Approximation of optimal stopping problems

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Abstract

We consider optimal stopping of independent sequences. Assuming that the corresponding imbedded planar point processes converge to a Poisson process we introduce some additional conditions which allow to approximate the optimal stopping problem of the discrete time sequence by the optimal stopping of the limiting Poisson process. The optimal stopping of the involved Poisson processes is reduced to a differential equation for the critical curve which can be solved in several examples. We apply this method to obtain approximations for the stopping of iid sequences in the domain of max-stable laws with observation costs and with discount factors.

Keywords: optimal stopping, Poisson processes, max stable distributions, critical curve

1 Introduction

The aim of this paper is to approximate the optimal stopping problem for a sequence $X_1, \dots, X_n$ by an optimal stopping problem for a limiting Poisson process N under the assumption that for some normalization constants a_n, b_n the imbedded planar point process N_n converges in distribution to N

$$N_n = \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, \frac{X_i - b_n}{a_n}\right)} \xrightarrow{\mathcal{D}} N \tag{1.1}$$

More precisely our aim is to determine the asymptotic distribution of the optimal stopping value $\frac{X_{T_n}-b_n}{a_n}$, the asymptotic expected stopping value $v_n = E\frac{X_{T_n}-b_n}{a_n}$ of the stopping problem and to construct explicit asymptotically optimal stopping rules T_n' based on the corresponding optimal stopping problem for the limiting Poisson process. Point process convergence in (1.1) alone is not enough to approximate the stopping problem. So our task is to introduce additional assumptions which together with (1.1) imply convergence of the stopping problems.

Some related ideas can be found in the literature. An approximation of the optimal stopping of max-sequences by the optimal stopping of corresponding extremal processes has been observed in Flateau and Irle (1983). In this paper both problems are of monotone kind and could be solved explicitely. The approximative optimal stopping of the max sequence by that of the (continuous) limiting process is not derived from some general approximation argument but is proved directly. It has also been observed in some papers that an optimal stopping problem has an easier solution in a related form with a Poisson-number of points. Bruss and Rogers (1991) and Gnedin (1996) use this idea in the context of an optimal selection problem. A famous example of this kind is the house selling problem due to Karlin (1962), Elfving (1967), and Siegmund (1967) who consider optimal stopping of a Poisson process with finite intensity of the form $\lambda \otimes \nu$ on $[0, 1] \times \mathbb{R}^1$. They derive in this context a differential equation for the critical curve which allows to calculate some examples explicitly.

In this paper we concentrate on the optimal stopping of independent sequences. The main source and starting point of this work are several papers of Kennedy and Kertz (1990, 1991, 1992) who determined the asymptotics of the optimal stopping of iid sequences directly. They also used point process convergence to derive asymptotics of several functionals of the optimally stopped sequences for the case of iid sequences in the domain of attraction (for maxima) of max stable distributions. In particular they proved convergence of the optimal stopping value and convergence of the normalized optimal stopping times to certain threshold stopping times in the limiting Poisson process. Our approach will allow to derive approximate optimality also in cases which can't be handled in a direct way.

In section two we state a characterization of optimal stopping times for a Poisson process by a differential equation for the critical curve. For our application to stopping problems for sequences we need to consider general Poisson processes with possibly infinite intensities of general form. As consequence one cannot order the points and reduce this problem to the optimal stopping of stationary discrete sequences directly as is done in the above mentioned papers to the house selling problem.

In section three we state an approximation result for the optimal stopping of independent sequences by the optimal stopping of the limiting Poisson process. We discuss an application to the optimal stopping of iid sequences with observation costs resp. discount factors in the domain of attraction of max-stable distributions in detail in section 4. This extends results of Kennedy and Kertz (1991) for the iid case without observation costs or discounts. For several details we refer to the dissertation of R. Kühne (1997) on which this paper is based.

2 Optimal stopping of Poisson processes

Optimal stopping of a Poisson process has been considered in Karlin (1962), Elfving (1967), Siegmund (1967), and Chow Robbins and Siegmund (1971), in the case where the intensity measure is finite and of product form $\lambda_{[0,\infty)} \otimes \nu$. Their motivating example was the house selling problem with a random number of iid offers Y_i at random times τ_i, $\tau_1 < \tau_2 < \ldots, (Y_i)$ iid. The value of the house at the time of the

n-th offer is $X_n = Y_n\, r(\tau_n)$ where r is a nonincreasing nonnegative discount function. By making use of the continuous time aspects of the problem a differential equation has been derived for the boundary of an optimal stopping region which can be solved in some cases explicitely.

The finiteness and product form of the intensity is used essentially in the derivation of the result by reducing the question of optimality to the discrete stationary Markov case.

In this section we consider the optimal stopping of Poisson processes for more general intensities allowing in particular infinitely many points. Therefore, it is not possible to arrange the points in increasing order $\tau_1 < \tau_2 < \cdots$ and to reduce the problem to the discrete case directly. This more general situation will be typical for applications to point processes which arise as the limit of point processes of normalized independent variables.

We consider two dimensional point processes on $[0,1] \times \mathbb{R}^1$ of the form $N = \sum_k \varepsilon_{(\tau_k, Y_k)}$ where the sum may be finite or countable infinite.

Definition 2.1 *Let $N = \sum_k \varepsilon_{(\tau_k, Y_k)}$ be a point process on $[0,1] \times \mathbb{R}^1$. A mapping $\tau : \Omega \to [0,1]$ is called (canonical) stopping time if:*

a) For P almost all $\omega \in \{\tau < 1\} : \exists k \in \mathbb{N}$ such that $\tau(\omega) = \tau_k(\omega)$.

b) $\{\tau \le t\} \in \sigma(N_s, s \le t) = \sigma(\{(\tau_k, Y_k); \tau_k \le t\})$ for $0 \le t \le 1$.

So canonical stopping times stop either at the points τ_k or in 1. For the optimal stopping problem we introduce the gain $\overline{Y}_\tau = \overline{Y}_{\tau,c} = \begin{cases} \sup\{Y_k, \tau_k = \tau\} \\ c \text{ if } \tau = 1 \text{ and } \tau \neq \tau_k, \forall k \end{cases}$. Here c is a guaranteed gain (which might be $-\infty$) in case of not stopping at all in $[0,1]$.

Definition 2.2 *Let $N = \sum \varepsilon_{(\tau_k, Y_k)}$ be a point process. A stopping time τ_0 is optimal if*

$$E\overline{Y}_{\tau_0} = \sup_\tau E\overline{Y}_\tau =: V. \tag{2.1}$$

Remark 2.3 In general there will be more than one point at a stopping time. In this case we would like to choose the maximum of these points. In the further part of this paper we usually will assume continuity conditions which imply that no multiple points arise in the point processes considered.

For general point processes it is not clear that the formulation of the optimal stopping problem as in Definitions 2.1 or 2.2 is suitable. Problems arise from accumulation points of the point measures. In this paper we consider point processes with accumulation points only at the lower boundary. For this class of processes the definition of stopping times is flexible enough to consider suitable threshold stopping times.

In the definition we assume that the expectation is welldefined. Finiteness of the value V will be a consequence of the boundedness assumption

(B) *Boundedness condition*

$$EM < \infty, \text{where } M := \sup_k Y_k$$

For functions $v : [0,1) \to \mathbb{R}^1$ we define

$$\tau = \tau^v = \inf\{\tau_k;\ Y_k \geq v(\tau_k)\}, \quad \inf \emptyset = 1. \tag{2.2}$$

For curves v such that there are only finitely many points above v τ^v defines a stopping time, the threshold stopping time associated with v. The independence properties of Poisson processes suggest the threshold stopping rule τ^u associated with the 'optimal' stopping curve

$$u(t) = \sup\{E\overline{Y}_\tau;\ \tau \text{ a stopping time} \geq t\}, \quad \overline{Y}_\tau = \overline{Y}_{\tau,c} \tag{2.3}$$

for some guarantee value c.

In the following we consider Poisson processes whose intensity is concentrated on

$$M_f := \{(t,y) \in [0,1] \times \overline{\mathbb{R}};\ y > f(t)\} \tag{2.4}$$

for some function $f : [0,1] \to \mathbb{R} \cup \{-\infty\}$ monotonically nonincreasing on $\{f > -\infty\}$. We allow clustering of the points of N at the lower boundary f. Formally, we consider on $S = M_f$ the topology which is induced on M_f by the usual topology on $[0,1] \times \overline{\mathbb{R}}$. We assume that the intensity measure μ is a Radon measure on M_f. This assumption implies that for any function $v > f$ separated from f there are only finitely many points in M_v. We generally assume $c \geq f(1)$.

