, for every* $(X_2, \mathcal{A}_2, P_2)$*, the duality (D) holds for all* $h \in \mathcal{A}_1 \otimes \mathcal{A}_2$*.*

A natural strengthening of the notion of a duality space arises if we postulate additionally that for any sub σ-algebra $\mathcal{C}_2 \subset \mathcal{A}_2$ such that $h \in \mathcal{A}_1 \otimes \mathcal{C}_2$ the optimal value $S_{\mathcal{A}_1 \otimes \mathcal{A}_2}(h) = S_{\mathcal{A}_1 \otimes \mathcal{C}_2}(h)$ (or equivalently, $I_{\mathcal{A}_1 \otimes \mathcal{A}_2}(h) = I_{\mathcal{A}_1 \otimes \mathcal{C}_2}(h)$), i.e. the optimal value remains the same with refinements on the second space as long as the measurability conditions on h are fulfilled. In other words the value of the transportation problem remains the same for only 'technically' different formulations of the problem. The following proposition shows that perfect spaces have this property.

Proposition 1 *Let* $(X_i, \mathcal{A}_i, P_i)$*,* $i = 1, 2$ *be two probability spaces where* P_1 *is perfect. If* $h \in \mathcal{A}_1 \otimes \mathcal{C}_2 \subset \mathcal{A}_1 \otimes \mathcal{A}_2$ *then*

$$I_{\mathcal{A}_1 \otimes \mathcal{A}_2}(h) \quad = \quad I_{\mathcal{A}_1 \otimes \mathcal{C}_2}(h) \ .$$

Proof: See Corollary 1 in Ramachandran and Rüschendorf(1995).

This motivates the following

Definition 2 *A probability space* $(X_1, \mathcal{A}_1, P_1)$ *is called a strong duality space if*

(i) it is a duality space, and
(ii) for every sub σ*-algebra* $\mathcal{C}_2 \subset \mathcal{A}_2$ *and for every* $h \in \mathcal{A}_1 \otimes \mathcal{C}_2$ *the condition*

$$I_{\mathcal{A}_1 \otimes \mathcal{A}_2}(h) \quad = \quad I_{\mathcal{A}_1 \otimes \mathcal{C}_2}(h) \qquad\qquad (SD)$$

holds.

Theorem 1 and Corollary 1 together show that every perfect probability space is a strong duality space. Since every measure on a standard Borel space is perfect, the class of strong duality spaces is rich.

Proposition 2 *Every strong duality space is perfect.*

Proof: Suppose $(X_1, \mathcal{A}_1, P_1)$ is nonperfect. Then there exists a function $f : X_1 \longrightarrow [0,1]$ such that $f \in \mathcal{A}_1$ and $Q_{2*}(f(X_1)) < 1$ where $Q_2 = P_1 f^{-1}$ on $([0,1], \mathcal{B})$. Let $X_2 = [0,1], \mathcal{C}_2 = \mathcal{B}$. Since

$$G \;=\; \text{Graph of } f \;=\; \{(x_1, f(x_1)) \;:\; x_1 \in X_1\} \;\in\; \mathcal{A}_1 \otimes \mathcal{C}_2$$

if we define for $C \in \mathcal{A}_1 \otimes \mathcal{C}_2$

$$\mu(C) \;=\; P_1(\pi_1(C \cap G)) \;=\; P_1(\{x_1 \in X_1 \;:\; (x_1, f(x_1)) \in C\})$$

then $\mu \in \mathcal{M}(P_1, Q_2)$ and $\mu(G) = 1$. Hence $I_{\mathcal{A}_1 \otimes \mathcal{C}_2}(G) \geq S_{\mathcal{A}_1 \otimes \mathcal{C}_2}(G) = 1$.

Now let

$$\mathcal{A}_2 = \sigma(\{\mathcal{C}_2, f(X_1)\}) = \{(D_1 \cap f(X_1)) + (D_2 \cap (f(X_1)^c) : D_i \in \mathcal{C}_2, i = 1, 2\}.$$

Choose $D_0 \subset f(X_1)$, $D_0 \in \mathcal{C}_2$ with $Q_2(D_0) = Q_{2*}(f(X_1)) < 1$ and define P_2 on $\mathcal{A}_2$ by

$$P_2((D_1 \cap f(X_1)) + (D_2 \cap (f(X_1)^c)) \stackrel{\text{def}}{=} Q_2(D_1 \cap D_0) + \frac{1}{2} Q_2(D_1 \cap D_0^c) + \frac{1}{2} Q_2(D_2 \cap D_0^c).$$