Point process convergence on a metric space S is defined in the usual sense and $N_n \xrightarrow{\mathcal{D}} N_0$ if for all $g \in C_k^+(S)$ holds $EN_n(g) \to EN_0(g)$. We consider throughout point process convergence to a point process N_0 supported by M_f; the convergence takes place either in $S = [0,1] \times \left(\overline{\mathbb{R}} \setminus \text{graph}(f)\right) =: \overline{M}_f$ or in $S = M_f$ supplied in each case with the relative topology. Convergence in $\overline{M}_f$ implies convergence of the restrictions to M_f. For our main theorem convergence in M_f together with an additional condition (condition (L)) will be sufficient to imply approximation of the stopping problem. For the relevant facts on point process convergence we refer to Resnick (1987, chapter 3). Applying the Skorohod theorem to point process convergence $N_n \xrightarrow{\mathcal{D}} N_0$ we obtain versions which converge a.s. in the vague topology. Therefore, a wellknown result (see Prop. 3.13 in Resnick (1987)) implies that for each ω and any compact set K which may depend on ω with $N_0(\omega, \partial K) = 0$ there is a labeling of the points of $N_n(\omega, \cdot \cap K)$ for $n \geq n(\omega, K)$ such that the relabeled points converge pointwise to the points of $N_0(\omega, \cdot \cap K)$. In this sense we may assume a.s. convergence of the points on compact sets.

In order to describe the optimal stopping curve by a differential equation we introduce

(D) *Differentiability condition*
${}^{\mu}/_{M_f}$ is continuous w.r.t. Lebesgue-measure $\lambda_f = {}^{\lambda^2}/_{M_f}$ such that with h_f a version of the density $\frac{d\mu/M_f}{d\lambda_f}$

$$(t,z) \to \int_z^\infty \int_x^\infty h_f(t,y)\,\mathrm{d}y\,\mathrm{d}x \tag{2.5}$$

is continuous on M_f.

In particular

$$z \to \int_z^\infty \left(\int_x^\infty h_f(t,y)\,\mathrm{d}y \right)\,\mathrm{d}x \tag{2.6}$$

is continuously differentiable for any t.

Under the differentiability condition no multiple time points arise. Therefore, there is a uniquely determined stopping index $k = K^\tau(\omega)$ for any stopping time and $\omega = \{\tau < 1\}$, such that

$$E\overline{Y}_\tau = EY_{K^\tau}, \tag{2.7}$$

here $Y_{K^\tau} := c$ if $\tau = 1$ and $\tau \neq \tau_k$ for all k.

For technical reasons we need that the distance of the optimal stopping curve u to the lower boundary is bounded away from zero on intervals $[0,t]$ for $t < 1$. We introduce the following

(S) *Separation condition*
Let v be a monotonically nonincreasing function on $[0,1]$. v satisfies the separation condition (w.r.t. N) if for all $t < \omega_1 := \inf\{t \leq 1;\ \mu([t,1] \times (c,\infty)) = 0\}$ there exists a constant $c_t > 0$ such that

$$(v - f)/[0,t] > c_t. \tag{2.8}$$

(We define $(-\infty) - (-\infty) := 0$.)

This condition will be obviously fulfilled in the case when f is constant and $v > f$.

The following proposition concerns convergence of threshold stopping times under the assumption that $N_n \xrightarrow{\mathcal{D}} N$ a Poisson point process which satisfies the differentiability assumption. Let $N_n = \sum_{i=1}^n \varepsilon_{(\frac{i}{n}, X_{n,i})}$, $N = \sum_i \varepsilon_{(\tau_i, Y_i)}$ and note that N has a.s. no point on the line $\{1\} \times \overline{\mathbb{R}}$. From the convergence in distribution we conclude that $\overline{\lim} X_{n,n} \leq f(1)$ which implies by the boundedness assumption for the optimal stopping boundary u_n that $\limsup u_n(1) \leq f(1)$. To obtain convergence of threshold stopping times at time point $t = 1$ we set the guarantee value $c \geq f(1)$ and so $Y_{K^\tau} = c \geq f(1)$ on $\{\tau = 1\}$.

Proposition 2.4 *Let $(X_{n,i})_{1\le i\le n}$ be real random variables, $n \in \mathbb{N}$, such that*

$$N_n := \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, X_{n,i}\right)} \xrightarrow{\mathcal{D}} N = \sum_i \varepsilon_{(\tau_i, Y_i)} \text{ on } M_f. \tag{2.9}$$

Let N satisfy (D) *and let $v_n, v : [0,1] \to \mathbb{R}$ be monotonically nonincreasing functions, such that $v_n \to v$ pointwise, v a continuous function fulfilling* (S). *Let*

$$\begin{aligned} T_n &:= n\tau^{v_n} = \inf\left\{1 \le i \le n;\ X_{n,i} \ge v_n\left(\tfrac{i}{n}\right)\right\} && \text{where } \inf \emptyset := n \\ T &:= \tau^v = \inf\{\tau_i;\ Y_i \ge v(\tau_i)\} && \text{where } \inf \emptyset := 1. \end{aligned}$$

Then

$$\left(\frac{T_n}{n}, X_{n,T_n}\right) \xrightarrow{\mathcal{D}} (T, Y_{K^T}). \tag{2.10}$$

If the point process convergence is a.s., then also the convergence in (2.10) *is a.s.*

Proof: Define $X'_{n,i} := X_{n,i} - v_n\left(\frac{i}{n}\right) + v\left(\frac{i}{n}\right)$ and $N'_n := \sum_{i=1}^n \varepsilon_{\left(\frac{i}{n}, X'_{n,i}\right)}$. N'_n is no longer supported by M_f. Let $\overline{N}_n$ denote the restriction of N'_n on M_f. Since $v_n - v \to 0$, v_n are monotonically nonincreasing and v is continuous it follows by (S) that $v_n - v \to 0$ uniformly on $[0,t], t < 1$. Therefore, also $\overline{N}_n \xrightarrow{\mathcal{D}} N$ on M_f and w.l.g. let $\overline{N}_n = \sum_{i=1}^n \varepsilon_{\left(\frac{i}{n}, X'_{n,i}\right)}$. Otherwise, replace for $X'_{n,i} \le f\left(\frac{i}{n}\right)$ $X'_{n,i}$ by $\overline{X}_{n,i} = f\left(\frac{i}{n}\right) + \frac{1}{2n}\left(v\left(\frac{i}{n}\right) - f\left(\frac{i}{n}\right)\right)$. Then, using Skorohod's theorem w.l.g. $\overline{N}_n \to N$ a.s. and by definition $T_n = \inf\left\{1 \le i \le n;\ X'_{n,i} \ge v\left(\frac{i}{n}\right)\right\}$ noting that $v\left(\frac{i}{n}\right) > f\left(\frac{i}{n}\right)$ and so points $X'_{n,i} \le f\left(\frac{i}{n}\right)$ do not cross the boundary $v\left(\frac{i}{n}\right)$.

Define $N''_n := \overline{N}_n(\cdot \cap \overline{M}_v) = \sum_{X'_{n,i} \ge v\left(\frac{i}{n}\right)} \varepsilon_{\left(\frac{i}{n}, X'_{n,i}\right)}$. With the labeling $N''_n = \sum_{i=1}^{m_n} \varepsilon_{\left(\frac{k_i^n}{n}, X'_{n,k_i^n}\right)}$, $k_1^n < k_2^n < \cdots$ it follows that $T_n = k_1^n$. Since μ is Lebesgue continuous on $M_f \supset M_v$ it follows that $\mu(\partial M_v) = 0$ and $N''_n = \overline{N}_n(\cdot \cap \overline{M}_v) \to N(\cdot \cap \overline{M}_v) =: \overline{N} = \sum \varepsilon_{(\tau'_i, Y'_i)}$. The separation condition (S) implies compactness of $\overline{M}_v$ in $[0,t] \times \overline{\mathbb{R}}$ and, therefore, $\mu(([0,t] \times \overline{\mathbb{R}}) \cap \overline{M}_v) < \infty$, $\forall t < 1$. This implies that $\overline{N}$ has a.s. finitely many points on $[0,t] \times \mathbb{R}$ which we rearrange w.l.g. as $T = \tau'_1 < \tau'_2 < \cdots$. For $\omega \in \Omega$ with $\tau'_1(\omega) < 1$ there exists $t = t(\omega) \in (\tau'_1(\omega), 1)$ such that $\tau'_i \ne t(\omega), i \in \mathbb{N}$. The set $M^t := \overline{M}_v \cap ([0, t(\omega)] \times \mathbb{R})$ is compact. Since convergence of pointmeasures implies convergence of the points in M^t after relabeling we conclude $\left(\frac{T_n}{n}, X'_{n,T_n}\right) = \left(\frac{k_1^n}{n}, X'_{n,k_1^n}\right)(\omega) \to (\tau'_1, Y'_1)(\omega) = (T, Y_{K^T})(\omega)$ and X'_{n,T_n} can be replaced by X_{n,T_n} to yield the same convergence. $\square$

For a threshold stopping time $\tau = \tau^v$ where $v : [0,1] \to \mathbb{R} \cup \{-\infty\}$ is monotonically nonincreasing define

$$\tau^{\ge t} = \inf\{\tau_k;\ Y_k \ge v(\tau_k),\ \tau_k \ge t\}. \tag{2.11}$$

Optimality of a stopping curve v will be related to the equation

$$EY_{K^{\tau^{\ge t}}} = v(t),\ t < 1. \tag{2.12}$$

In the following theorem we consider the optimal stopping for a Poisson process on M_f with guarantee value $c = f(1)$. The case $c > f(1)$ can be reduced to this case by restricting the point process to $M_{f \vee c}$.

Theorem 2.5 (Optimal stopping of Poisson processes)
Let N be a Poisson process, fulfilling the boundedness assumption (B) *and the differentiability condition (D).*

a) *Under the separation condition* (S) *for the optimal stopping curve u*

$$T := \tau^u := \inf\{\tau_i;\ \ Y_i \geq u(\tau_i)\}, \quad \inf \emptyset = 1 \tag{2.13}$$

is an optimal stopping time for N. Any optimal stopping time is a.s. identical to T.

b) *Under condition* (S) *u solves the differential equation*

$$\begin{aligned} u'(t) &= -\int_{u(t)}^{\infty}\int_{x}^{\infty} h_f(t,y)\,\mathrm{d}y\,\mathrm{d}x, \quad 0 \leq t < 1 \\ u(1) &= c = f(1). \end{aligned} \tag{2.14}$$

If $c > -\infty$, then (2.14) *has a unique solution.*

c) *Assume $c > -\infty$ and let a monotonically nonincreasing function v satisfy* (S) *and solve the differential equation* (2.14) *then v is the optimal stopping curve of N (i.e. $T = \tau^v$ is optimal).*

d) *Let $v : [0,1] \to \mathbb{R} \cup \{-\infty\}$ satisfy* (S) *and solve equation* (2.12) *for $t \leq 1$ then v solves the differential equation* (2.14).

e) *Let $c = -\infty, f \equiv -\infty$. If the differential equation* (2.14) *has a unique solution and $u(t) > -\infty$ for all $t < 1$, then u is a solution of* (2.14).

Proof:

a) To prove optimality of T we reduce the stopping problem of N to discrete time stopping problems. Let $M_{s,t} := \sup_{s<\tau_k \leq t} Y_k$ where for $s = 0$ we define $M_{0,t} = \sup_{0 \leq \tau_k \leq t} Y_k$ and

$$\begin{aligned} Z_{n,i} &:= M_{\frac{i-1}{2^n},\frac{i}{2^n}}, \qquad 1 \leq i \leq 2^n, n \in \mathbb{N} \\ \mathcal{G}_n &:= (\mathcal{G}_{n,k})_{1 \leq k \leq 2^n}, \quad \mathcal{G}_{n,k} := \sigma\Big((\tau_i, Y_i)1_{\{\tau_i \leq \frac{k}{2^n}\}}, i \in \mathbb{N}\Big). \end{aligned} \tag{2.15}$$

The sup over the empty set is defined as $-\infty$.

Claim 1 *For all stopping times τ for N there exists a $\mathcal{G}_n$ stopping time τ' such that*

1) $Z_{n,\tau'} \geq Y_{K^\tau}$ *a.s.*

2) If $\tau \geq \frac{i}{2^n}$ *a.s., then* $\tau' \geq i$ *a.s.*

Proof of claim 1: Let $h_n(x) := \frac{1}{2^n} \inf\{i \in \mathbb{N};\ \frac{i}{2^n} \geq x\} = \frac{\lceil 2^n x \rceil}{2^n}$, $\tau^m := \inf\Big\{\tau_i \geq \frac{i}{2^n};\ Y_i = M_{h_n(\tau)-\frac{1}{2^n},\, h_n(\tau)}\Big\}$, then with $\tau' := 2^n h_n(\tau^m)$ holds $Y_{K^\tau} \leq Y_{K^{\tau^m}} = Z_{n,2^n h_n(\tau^m)} = Z_{n,\tau'}$. It can be checked easily that τ' is a stopping time w.r.t. the filtration $\mathcal{G}_n$. Furthermore, $\tau \geq \frac{i}{2^n}$ implies $\frac{i}{2^n} \leq \tau^m \leq h_n(\tau^m) = \tau'$. This finishes the proof of claim one.

The σ-algebras $\sigma\Big((\tau_i, Y_i)1_{\left(\frac{k-1}{2^n} < \tau_i \leq \frac{k}{2^n}\right)}, i \in \mathbb{N}\Big)_{1 \leq k \leq 2^n}$ (where for $k = 1$ $\tau_i = 0$ is to be included), are independent. Therefore, we may consider the stopping problem for $Z_{n,1}, \ldots, Z_{n,2^n}$ w.r.t. the canonical filtration $\mathcal{H}_n$ as stopping problem of independent sequences.

Let $w_{n,i} := V(Z_{n,i}, \ldots, Z_{n,2^n})$ be the value of the stopping problem of $Z_{n,1}, \ldots$, $Z_{n,2^n}$, $w_n(t) := w_{n,[2^n t]}$, $T_n := \inf\{i \leq 2^n; Z_{n,i} \geq w_{n,i+1}\}$, $\inf \emptyset = n$.

Claim 2 *There exists a function* $w \geq u$ *such that*

$$w_n \to w. \tag{2.16}$$

Proof of claim 2: By claim one we have

$$\begin{aligned}
w_n(t) &= w_{n,[2^n t]} = V(Z_{n,[2^n t]}, \ldots, Z_{n,2^n}) \\
&\geq \sup\{EZ_{n,\tau'};\ \tau' \geq [t2^n], \tau' \text{ a } \mathcal{G}_n \text{ stopping time}\} \\
&\geq \sup\left\{EY_{K^\tau};\ \tau \text{ a stopping time for } N, \tau \geq \frac{[t2^n]}{2^n}\right\} = u\left(\frac{[t2^n]}{2^n}\right) \\
&\geq u(t) \quad \text{ since } \frac{[t2^n]}{2^n} \leq t.
\end{aligned}$$

Furthermore, (w_n) is monotonically nonincreasing in n. Consider the filtration $\mathcal{H}_n^k$ defined by

$$\mathcal{H}_{n,i}^k = \begin{cases} \sigma(Z_{n+1,1}, \ldots, Z_{n+1,i+1}) & \text{if } i \text{ is odd} \\ \sigma(Z_{n+1,1}, \ldots, Z_{n+1,i}) & \text{if } i \text{ is even, } k \leq i \leq 2^{n+1}, \quad \text{then} \end{cases}$$

$$\begin{aligned}
w_{n+1}(t) &\leq w_{n+1}\left(\frac{[2^n t]}{2^n}\right) = V(Z_{n+1,2[2^n t]}, \ldots, Z_{n+1,2^{n+1}}) \\
&\leq V_{\mathcal{H}_n^{[2^n t]}}(Z_{n+1,2[2^n t]}, \ldots Z_{n+1,2^{n+1}}) \\
&= V(Z_{n+1,2[2^n t]} \vee Z_{n+1,2[2^n t]+1}, \ldots, Z_{n+1,2^{n+1}-1} \vee Z_{n+1,2^{n+1}})
\end{aligned}$$

since using the filtration $\mathcal{H}^k_{n,i}$ it is possible at odd time points to foresee the next random variable. So optimal stopping times stop at the maximum of these point pairs. Since $Z_{n+1,[2^n t]} \vee Z_{n+1,[2^n t]+1} = Z_{n,[2^n t]}, \dots$ we obtain $w_{n+1}(t) \leq V(Z_{n,[2^n t]}, \dots, Z_{n,2^n}) = w_n(t)$. This implies claim 2.

Next observe that

$$\sum_{i=1}^{2^n} \varepsilon_{\left(\frac{i}{2^n}, Z_{n,i}\right)} \xrightarrow{\mathcal{D}} N.$$

N has only finitely many points in compact subsets of M_f. Therefore, it is enough to prove convergence on subsets $[a,b] \times [d,\infty) \subset M_f$ with $0 \leq a < b \leq 1$, $d \in \mathbb{R}^1$ and the above convergence is checked elementary.

Now by Proposition 2.4 we obtain

$$\left(\frac{T_n}{n}, Z_{n,T_n}\right) \to \left(\widetilde{T}, Y_{K^{\widetilde{T}}}\right), \tag{2.17}$$

where $\widetilde{T} = \tau^w = \inf\{\tau_i;\ Y_i \geq w(\tau_i)\}$. Note that $w \geq u$ satisfies the separation condition (S) and so Proposition 2.4 applies. By the boundedness assumption (B) it follows from Fatou's lemma that

$$\limsup_{n\to\infty} EZ_{n,T_n} \leq EY_{K^{\widetilde{T}}} \leq u(0).$$

On the other hand $EZ_{n,T_n} = w_n(0) \geq u(0)$. Therefore, $\lim w_n(0) = \lim\limits_{n\to\infty} EZ_{n,T_n} = u(0) = EY_{K^{\widetilde{T}}}$. Similarly, by considering the stopping problem of N restricted to the interval $[t,1]$ we obtain $w_n(t) \to u(t), \forall t < 1$. The separation condition (S) implies that $Z_{n,n} \xrightarrow{\mathcal{D}} f(1)$. Restricting the point processes N to M_f and using condition (B) we obtain $w_n(1) = EZ_{n,n} \to f(1) = c = u(1)$. This implies $w = u$ and $\widetilde{T} = T$ is optimal.