It is easy to check that P_2 is a probability on $\mathcal{A}_2$ such that $P_2|_{\mathcal{C}_2} = Q_2$ and $P_2(f(X_1)) = Q_2(D_0) + \frac{1}{2} Q_2(D_0^c) < 1$. Since $G \subset (X_1 \times f(X_1)) \in \mathcal{A}_1 \otimes \mathcal{A}_2$ we have

$$I_{\mathcal{A}_1 \otimes \mathcal{A}_2}(G) \;\leq\; P_2(f(X_1)) \;<\; 1 \;\leq\; I_{\mathcal{A}_1 \otimes \mathcal{C}_2}(G)$$

and so $(X_1, \mathcal{A}_1, P_1)$ is not a strong duality space. □

Perfect spaces have the following extension property for measures (see Theorem 9, Ramachandran(1996)) which arises naturally in the marginal problem.

Definition 3 *We say that $(X_1, \mathcal{A}_1, P_1)$ has the extension property if for every $(X_2, \mathcal{A}_2, P_2)$ and for every sub σ-algebra $\mathcal{C}_2 \subset \mathcal{A}_2$, if $\mu \in \mathcal{M}(P_1, P_2|_{\mathcal{C}_2})$ then $\mu \nearrow \bar{\mu} \in \mathcal{M}(P_1, P_2)$,* i.e., *$\mu$ extends to a measure $\bar{\mu}$ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ with marginals P_1 and P_2.*

Proposition 3 *If $(X_1, \mathcal{A}_1, P_1)$ has the extension property then it is a strong duality space.*

Proof: 1^0. We first show that $(X_1, \mathcal{A}_1, P_1)$ is a duality space. To this end, let $(Y_2, \mathcal{D}_2, Q_2)$ be an arbitrary probability space. There exists a perfect probability space $(X_2, \mathcal{C}_2, \overline{Q_2}) \supset (Y_2, \mathcal{D}_2, Q_2)$ as a thick subspace ($(X_2, \mathcal{C}_2, \overline{Q_2})$ can be constructed using the Stone space of $\mathcal{D}_2$ (see Sikorski(1960)) or by using the map $\Psi : Y_2 \to \{0,1\}^{\mathcal{D}_2}$ defined by $\Psi(y_2) = \{1_D(y_2)\}_{D \in \mathcal{D}_2}$ and taking $X_2 = \{0,1\}^{\mathcal{D}_2}$, $\mathcal{C}_2$= product σ-algebra on X_2 and $\overline{Q_2} = P_2 \Psi^{-1}$). Let

$$\begin{aligned} \mathcal{A}_2 &= \sigma(\{\mathcal{C}_2, Y_2\}) \\ P_2(A_2) &= Q_2(A_2 \cap Y_2), \quad A_2 \in \mathcal{A}_2. \end{aligned}$$

Note that if $C_2 \in \mathcal{C}_2$ then $P_2(C_2) = Q_2(C_2 \cap Y_2) = \overline{Q_2}(C_2)$ and so $P_2 \mid_{\mathcal{C}_2} = \overline{Q_2}$.

Suppose $\mu \in \mathcal{M}(P_1, \overline{Q_2})$. Then, by the extension property $\mu \nearrow \overline{\mu} \in \mathcal{M}(P_1, P_2)$. So, $\overline{\mu}(X_1 \times Y_2) = P_2(Y_2) = Q_2(Y_2) = 1$ implying that $\overline{\mu} \mid_{X_1 \times Y_2} \in \mathcal{M}(P_1, Q_2)$. Thus to every $\mu \in \mathcal{M}(P_1, \overline{Q_2})$ corresponds $\overline{\mu} \in \mathcal{M}(P_1, Q_2)$. Conversely, if $\mu \in \mathcal{M}(P_1, Q_2)$ define $\overline{\mu}(C) = \mu(C \cap (X_1 \times Y_2)), C \in \mathcal{A}_1 \otimes \mathcal{C}_2$ to obtain $\overline{\mu} \in \mathcal{M}(P_1, \overline{Q_2})$ extending μ. Hence we have established that $\mathcal{M}(P_1, Q_2) \longleftrightarrow \mathcal{M}(P_1, \overline{Q_2})$.

Let $h \in \mathcal{A}_1 \otimes \mathcal{D}_2$. Since $(Y_2, \mathcal{D}_2, Q_2) \subset (X_2, \mathcal{C}_2, \overline{Q_2})$, using standard measure theoretic arguments, we can find a $\overline{h} \in \mathcal{A}_1 \otimes \mathcal{C}_2$ on $X_1 \times X_2$ with $\int h d\mu = \int \overline{h} d\overline{\mu}$ for all $\mu \in \mathcal{M}(P_1, Q_2)$. Thus $S_{\mathcal{A}_1 \otimes \mathcal{D}_2}(h) = S_{\mathcal{A}_1 \otimes \mathcal{C}_2}(\overline{h})$. Since $(X_2, \mathcal{C}_2, \overline{Q_2})$ is a duality space being perfect, by the definitions of $S(.)$ and $I(.)$, we have

$$I_{\mathcal{A}_1 \otimes \mathcal{C}_2}(\overline{h}) = S_{\mathcal{A}_1 \otimes \mathcal{C}_2}(\overline{h}) = S_{\mathcal{A}_1 \otimes \mathcal{D}_2}(h) \leq I_{\mathcal{A}_1 \otimes \mathcal{D}_2}(h) \leq I_{\mathcal{A}_1 \otimes \mathcal{C}_2}(\overline{h}).$$

Hence $(X_1, \mathcal{A}_1, P_1)$ is a duality space.

2^0. Now, if $(X_2, \mathcal{A}_2, P_2)$ is an arbitrary probability space, $\mathcal{C}_2 \subset \mathcal{A}_2$ is a sub σ-algebra and $h \in \mathcal{A}_1 \otimes \mathcal{C}_2$ then

$$\begin{aligned} I_{\mathcal{A}_1 \otimes \mathcal{C}_2}(h) &= S_{\mathcal{A}_1 \otimes \mathcal{C}_2}(h) \quad \text{(by } 1^0 \text{ above)} \\ &= S_{\mathcal{A}_1 \otimes \mathcal{A}_2}(h) \quad \text{(by the extension property)} \\ &= I_{\mathcal{A}_1 \otimes \mathcal{A}_2}(h) \quad \text{(by } 1^0 \text{ above)} \end{aligned}$$

and so (SD) holds for $(X_1, \mathcal{A}_1, P_1)$. □

It is well known that the projection of a Borel set in the product of standard Borel spaces is an analytic set and therefore universally measurable (see Hoffmann-Jϕrgensen (1970), Cohn (1980)). This property is useful in descriptive set theory. There is no measure theoretic analogue for the projection of a measurable set in the product of two probability spaces. The following gives a suitable measure theoretic definition of the projection property.

Definition 4 *A probability space $(X_1, \mathcal{A}_1, P_1)$ is said to have the projection property if for every $(X_2, \mathcal{A}_2, P_2)$ and for every $C \in \mathcal{A}_1 \otimes \mathcal{A}_2$ there exists $A_1 = A_1(C) \in \mathcal{A}_1$ with $P_1(A_1) = 1$ such that $\pi_2(C \cap (A_1 \times X_2)) \in \overline{\mathcal{A}_2}^{P_2}$.*

Proposition 4 *Every perfect probability space has the projection property.*

<u>Proof:</u> Let $(X_1, \mathcal{A}_1, P_1)$ be perfect. Let $(X_2, \mathcal{A}_2, P_2)$ be arbitrary and let $C \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Then $C \in \mathcal{D}_1 \otimes \mathcal{D}_2$ where $\mathcal{D}_i \subset \mathcal{A}_i$ are c.g. sub σ-algebras for $i = 1, 2$. Hence, using the Marczewski function, we can assume that $X_i \subset [0,1], \mathcal{D}_i = \mathcal{B} \cap X_i, i = 1, 2$. Since P_1 is perfect there exists $A_1 \subset X_1, A_1 \in \mathcal{B}$ with $P_1(A_1) = 1$. Since $C \in \mathcal{D}_1 \otimes \mathcal{D}_2$, $C = \overline{C} \cap (X_1 \times X_2), \overline{C} \in \mathcal{B} \otimes \mathcal{B}$. $D = \bar{C} \cap (A_1 \times [0,1]) \in \mathcal{B} \otimes \mathcal{B}$ and so $\pi_2(D)$ is an analytic set. Hence there exists $B_{2*}, B_2^* \in \mathcal{B}$ such that $B_{2*} \subset \pi_2(D) \subset B_2^*$ with $\overline{P_2}(B_{2*}) = \overline{P_2}(B_2^*)$ (where $\overline{P_2}$ is the completion of the probability on $\mathcal{B}$ induced by $P_2|_{\mathcal{D}_2}$). Since $\pi_2(D) \cap X_2 = \pi_2(C \cap (A_1 \times X_2))$, we have $B_{2*} \cap X_2 \subset \pi_2(C \cap (A_1 \times X_2)) \subset B_2^* \cap X_2$. Note that $B_{2*} \cap X_2, B_2^* \cap X_2 \in \mathcal{D}_2$ and $P_2(B_{2*} \cap X_2) = \overline{P_2}(B_{2*}) = \overline{P_2}(B_2^*) = P_2(B_2^* \cap X_2)$. Hence $\pi_2(C \cap (A_1 \times X_2)) \in \overline{\mathcal{D}_2}^{P_2} \subset \overline{\mathcal{A}_2}^{P_2}$. □