To prove uniqueness of the optimal stopping time we first state that any optimal stopping time does not use points below u. Suppose T_1 is optimal and for some $j \in \mathbb{N}$

$$P(\tau_j = T_1, Y_j < u(\tau_j)) > 0.$$

Then define

$$T_1^* := T^* := \inf\{\tau_i \geq T_1;\ Y_i \geq u(\tau_i)\} \text{ on } \bigcup_j \{\tau_j = T_1, Y_j < u(\tau_j)\}$$

and $T_1^* = T_1$ else. Conditionally by the strong markov property under $T_1 = t$ $N^{\geq T_1} = \sum_{\tau_i \geq T_1} \varepsilon_{(\tau_i, Y_i)}$ is a Poisson process with intensity $\mu(\cdot \cap [t,1] \times \mathbb{R})$ with optimal stopping value $u(t) = E(Y_{K^{T^*}} | T_1 = t))$.

Then

$$EY_{K^{T_1}} = EY_{K^{T_1}} 1_{\{T_1 = T_1^*\}} + \sum_j EY_{K^{T_1}} 1_{\{T_1 = \tau_j, Y_j < u(\tau_j)\}}. \tag{2.18}$$

On $\{T_1 = \tau_j, Y_j < u(\tau_j)\}$ we have conditionally under $T_1 = t$

$$u(t) = E(Y_{K^{T^*}} | T_1 = t) > Y_{K^{T_1}}, \tag{2.19}$$

and, therefore,

$$\begin{aligned} EY_{K^{T_1}} 1_{\{T_1=\tau_j, Y_j<u(\tau_j)\}} &= \int E(Y_{K^{T_1}} 1_{\{T_1=\tau_j, Y_j<u(\tau_j)\}} | T_1 = t)\, \mathrm{d}P^{T_1}(t) \\ &< \int E(u(t) 1_{\{T_1=\tau_j, Y_j<u(\tau_j)\}} | T_1 = t)\, \mathrm{d}P^{T_1}(t) \\ &= EY_{K^{T_1^*}} 1_{\{T_1=\tau_j, Y_j<u(\tau_j)\}} \end{aligned}$$

This implies $EY_{K^{T_1}} < EY_{K^{T_1^*}}$.

Similarly, any stopping time T_1 can be improved on $\{T_1 > T\}$ by replacing it on this set by T and arguing as above.

b) Assume that $P(T < 1) = 1$ and define $N_u := N(\cdot \cap \overline{M}_u)$, $\mu_u := \mu(\cdot \cap \overline{M}_u)$. Let $N^1 = \sum_k \varepsilon_{\tau'_k}$ be a Poisson process on $[0,1]$ with intensity $\mu^1 := \mu_u^{\pi_1}$, where $\pi^1(s,z) = s$ is the first projection, which is well defined since μ_u is a finite measure. Let $\{Y_i\}$ be random variables conditional independent given N^1 with distribution $P(Y_1 \in \cdot | N^1) = K(\cdot, \tau'_i)$ where $K([x,\infty),t) := \frac{\int_{x \vee u(t)}^{\infty} \frac{d\mu}{d\lambda^2}(t,y)\,\mathrm{d}y}{\int_{u(t)}^{\infty} \frac{d\mu}{d\lambda^2}(t,y)\,\mathrm{d}y}$ if the denominator is $\neq 0$ and identical $\varepsilon_{\{0\}}$ else. Then $N^2 := \sum_k \varepsilon_{(\tau'_k, Y_k)} \overset{d}{=} N_n$ so we use w.l.g. N^2 for our calculations and assume w.l.g. $\tau'_1 < \tau'_2 < \cdots$. Let for fixed $t' \in [0,1)$ $\mathcal{T} = \mathcal{T}^{\geq t'} := \inf\{\tau'_i \geq t';\ Y_i \geq u(\tau'_i)\}$. Then for $P^{\mathcal{T}}$ a.a. t holds

$$\begin{aligned} E(Y_{K^{\mathcal{T}}} | \mathcal{T} = t) &= \int_{u(t)}^{\infty} x\, \mathrm{d}P^{Y_{K^{\mathcal{T}}} | \mathcal{T} = t}(x) \\ &= u(t) + \int_{u(t)}^{\infty} P(Y_{K^{\mathcal{T}}} \geq x | \mathcal{T} = t)\, \mathrm{d}x \\ &= u(t) + \int_{u(t)}^{\infty} K([x,\infty),t)\, \mathrm{d}x. \end{aligned}$$

From $P(\mathcal{T} \geq t) = e^{-\mu_u([t',t]\times\mathbb{R})} = e^{-\mu^1([t',t])}$ we obtain $P^{\mathcal{T}}(dt) = \frac{d\mu^1}{d\lambda}(t)e^{-\mu^1([t',t])}$ and, therefore,

$$\begin{aligned}
u(t') &= \int Y_{K^{\mathcal{T}}}\,\mathrm{d}P = \int_{t'}^{1} E(Y_{K^{\mathcal{T}}}|\mathcal{T}=t)\,\mathrm{d}P^{\mathcal{T}}(t) \\
&= \int_{t'}^{1}\left(u(t) + \int_{u(t)}^{\infty} K([y,\infty),t)\,\mathrm{d}y\right)\frac{d\mu^1}{d\lambda}(t)e^{-\mu^1([t',t])}\,\mathrm{d}t \\
&= \int_{t'}^{1}\left(u(t) + \int_{u(t)}^{\infty} \frac{\int_y^\infty \frac{d\mu}{d\lambda^2}(t,z)\,\mathrm{d}z}{\int_{u(t)}^\infty \frac{d\mu}{d\lambda^2}(t,z)\,\mathrm{d}z} 1_{\left\{\int_{u(t)}^\infty \frac{d\mu}{d\lambda^2}(t,z)\,\mathrm{d}z \neq 0\right\}}\,\mathrm{d}y\right)\frac{d\mu^1}{d\lambda}(t)e^{-\mu^1([t',t])}\,\mathrm{d}t \\
&= \int_{t'}^{1}\left(u(t)\frac{d\mu^1}{d\lambda}(t) + \int_{u(t)}^{\infty}\int_y^\infty \frac{d\mu}{d\lambda^2}(t,z)\,\mathrm{d}z\,\mathrm{d}y\right)e^{-\mu^1([t',t])}\,\mathrm{d}t.
\end{aligned}$$

This implies differentiability of u and the argument of the last integral is differentiable in t' and continuous in t. From the rule

$$\frac{d}{dt}\int_t^1 f_t(x)\,\mathrm{d}x = \int_t^1 \frac{d}{dt}f_t(x)\,\mathrm{d}x - f_t(t)$$

valid for $f_t(x)$ differentiable in t, continuous in x we obtain

$$\begin{aligned}
u'(t') &= \int_{t'}^{1}\left(u(t)\frac{d\mu^1}{d\lambda}(t) + \int_{u(t)}^{\infty}\int_y^\infty \frac{d\mu}{d\lambda^2}(t,z)\,\mathrm{d}z\,\mathrm{d}y\right)e^{-\mu^1([t',t])}\,\mathrm{d}t\frac{d\mu^1}{d\lambda}(t') \\
&\quad - \left(u(t')\frac{d\mu^1}{d\lambda}(t') + \int_{u(t')}^{\infty}\int_y^\infty \frac{d\mu}{d\lambda^2}(t',z)\,\mathrm{d}z\,\mathrm{d}y\right) \\
&= -\int_{u(t')}^{\infty}\int_y^\infty \frac{d\mu}{d\lambda^2}(t',z)\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}$$

To prove uniqueness of a solution of (2.14) for $c > -\infty$ assume that $v_1 \neq v_2$ are solutions of (2.14). Since $h(t,x) := -\int_x^\infty\int_y^\infty \frac{d\mu}{d\lambda^2}(t,z)\,\mathrm{d}z\,\mathrm{d}y$ is continuous on M_f and differentiable in x, the differential equation $v'(t) = h(t,v(t))$ with initial values $v(z) = c_0$ for some $z \in (0,1)$ has a unique solution. This implies $v_1(t) \neq v_2(t)$ for all $t \in [0,1)$ and from continuity we assume w.l.g. that $v_1 > v_2$ on $[0,1)$. Therefore, we conclude from (2.14) that $v_1'(t) \geq v_2'(t)$ $\forall t \in [0,1)$ which implies $v_1(1) > v_2(1)$ a contradiction to $v_1(1) = v_2(1) = c$. This proves b).

In the derivation of the differential equation (2.14) in a) we in fact used only equation (2.12). Therefore, d) is true.

c) In b) we proved that u satisfies (2.14) if it satisfies (S) and if $c > -\infty$ then (2.14) has a unique solution. For the proof of c) we modify the measure μ to a measure μ_0 in such a way, that the separation condition is valid for μ_0 and the stopping curve is the same. Define μ_0 on $M_{-\infty}$ by $\mu_0/M_u = \mu/M_u$ and for $-\infty < x < u(t)$, $\frac{d\mu_0}{d\lambda\!\!\!\lambda^2}(t,x) = h_f(t,u(t))$, where $h_f = \frac{d\mu}{d\lambda\!\!\!\lambda_f}$ is a version of the density. μ_0 satisfies (D). Let $N_0 = \sum \varepsilon_{(\tau_i^0, Y_i^0)}$ be a Poisson process with intensity μ_0, optimal stopping curve u_0 and optimal stopping time T_0. Then $u_0(t) > c, \forall t < \omega_1$, u_0 satisfies the separation condition for $t < 1$. Therefore, by a), u_0 is the unique solution of the differential equation (2.14) and, therefore, $u_0 = v$ and $EY^0_{K^{T_0}} = u_0(t), \forall t \leq 1$. Since $N/M_u \overset{d}{=} N_0/M_u$ we have $EY_{K^{\tau^u}} = u(t), t \leq 1$. Therefore, the optimal stopping curve u_1 of N satisfies $u_1 \geq u$ and thus satisfies (S). Again from a) we conclude $u_1 = u$.

Finally e) is proved similarly to d). □

The uniqueness of solutions of (2.14) holds in the case of a differential equation in separate variables.

Proposition 2.6 *Let $f : [0,1] \to \mathbb{R}^1, g : \mathcal{I} \to \mathbb{R}^1$ continuous functions, $\mathcal{I} \subset \mathbb{R}^1$ an open interval. Assume that $g(y) \neq 0, \forall y \in \mathcal{I}$ and for some $y_0 \in \overline{\mathcal{I}}$, $G(y) := \int\limits_{y_0}^{y} \frac{1}{g(s)}\,\mathrm{d}s$ exists and $F([0,1]) \subset \overline{\mathcal{I}}$ where $F(x) = \int\limits_{1}^{x} f(t)\,\mathrm{d}t$. Then the differential equation*

$$\begin{aligned} \varphi'(t) &= f(t)g(\phi(t)), \quad t \in [0,1) \\ \varphi(1) &= y_0 \end{aligned} \tag{2.20}$$

has a unique continuous solution and

$$G(\varphi(t)) = F(t). \tag{2.21}$$

Proof: The case $y_0 > -\infty$ can be found in text books. In the case $y_0 = -\infty$ we have to assume existence of G. □

3 Approximate optimal stopping of independent sequences

Let $(X_{n,i})_{1\leq i\leq n}$ be independent sequences for $n \in \mathbb{N}$ with associated planar point processes $N_n = \sum_{i=1}^{n} \varepsilon_{(\frac{i}{n}, X_{n,i})}$ converging to some Poisson processes N on M_f with intensity μ, optimal stopping curve u and optimal stopping time $T = \inf\{\tau_i;\ Y_i \geq u(\tau_i)\}$ fulfilling (S). In general it is not possible without further conditions to approximate the optimal stopping behaviour of N_n by that of N.

Example 3.1 Let (X_i) be independent, $P(X_i \geq x) = e^{-x}, x \geq e^{-i}, P(X_i = a_i) = 1 - e^{-e^{-i}}$ where a_i are choosen such that $EX_1 = 0, EX_2 = a_1, EX_3 = a_2, \dots$. Consider (Y_i) iid exp(1)-distributed, then $N_n = \sum_{i>1}^{n} \varepsilon_{(\frac{i}{n}, X_i - \log n)}$ and $N'_n = \sum_{i>1}^{n} \varepsilon_{(\frac{i}{n}, Y_i - \log n)}$ both converge to a Poisson process with intensity $\lambda_{[0,1]} \otimes v, v([x,\infty)) = e^{-x}$ (see Resnick (1987), section 4). But both sequences have quite different stopping behaviour. For the optimal stopping of $X_1, \dots X_n$ the optimal stopping curve is given by

$$
\begin{aligned}
u_{n,n-1} &= EX_n \qquad \text{and} \qquad X_{n-1} \geq a_{n-1} = u_{n,n-1} \\
u_{n,n-2} &= E(X_{n-1} \vee u_{n,n-1}) = EX_{n-1} = a_{n-2} \\
u_{n,n-3} &= E(X_{n-2} \vee u_{n,n-2}) = EX_{n-2} = a_{n-3}
\end{aligned}
$$

as $X_{n-2} \geq a_{n-2}$. Finally, $u_{n,1} = EX_2 = a_1$.

$T_n = \inf\{1 \leq i \leq n;\ X_i \geq u_{n,i}\}$ is an optimal stopping time. As $X_1 \geq a_1 = u_{n,1}$ we have $T_n \equiv 1$ and $EX_{T_n} = 0, \forall n$. This implies $E(X_{T_n} - \log n) \to -\infty, \frac{T_n}{n} \to 0$. On the other hand by Kennedy and Kertz (1991) the stopping problem for the exponential sequence has a nondegenerate limiting distribution.

It is also easy to construct examples with point process convergence but no convergence of the stopping problem.

With

$$M_{n,\ell,m} := \max\{X_{n,\ell}, \dots, X_{n,m}\}, \quad M_n := M_{n,1,n} \tag{3.1}$$

we introduce the following condition

(G) *Uniform integrability*

$\{(M_n)_+;\ n \in \mathbb{N}\}$ is uniformly integrable.

Let $u_{n,1}, \dots, u_{n,n}$ be the optimal stopping curve of $X_{n,1}, \dots X_{n,n}$ and define

$$u_n(s) := u_{n,[ns+1]\wedge n}, \quad s \in [0,1]. \tag{3.2}$$

(L) *Lower curve condition*

$$\liminf_{n\to\infty} u_n(1-\varepsilon) > -\infty, \quad \forall \varepsilon \in (0,1].$$

Theorem 3.2 (Approximation of optimal stopping) *Let (X_{ni}) be an independent sequence satisfying* (G) *such that the associated point processes converge*

$$N_n = \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, X_{n,i}\right)} \xrightarrow{\mathcal{D}} N \ \ on\ M_f, \tag{3.3}$$

where N is a Poisson process with optimal stopping curve u, optimal stopping time $T = \tau^u$ satisfying (S) *and* (D). *Let $(u_{n,j})$ denote the optimal stopping curve for $(X_{n,i})$ and let T_n denote the corresponding optimal stopping time.*

a) If $\lim_u u_n(1) = c = f(1) \in \mathbb{R}$ exists, then $u_n(t) \to u(t)$ uniformly on $[0,t]$ for $t < 1$ and

$$\left(\frac{T_n}{n}, X_{n,T_n}, M_{n,1,T_n-1}, M_{n,T_n+1,n}\right) \xrightarrow{\mathcal{D}} (T, Y_{K^T}, M_{0,T-}, M_{T+,1}). \tag{3.4}$$

u is a solution of the differential equation

$$u'(t) = -\int_{u(t)}^{\infty}\int_{x}^{\infty} h_f(t,y)\,\mathrm{d}y\,\mathrm{d}x, \quad u(1) = c. \tag{3.5}$$

b) If $\lim_{\varepsilon\to 0}\lim_{n\to\infty} u_n(1-\varepsilon) = -\infty = f(1)$ and (L) holds then for any pointwise convergent subsequence $u_{n'} \to \widehat{u}$, convergence is uniform on $[0,t]$ for $t < 1$, and the limit $\widehat{u}$ satisfies

$$\widehat{u}'(t) = -\int_{\widehat{u}(t)}^{\infty}\int_{x}^{\infty} h_f(t,y)\,\mathrm{d}y\,\mathrm{d}x, \quad \widehat{u}(1) = -\infty. \tag{3.6}$$

If (3.6) has a unique solution, then $u_n(t) \to u(t), t \in [0,1)$ and (3.4) holds.