Charges on $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ with given marginals P_1 and P_2 play a crucial role in establishing duality theorems (see Rüschendorf (1981), Ramachandran and Rüschendorf (1997)). The following charge extension property is the finitely additive analogue of the extension property given in Definition 3.

Definition 5 *We say that $(X_1, \mathcal{A}_1, P_1)$ has the charge extension property if for every $(X_2, \mathcal{A}_2, P_2)$ and for every sub σ-algebra $\mathcal{C}_2 \subset \mathcal{A}_2$, if $\mu \in \tilde{\mathcal{M}}(P_1, P_2|_{\mathcal{C}_2})$ then $\mu \nearrow \bar{\mu} \in \tilde{\mathcal{M}}(P_1, P_2)$,* i.e., *$\mu$ extends to a charge $\bar{\mu}$ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ with marginals P_1 and P_2.*

Proposition 5 *If $(X_1, \mathcal{A}_1, P_1)$ has the projection property then it has the charge extension property.*

Proof: Let $(X_2, \mathcal{A}_2, P_2)$ be arbitrary, let $\mathcal{C}_2 \subset \mathcal{A}_2$ be a sub σ-algebra and let $\mu \in \tilde{\mathcal{M}}(P_1, Q_2)$ where $Q_2 = P_2|_{\mathcal{C}_2}$. Let $\mathcal{D}_1 = \mathcal{A}_1 \otimes \mathcal{C}_2, \mu_1 = \mu, \mathcal{D}_2 = X_1 \otimes \mathcal{A}_2$ and $\mu_2(X_1 \times A_2) = P_2(A_2), A_2 \in \mathcal{A}_2$. μ_i are charges on $\mathcal{D}_i$ and if $C_i \in \mathcal{D}_i, i = 1, 2$ are such that if $C_1 \subset C_2 = X_1 \times A_2$ then, by the projection property, there exists $A_1 \in \mathcal{A}_1$ with $P_1(A_1) = 1$ such that $D_2 = \pi_2(C_1 \cap (A_1 \times X_2)) \in \overline{\mathcal{C}_2}^{Q_2}$. Let $C_{2*}, C_2^* \in \mathcal{C}_2$ be such that $C_{2*} \subset D_2 \subset C_2^*$ with $Q_2(C_{2*}) = Q_2(C_2^*)$. Then

$$
\begin{aligned}
\mu(C_1) &= \mu(C_1 \cap (A_1 \times X_2)) \\
&\leq \mu_*(X_1 \times D_2) \\
&\leq Q_2^*(D_2) \\
&= Q_2(C_2^*) \\
&= Q_2(C_{2*}) \\
&\leq P_2(A_2) \quad (\text{since } X_1 \times C_{2*} \subset X_1 \times A_2) \\
&= \mu_2(X_1 \times A_2)
\end{aligned}
$$

By a well known result of Guy(1961) there is a charge $\bar{\mu}$ on $alg(\{\mathcal{D}_1, \mathcal{D}_2\})$ extending μ_1 and μ_2. $\bar{\mu}$ can be extended as a charge to $\sigma(\{\mathcal{D}_1, \mathcal{D}_2\}) = \mathcal{A}_1 \otimes \mathcal{A}_2$. By construction, $\bar{\mu} \in \tilde{\mathcal{M}}(P_1, P_2)$. □

Surprisingly the charge extension property already implies the(σ-additive) extension property.