Proof: Note that by assumption $(M_n)_+ \xrightarrow{\mathcal{D}} M_+$ and we assume w.l.g. that convergence is pointwise. (G) implies $(M_n)_+ \xrightarrow{L^1} M_+$; and $E(M_n)_+ \to EM_+$, in particular $\sup_n E(M_n)_+ < \infty$ and condition (B) is satisfied as well as the further conditions in Theorem 2.5.

a) For the proof of a) and b) we choose $\widetilde{t} \in (0,1)$ and consider convergence of the stopping curves on $[0,\widetilde{t}\,]$. Then for the proof of a) we take $\widetilde{t} = 1$. By (L) and (G) there exist a subsequence $(n') \subset \mathbb{N}$ and $d \in \mathbb{R}$ such that $\lim u_{n'}(\widetilde{t}\,) = d \in \mathbb{R}$. Let $u_{\widetilde{t}}$ be the optimal stopping curve of $N_{\widetilde{t},d}$ where for points $\geq \widetilde{t}$, the values are set to be d and let $\widehat{T}^{\widetilde{t}}$ denote the optimal stopping time for $N_{\widetilde{t},d}$. The separation condition is fulfilled for $u_{\widetilde{t}}, N_{\widetilde{t},d}$. Define new stopping curves

$$\widehat{u}_{n'}(t) = \begin{cases} u_{\widetilde{t}}(t), & 0 \leq t \leq \widetilde{t} \\ u_{n'}(t), & \widetilde{t} < t \leq 1 \end{cases} \tag{3.7}$$

and $\widehat{T}_{n'}$ the corresponding threshold stopping times. Since $\widehat{u}_{n'}(t) \to u_{\widetilde{t}}(t)$, $t \leq \widetilde{t}$ we conclude from Proposition 2.4 modified for $N_{\widetilde{t},d}$ that

$$X_{n',\widehat{T}_{n'}} 1_{\left\{\frac{\widehat{T}_{n'}}{n'} \leq \widetilde{t}\right\}} \xrightarrow{\mathcal{D}} Y_{K^{\widehat{T}^{\widetilde{t}}}} 1_{\left\{\widehat{T}^{\widetilde{t}} \leq \widetilde{t}\right\}} \tag{3.8}$$

and convergence of expectations in (3.8) holds. For the proof note on one hand side $X_{n',\widehat{T}_{n'}} 1_{\left\{\frac{\widehat{T}_{n'}}{n'} \leq \widetilde{t}\right\}} \geq u_{\widetilde{t}}(\widetilde{t}\,)$.

Since $u_{n'}(\widetilde{t}\,) \to d$ we have lower bounds. Also $X_{n',\widehat{T}_{n'}} \leq (M_{n'})_+$ and (G) imply uniform integrability of $\{X_{n',\widehat{T}_{n'}} 1_{\left\{\widehat{T}^{\widetilde{t}} \leq \widetilde{t}\right\}};\ n \in \mathbb{N}\}$.

For $i \leq n$ holds by the independence assumption

$$\begin{aligned} EX_{n,\widehat{T}_n} 1_{\left\{\frac{\widehat{T}_n}{n} > \widetilde{t}\right\}} &= E\left(X_{n,\widehat{T}_n} \middle| \frac{\widehat{T}_n}{n} > \widetilde{t}\right) P\left(\frac{\widehat{T}_n}{n} > \widetilde{t}\right) \\ &= E\left(X_{n,\widehat{T}_n} \middle| X_{n,j} < u_{n,j+1}, \frac{j}{n} \leq \widetilde{t}\right) P\left(\frac{\widehat{T}_n}{n} > \widetilde{t}\right) \\ &= EX_{n,\widehat{T}_n^{>[\widetilde{t}n]}} P\left(\frac{\widehat{T}_n}{n} > \widetilde{t}\right) = u_n(\widetilde{t}\,) P\left(\frac{\widehat{T}_n}{n} > \widetilde{t}\right). \end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned} EX_{n',\widehat{T}_{n'}} &= EX_{n',\widehat{T}_{n'}} 1_{\left\{\frac{\widehat{T}_{n'}}{n} \leq \widetilde{t}\right\}} + EX_{n',\widehat{T}_{n'}} 1_{\left\{\frac{\widehat{T}_{n'}}{n} > \widetilde{t}\right\}} \\ &\to EY_{K^{\widehat{T}}} 1_{\{\widehat{T}^{\widetilde{t}} \leq \widetilde{t}\}} + cP(\widehat{T}^{\widetilde{t}} > \widetilde{t}\,) \\ &= u_{\widetilde{t}}(0). \end{aligned}$$

This implies

$$\liminf_{n'} u_{n'}(0) \geq \liminf_{n'} EX_{n',\widehat{T}_{n'}} = EY_{K^{\widehat{T}}} = u_{\widetilde{t}}(0).$$

Similarly, restricting to stopping times $\geq [nt]$ resp. $\geq t$ we obtain

$$\liminf_{n'} u_{n'}(t) \geq u_{\widetilde{t}}(t), \quad 0 \leq t < \widetilde{t}. \tag{3.10}$$

For the converse inequality let $(n'') \subset (n')$ such that $u_{n''} \to u''$ on $[0, \widetilde{t}\,]$. A subsequence (n'') with this property exists as $(u_n)_n$ are monotonically decreasing functions bounded below by the function f which is bounded on $[0, \widetilde{t}\,]$ and u'' is easily shown to be continuous. Furthermore, by (3.10) $u'' \geq u_{\widetilde{t}}$ on $[0, \widetilde{t}\,)$ and so condition (S) holds. By Proposition 2.4

$$\left(\frac{T_{n''}}{n''}, X_{n'',T_{n''}}\right) \to (\tau^{u''}, Y_{K^{\tau^{u''}}}).$$

As above, therefore, by Proposition 2.4 again

$$\begin{aligned} u_{n''}(0) &= EX_{n'',T_{n''}} \\ &= EX_{n'',T_{n''}} 1_{\left\{\frac{T_{n''}}{n''} \le \tilde{t}\right\}} + EX_{n'',T_{n''}} 1_{\left\{\frac{T_{n''}}{n''} > \tilde{t}\right\}} \\ &\to EY_{K^{\tau^{u''}}} 1_{\{\tau^{u''} \le \tilde{t}\}} + cP(\{\tau^{u''} > \tilde{t}\}) \\ &\le u_{\tilde{t}}(0). \end{aligned} \tag{3.11}$$

(3.10) and (3.11) together imply $\lim u_{n'}(0) = u_{\tilde{t}}(0)$ and similarly one obtains

$$u_{n'}(t) \to u_{\tilde{t}}(t) \tag{3.12}$$

for all $t \le \tilde{t}$. This being true for any converging subsequences (n') we conclude convergence of the joint distribution of the optimal stopping time and the stopping variable choosing $\tilde{t} = 1$.

Assuming a.s. convergence $N_n \to N$ we conclude as in Kennedy and Kertz (1990)

$$\sum_{i<T_n} \varepsilon_{\left(\frac{i}{n}, X_{n,i}\right)} \to \sum_{\tau_i<T} \varepsilon_{(\tau_i, Y_i)}, \tag{3.13}$$

and we obtain $M_{n,1,T_n-1} \to M_{0,T-}$ and $M_{n,T_n+1,n} \to M_{T+,1}$ and so (3.4).

b) Assumption (L) implies for any convergent subsequence $(u_{n'})$ that

$$c_\varepsilon = \lim_{n'} u_{n'}(1-\varepsilon) > -\infty, \quad \forall \varepsilon > 0.$$

Let $u_{1-\varepsilon}$ be the optimal stopping curve of $N_{1-\varepsilon,c_\varepsilon}$. For $t \le 1-\varepsilon$ by the argument in a) with $\tilde{t} = 1-\varepsilon$

$$u_{n'}(t) \to u_{1-\varepsilon}(t), \quad t \le 1-\varepsilon$$

and $u_{1-\varepsilon} = \hat{u}/[0, 1-\varepsilon]$ solves by Theorem 2.5 the differential equation

$$u'_{1-\varepsilon}(t) = -\int_{u_{1-\varepsilon}(t)}^{\infty} \int_x^{\infty} h_f(t,y)\,\mathrm{d}y\,\mathrm{d}x \text{ on } [0,1-\varepsilon], \quad u_{1-\varepsilon(1-\varepsilon)} = \hat{u}(1-\varepsilon) = c_\varepsilon.$$

Therefore, $\hat{u}(t) = u_{1-\varepsilon}(t)$ on $[0, 1-\varepsilon]$ and by assumption $\hat{u}(1-) = \hat{u}(1) = -\infty$. This implies that $c_\varepsilon \to -\infty$ and $\hat{u}(t)$ solves the differential equation (3.6). Since this holds true for any converging subsequence we conclude convergence of u_n in the case that the differential equation has a unique solution $\tilde{u}$. Since (L), (G) and Fatou imply for $t < 1$ that $EY_{K\left(\tau^{\tilde{u}}\right)\ge t} > -\infty$ we conclude from Theorem 2.5 that $\tilde{u}$ is the optimal stopping curve of N. □

Remark 3.3 The assumption $u_n(1) \to c = f(1) \in \mathbb{R}$ in part a) of Theorem 3.2 can be replaced by the weaker assumption $\lim_{\varepsilon\to 0} \overline{\lim}_{n\to\infty} u_n(1-\varepsilon) = c = f(1)$ as in part b). But in the examples considered it is easier to establish the condition on $u_n(1)$.

4 IID sequences with observation or discounted costs

In this section we extend approximative optimal stopping results from Kennedy and Kertz (1991) for the iid case to include observation costs or discount factors. Approximative optimal stopping results for infinitely many iid observations (Y_i) with linear costs of the form $Y_1 - \delta, Y_2 - 2\delta, \dots$ or for $Y_1 e^{-\delta}, Y_2 e^{-2\delta}, \dots$ as the costs $\delta \to 0$ are given in Kennedy and Kertz (1992). We consider the finite stopping problem for fixed costs δ as $n \to \infty$. We apply our general approximation result for the optimal stopping problem in section 3. Thus we also obtain an interpretation of the limiting stopping times, distributions and stopping values as optimal stopping times and values in the limiting Poisson process stopping problem. Some further examples and an extension to some dependent sequences will be investigated in a subsequent paper.

Consider iid sequences (Y_i) in the domain of an extreme value distribution, i.e. of type

$$\begin{cases} \Lambda(x) &= \quad e^{-e^{-x}}, \quad x \in \mathbb{R}^1 \\ \Phi_\alpha(x) &= \begin{cases} e^{-x^{-\alpha}}, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad \alpha > 0, \quad \text{or} \\ \Psi_\alpha(x) &= \begin{cases} e^{-(-x)^\alpha}, & x < 0, \\ 1, & x \geq 0, \end{cases} \quad \alpha \geq 0. \end{cases} \tag{4.1}$$

Then with suitable normalizing constants a_n, b_n

$$N_n = \sum_{i=1}^n \varepsilon_{\left(\frac{i}{n}, \frac{Y_i - b_n}{a_n}\right)} \xrightarrow{\mathcal{D}} N \text{ on } M_f \tag{4.2}$$

where N is a Poisson process with intensity $\mu = \lambda\!\!\lambda_{[0,1]} \otimes v$ where

$$\begin{cases} v([x,\infty)) = e^{-x}, & x \in \mathbb{R}^1, \quad f \equiv -\infty \quad if\ F \in D(\Lambda), \\ v([x,\infty)) = x^{-\alpha}, & x > 0, \quad f \equiv \ \ 0 \quad if\ F \in D(\Phi_\alpha), \text{ and} \\ v([x,\infty)) = (-x)^\alpha, & x < 0, \quad f \equiv -\infty \quad if\ F \in D(\Psi_\alpha) \end{cases} \tag{4.3}$$

(see Resnick (1987, p. 210)).

To establish the uniform integrability condition (G) we shall make use of the following proposition.

Proposition 4.1 *Let (Y_i) be an iid sequence with df F in the domain of attraction of an extreme value distribution with $\alpha > 1$ in case $F \in D(\Phi_\alpha)$. Then*

$$\left\{ \frac{(L_n - b_n)_+}{a_n};\ n \in \mathbb{N} \right\} \text{ is uniformly integrable,} \tag{4.4}$$

where $L_n = \max\{Y_1, \dots, Y_n\}$.

Proof: This follows from Resnick (1987, p. 80–82), using the following modification of the uniform integrability condition: A sequence $(Z_n) \geq 0$ of nonnegative real random variables is uniformly integrable if and only if

$$\lim_{L\to\infty} \limsup_{n\to\infty} E Z_n 1_{\{Z_n \geq L\}} = 0. \tag{4.5}$$

□

The extreme value theory of sequences with observation or with discount costs has been dealt with in de Haan and Verkade (1987). The following gives a related point process result.

Theorem 4.2 *Let (Y_i) be an iid sequence with df F.*

a) If $F \in D(\Lambda)$ and $(c_i) \subset \mathbb{R}$ such that $c_i \leq c_{i+1}$ and

$$0 \leq \gamma_t := \lim_{n\to\infty} \frac{c_n - c_{[nt]}}{a_n} < \infty, \quad 0 < t \leq 1, \tag{4.6}$$

then $\gamma_t = -c \log t$ for some $c \geq 0$ and with $X_i := Y_i + c_i$, $\widehat{b}_i := b_i + c_i$, $i \in \mathbb{N}$, holds

$$N_n := \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, \frac{X_i - \widehat{b}_n}{a_n}\right)} \xrightarrow{\mathcal{D}} N; \tag{4.7}$$

N a Poisson process with intensity

$$\frac{d\mu(\cdot \times [y,\infty))}{d\lambda\!\!\lambda_{[0,1]}}(t) = e^{-y} t^c. \tag{4.8}$$

b) If $F \in D(\Phi_\alpha)$, $\alpha > 1$ and $(c_i) \subset \mathbb{R}$ such that $c_i \geq 1$, $c_{i+1} \geq c_i$, $\gamma_t := \lim_{n\to\infty} \frac{c_{[nt]}}{c_n}$ exists, then $\gamma_t = t^c$ for some $c \geq 0$ and with $X_i := c_i Y_i$, $\widehat{a}_i := c_i a_i$, $i \in \mathbb{N}$ holds

$$N_n := \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, \frac{X_i}{\widehat{a}_n}\right)} \xrightarrow{\mathcal{D}} N; \tag{4.9}$$

N a Poisson process with intensity given by

$$\frac{d\mu(\cdot \times [y,\infty])}{d\lambda\!\!\lambda_{[0,1]}}(t) = y^{-\alpha} t^{c\alpha}, \quad y > 0. \tag{4.10}$$

c) *If* $F \in D(\Psi_\alpha), \alpha > 0, (c_i) \subset \mathbb{R}$ *such that* $0 < c_i \le 1$, $c_i \ge c_{i+1}$, $\gamma_t := \lim_{n\to\infty} \frac{c_{[nt]}}{c_n}$ *exists, then* $\gamma_t = t^{-c}$ *for some* $c \ge 0$ *and with* $X_i := c_i Y_i$, $\widehat{a}_i := c_i a_i$ *holds*

$$N_n := \sum_{i=1}^{n} \varepsilon_{\left(\frac{i}{n}, \frac{X_i}{\widehat{a}_n}\right)} \xrightarrow{\mathcal{D}} N; \tag{4.11}$$

N a Poisson process with intensity given by

$$\frac{d\mu(\cdot \times [y,\infty])}{d\lambda^1_{[0,1]}}(t) = (-y)^\alpha t^{c\alpha}, \quad y < 0. \tag{4.12}$$

Proof:

a) For $0 \le s < t \le 1$ holds

$$\begin{aligned} \gamma_t - \gamma_s &= \lim_n \left(\frac{c_n - c_{[nt]}}{a_n} - \frac{c_n - c_{[ns]}}{a_n} \right) \\ &= -\lim_n \frac{c_{[nt]} - c_{[ns]}}{a_{[nt]}} \frac{a_{[nt]}}{a_n} \\ &= -\lim_n \frac{c_n - c_{[n\frac{s}{t}]}}{a_n} \;=\; -\gamma_{\frac{s}{t}}, \quad \text{as } \frac{a_{[nt]}}{a_n} \to 1. \end{aligned}$$

This implies that $\gamma_t = -c \log t$ for some $c \ge 0$ since $\gamma_t \ge 0$.

Consider the mapping $R_n : [0,1] \times \mathbb{R}^1 \to [0,1] \times \mathbb{R}^1$, $R_n(t,y) = (t, y - \gamma_{n,t})$, $n \ge 0$, where $\gamma_{n,t} := \frac{c_n - c_{[nt]}}{a_n}$ for $n \ge 1$, $\gamma_{0,t} := \gamma_t$. R_n induces a mapping on point processes given by

$$R_n\left(\sum_i \varepsilon_{(s_i,z_i)}\right) := \sum_i \varepsilon_{R_n(s_i,z_i)}. \tag{4.13}$$

By (4.2) $\widehat{N}_n := \sum_{i=1}^n \varepsilon_{\left(\frac{i}{n}, \frac{Y_i - b_n}{a_n}\right)} \xrightarrow{\mathcal{D}} \widehat{N}$, where $\widehat{N}$ is a Poisson process with intensity measure $\widehat{\mu} = \lambda_{[0,1]} \otimes v$, $v([x,\infty)) = e^{-x}$, $x \in \mathbb{R}^1$. Since $\frac{X_i - \widehat{b}_n}{a_n} = \frac{Y_i + c_i - b_n - c_n}{a_n} = \frac{Y_i - b_n}{a_n} - \gamma_{n,\frac{i}{n}}$, we obtain

$$R_n \widehat{N}_n = N_n, \quad n \ge 1, \quad R_0 \widehat{N} = N. \tag{4.14}$$

To prove that R_n operates continuously on the set of point measures first observe that $(t_n, y_n) \to (t,y), t \ne 0$, implies

$$R_n(t_n, y_n) = (t_n, y_n - \gamma_{n,t_n}) \to (t, y - \gamma_t) = R_0(t,y).$$

Then, for deterministic point measures $Q_n := \sum_i \varepsilon_{(s_i^n, z_i^n)}, Q = \sum \varepsilon_{(s_i, z_i)}$ on M_f with $s_i \neq 0$, $\forall i$, $Q_n \to Q$ implies

$$R_n Q_n \to R_0 Q. \tag{4.15}$$

For the proof of (4.15) note that $\gamma_{n,\cdot}$ is monotonically nondecreasing and for $t > 0$ holds $\gamma_{n,\cdot} \to \gamma_\cdot$ uniformly on $[t, 1]$. Furthermore, for compact sets $[a, b] \times [c, d] \subset M_f$, $0 < a < b \leq 1$, $-\infty < c < d$ with $Q(\partial([a, b] \times [c, d])) = 0$ holds $Q_n([a, b] \times [c, d]) \to Q([a, b] \times [c, d])$. Then using that the points converge we obtain $R_n Q_n \to R_0 Q$. By the continuous mapping principle, therefore, $N_n = R_n \widehat{N}_n \xrightarrow{\mathcal{D}} N = R_0 \widehat{N}$.