Proposition 6 *If $(X_1, \mathcal{A}_1, P_1)$ has the charge extension property it has the extension property.*

Proof:Suppose that $(X_1, \mathcal{A}_1, P_1)$ has the charge extension property. Let $(X_2, \mathcal{A}_2, P_2)$ be arbitrary, $\mathcal{C}_2 \subset \mathcal{A}_2$ be a sub σ-algebra and let $\mu \in \mathcal{M}(P_1, P_2|_{\mathcal{C}_2})$. Then $\mu \nearrow \tilde{\mu} \in \tilde{\mathcal{M}}(P_1, P_2)$. Let $(Y_2, \mathcal{D}_2, Q_2)$ be a perfect probability space such that $(X_2, \mathcal{A}_2, P_2) \subset (Y_2, \mathcal{D}_2, Q_2)$. Extend $\tilde{\mu}$ to $\tilde{\mu}_0 \in \tilde{\mathcal{M}}(P_1, Q_2)$ by defining $\tilde{\mu}_0(D) = \tilde{\mu}(D \cap (X_1 \times X_2)), D \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Since Q_2 is perfect, $\mu_1 = \tilde{\mu}_0|_{alg(\mathcal{R})}$ is σ-additive where $\mathcal{R}$ is the semialgebra of measurable rectangles in $\mathcal{A}_1 \otimes \mathcal{D}_2$ (see Marczewski and Ryll-Nardzewski (1953), Corollary 3.2.2 of Ramachandran (1979)). Let $\bar{\mu}_1$ be the unique extension of μ_1 as a measure to $\sigma(\mathcal{R}) = \mathcal{A}_1 \otimes \mathcal{D}_2$. Then $\bar{\mu}_1 \in \mathcal{M}(P_1, Q_2)$. Let $\overline{\mathcal{D}_2} = \sigma(\{\mathcal{D}_2, X_2\})$ and let $\overline{Q_2}(\overline{D_2}) = P_2(\overline{D_2} \cap X_2), \overline{D_2} \in \overline{\mathcal{D}_2}$. Again, by the charge extension property, $\bar{\mu}_1 \nearrow \tilde{\mu}_2 \in \tilde{\mathcal{M}}(P_1, \overline{Q_2})$. $\tilde{\mu}_2(X_1 \times X_2) = \overline{Q_2}(X_2) = P_2(X_2) = 1$. Also $D \in \mathcal{A}_1 \otimes \mathcal{D}_2, D \supset (X_1 \times X_2)$ implies $\bar{\mu}_1(D) = \tilde{\mu}_2(D) \geq \tilde{\mu}_2(X_1 \times X_2) = 1$. Hence $\bar{\mu}_1^*(X_1 \times X_2) = 1$. Thus $\nu = \bar{\mu}_1^*|_{\mathcal{A}_1 \otimes \mathcal{A}_2} \in \mathcal{M}(P_1, P_2)$ is an extension of $\mu \in \mathcal{M}(P_1, P_2|_{\mathcal{C}_2})$ and so $(X_1, \mathcal{A}_1, P_1)$ has the extension property. □

Combining the above propositions we obtain the following characterization theorem.

Theorem 2 (Equivalence Theorem) *Let $(X_1, \mathcal{A}_1, P_1)$ be a probability space. Then the following are equivalent:*

(a) $(X_1, \mathcal{A}_1, P_1)$ is a strong duality space
(b) $(X_1, \mathcal{A}_1, P_1)$ is perfect
(c) $(X_1, \mathcal{A}_1, P_1)$ has the extension property
(d) $(X_1, \mathcal{A}_1, P_1)$ has the projection property
(e) $(X_1, \mathcal{A}_1, P_1)$ has the charge extension property.

4 Consequences of the Equivalence Theorem and Comments

The equivalence of the notions of strong duality and perfectness enables us to obtain several new properties that strong duality spaces inherit from being perfect spaces (see Ramachandran (1979) for details). These properties are not easily established by direct arguments.

*D*1. $(X_1, \mathcal{A}_1, P_1)$ is a strong duality space if $(X_1, \mathcal{D}_1, P_1|_{\mathcal{D}_1})$ is a strong duality space for every c.g. sub σ-algebra $\mathcal{D}_1$ of $\mathcal{A}_1$.

*D*2. $(X_1, \mathcal{A}_1, P_1)$ is a strong duality space iff $(X_1, \overline{\mathcal{A}_1}, \overline{P_1})$, the completion of $(X_1, \mathcal{A}_1, P_1)$ is a strong duality space.

*D*3. $(X_1, \mathcal{A}_1, P_1)$ is a strong duality space whenever we can find $Y_1 \subset X_1$ with $P_1^*(Y_1) = 1$ such that $(Y_1, \mathcal{A}_1 \cap Y_1, P_1^*)$ is a strong duality space.

*D*4. Let $\{(X_i, \mathcal{A}_i), i \in I\}$ be a family of measurable spaces and let P be a probability on the product space $(\Pi_{i\in I} X_i, \otimes_{i\in I}\mathcal{A}_i)$. Then $(\Pi_{i\in I} X_i, \otimes_{i\in I}\mathcal{A}_i, P)$ is a strong duality space iff every marginal space $(X_i, \mathcal{A}_i, P \circ \pi_i)$ is a strong duality space.

We recast a useful result of Marczewski and Ryll-Nardzewski(1953) as

> *D*5. A charge on the semialgebra of measurable rectangles of the product of two measurable spaces with countably additive marginals is itself countably additive if at least one marginal space is a strong duality space.
>
> *D*6. Let $(X_1, \mathcal{A}_1, P_1)$ be a strong duality space. Then for every Q_1 on $(X_1, \mathcal{A}_1)$ with $Q_1 \ll P_1, (X_1, \mathcal{A}_1, Q_1)$ is a strong duality space.
>
> *D*7. Let $(X_1, \mathcal{A}_1)$ be a measurable space. Let $\{P_n\}_{n\geq 1}$ be a sequence of probabilities on $(X_1, \mathcal{A}_1)$ such that $(X_1, \mathcal{A}_1, P_n)$ is a strong duality space for every n. Let
>
> $$P = \sum_{n=1}^{\infty} \alpha_n P_n \quad ; \alpha_n \geq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = 1.$$
>
> Then $(X_1, \mathcal{A}_1, P)$ is a strong duality space.

Concerning a subset of the real line which becomes a duality space with respect to every probability on its Borel sets we have

> *D*8. Let X_1 be a subset of the real line and let $\mathcal{B}$ be the class of Borel subsets of $\mathbb{R}$. Then $(X_1, \mathcal{B} \cap X_1, P_1)$ is a strong duality space for every probability P_1 on $\mathcal{B} \cap X_1$ iff X_1 is universally measurable.

Indeed, in connection with *D*8, we have the following general property(see Theorem 2.3.2 of Ramachandran(1979)):

> *D*9. Let $(X_1, \mathcal{A}_1)$ be a measurable space. Then the following are equivalent:
>
> (a) For every sub σ-algebra $\mathcal{D}_1$ of $\mathcal{A}_1$, and for every probability Q_1 on $\mathcal{D}_1, (X_1, \mathcal{D}_1, Q_1)$ is a strong duality space.
>
> (b) For every c.g. sub σ-algebra $\mathcal{D}_1$ of $\mathcal{A}_1$ and for every probability Q_1 on $\mathcal{D}_1, (X_1, \mathcal{D}_1, Q_1)$ is a strong duality space.
>
> (c). The range of every real-valued $\mathcal{A}_1$-measurable function f on $(X_1, \mathcal{A}_1)$ is universally measurable.

Comments: 1. Using the construction in the proof of Proposition 2 it can be shown that $S-$stability ($S_{\mathcal{A}_1 \otimes \mathcal{C}_2}(h) = S_{\mathcal{A}_1 \otimes \mathcal{A}_2}(h)$ for all $h \in \mathcal{A}_1 \otimes \mathcal{C}_2, \mathcal{C}_2 \subset \mathcal{A}_2$) implies perfectness and therefore $S-$stability alone implies strong duality. The same is true for $I-$stability. Note that the same construction yields a concrete example where the (charge) extension property fails for a probability with given marginals (since $\mu \in \mathcal{M}(P_1, Q_2)$ constructed in the proof cannot be extended as a charge to $\tilde{\mathcal{M}}(P_1, P_2)$).

2. In the counterexample constructed in Section 4 of Ramachandran and Rüschendorf (1995), contrary to the claim therein, $I(h) = S(h) = 0$ which leaves open the question of the existence of a probability space which is not a duality space (see $Q1$ below). As a result, the proof of Proposition 1 in Ramachandran and Rüschendorf (1996) is incorrect. The main result of Ramachandran and Rüschendorf (1996) (Theorem 3) is proved now if we replace "duality space" by "strong duality space" in statement (c).

3. In view of the results in this paper the following questions are still open:

> $Q1$. Is there a probability space which is not a duality space?
> $Q2$. Is there a duality space which is not a strong duality space?

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