As $N = R_0 \widehat{N}$ we obtain that the intensity measure μ of N satisfies $\mu = \widehat{\mu}^{R_0}$. Therefore, for $0 < t \leq 1$, $x \in \mathbb{R}$ and with $R_0^{-1}(t, y) = (t, y + \gamma_t)$ we have

$$\begin{aligned} \mu([0, t] \times [x, \infty)) &= \widehat{\mu}(R_0^{-1}([0, t] \times [x, \infty))) \\ &= \widehat{\mu}(\{(s, z);\ z \geq x - c \log s, 0 \leq s \leq 1\}) \\ &= \int_0^t e^{-(x - c \log s)}\, \mathrm{d}s \ = \ e^{-x} \int_0^t s^c \, \mathrm{d}s \\ &= e^{-x} \frac{t^{1+c}}{1+c}. \end{aligned} \tag{4.16}$$

This implies a).

b) For $0 < s < t \leq 1$ holds

$$\frac{\gamma_s}{\gamma_t} = \lim_{n \to \infty} \frac{c_{[ns]}}{c_{[nt]}} = \lim_{n \to \infty} \frac{c_{[n \frac{s}{t}]}}{c_n} = \gamma_{\frac{s}{t}}$$

which implies $\gamma_t = t^c$ for some $c \geq 0$ since γ_t is monotonically nondecreasing.

Defining $R_n(t, y) := (t, y\gamma_{nt})$ we obtain as in a)

$$N_n = R_n \widehat{N}_n \to R_0 \widehat{N} = N \text{ on } M_f, \quad f \equiv 0.$$

c) The proof of c) is analogous. □

We next apply the approximation result of optimal stopping in Theorem 3.2 to the optimal stopping problem for sequences $X_1, \ldots, X_n$ as in Theorem 4.2 with observation or discounted costs. Let (Y_i) be iid integrable random variables with df F.

We also construct an asymptotically optimal stopping sequence (T_n'), i.e. a sequence of stopping times which asymptotically (after normalization) yield the same stopping values as the optimal stopping times. This is of interest in the typical case where the exactly optimal stopping times cannot be evaluated explicitely. The modification of the 'natural' asymptotic stopping times τ^u, for the optimal stopping curves u, is necessary in order to be able to establish the lower boundary condition (L).

Theorem 4.3 *Assume that $F \in D(\Lambda)$, $(c_i) \subset \mathbb{R}$ such that $c_i \geq 0$, $c_{i+1} \geq c_i$ and $\lim_{n\to\infty} \frac{c_n - c_{[nt]}}{a_n} = -c \log t$. Let T_n be the optimal stopping time of $X_1, \dots, X_n$ where $X_i := Y_i + c_i$. Then with $\widehat{b}_i := b_i + c_i$, $u_c^\Lambda(t) = \log \frac{1-t^{1+c}}{1+c}$ holds:*

$$\frac{EX_{T_n} - \widehat{b}_n}{a_n} \quad \to \quad -\log(1+c) \ \ \textit{and} \tag{4.17}$$

$$P\left(\left\{\frac{X_{T_n} - \widehat{b}_n}{a_n} \leq x\right\}\right) \quad \to \quad \begin{cases} 1 - \dfrac{1}{2}\dfrac{e^{-x}}{1+c}, & x \geq -\log(1+c) \\ \dfrac{1}{2} e^x (1+c), & x < -\log(1+c). \end{cases} \tag{4.18}$$

Furthermore, for any $\varepsilon > 0$ and any sequence (w_n) with $n(1 - F(w_n)) \to 1$ the sequence (T_n'), where

$$\begin{aligned} T_n' \ \ := \ \ \inf \Bigg\{ i \leq n;\ & \left(i \geq n - [n\varepsilon] \ \textit{and} \ \frac{X_i - \widehat{b}_n}{a_n} > u_c^\Lambda\left(\frac{i}{n}\right) - u_0^\Lambda\left(\frac{i}{n}\right) + \frac{w_{n-1} - b_n}{a_n}\right) \\ & \textit{or} \ \left(i < n - [n\varepsilon] \ \textit{and} \ \frac{X_i - \widehat{b}_n}{a_n} \geq u_c^\Lambda\left(\frac{i}{n}\right)\right)\Bigg\} \end{aligned} \tag{4.19}$$

is an asymptotically optimal stopping sequence.

We remark, that for $F \in D(G)$ for some extreme value distribution G and for a sequence $(w_n) \subset \mathbb{R}^1$ holds (see Kennedy and Kertz (1990, p. 309)):

$$n(1 - F(w_n)) \to x \quad \Leftrightarrow \quad \frac{w_n - b_n}{a_n} \to -\log G(x). \tag{4.20}$$

Theorem 4.4 *Let $F \in D(\Phi_\alpha), \alpha > 1$ and let $(c_i) \subset \mathbb{R}^1, 1 \leq c_i, c_i \leq c_{i+1}$ and $\frac{c_{[nt]}}{c_n} \to t^c$ for some $c > 0$.*

Then with $X_i := c_i Y_i, \hat{a}_i := c_i a_i$ and $u_{c,\alpha}^\Phi(t) = \left(\frac{\alpha}{(1+c\alpha)(\alpha-1)}\right)^{1/\alpha} (1 - t^{1+c\alpha})^{1/\alpha}$ the optimal stopping times T_n for $X_1, \dots, X_n$ satisfy:

$$\frac{EX_{T_n}}{\hat{a}_n} \quad \to \quad \left(\frac{\alpha}{(1+c\alpha)(\alpha-1)}\right)^{1/\alpha} \qquad \textit{and} \tag{4.21}$$

$$P\left(\left\{\frac{X_{T_n}}{\hat{a}_n} \leq x\right\}\right) \quad \to \quad \begin{cases} 1 - x^{-\alpha} \dfrac{1}{1+c\alpha}\dfrac{\alpha}{2\alpha-1}, & x \geq c_\alpha \\ \dfrac{\alpha}{2\alpha-1}\left(\dfrac{\alpha-1}{\alpha}(1+c\alpha)\right)^{\frac{\alpha-1}{\alpha}} x^{\alpha-1}, & 0 < x < c_\alpha \\ 0, & x \leq 0 \end{cases} \tag{4.22}$$

$$\textit{where } c_\alpha := \left(\frac{\alpha}{(1+c\alpha)(1+\alpha)}\right)^{1/\alpha}.$$

$$T_n' := \inf\left\{1 \leq i \leq n;\ \ X_i \geq a_n u_{c,\alpha}^\Phi\left(\frac{i}{n}\right)\right\} \tag{4.23}$$

defines an asymptotically optimal stopping sequence.

Theorem 4.5 *Let $F \in D(\Psi_\alpha)$, $\alpha > 0$, $(c_i) \subset \mathbb{R}$, $0 < c_i \le 1$, $c_i \ge c_{i+1}$, and $\frac{c_{[nt]}}{c_n} \to t^{-c}$ for some $c \ge 0$.*

Then with $X_i := c_i Y_i$, $\widehat{a}_i := c_i a_i$ the optimal stopping times T_n for $X_1, \dots, X_n$ satisfy:

$$\frac{EX_{T_n}}{\widehat{a}_n} \to -\left(\frac{\alpha}{(1+c\alpha)(\alpha+1)}\right)^{-1/\alpha} \tag{4.24}$$

$$P\left(\frac{X_{T_n}}{\widehat{a}_n} \le x\right) \to \begin{cases} 1, & x \ge 0 \\ 1-(-x)^\alpha \dfrac{1}{1+c\alpha}\dfrac{1}{2+\frac{1}{\alpha}}, & x \ge -c_\alpha \\ \left(\dfrac{\alpha+1}{\alpha}(1+c\alpha)\right)^{\frac{\alpha+1}{\alpha}}(-x)^{-\alpha-1}, & x < -c_\alpha \end{cases} \tag{4.25}$$

$$\textit{where } c_\alpha := -\left(\frac{\alpha}{(1+c\alpha)(\alpha+1)}\right)^{-1/\alpha}.$$

Furthermore, for any $\varepsilon > 0$, for any sequence (w_n) with $n(1-F(w_n)) \to 1+\frac{1}{\alpha}$ and $u^\Psi_{c,\alpha}(t) := -\left(\frac{1-t^{1+c\alpha}}{1+c\alpha}\frac{\alpha}{1+\alpha}\right)^{-1/2}$ holds

$$T'_n := \inf\left\{ i \le n;\ \left(i \ge n-[n\varepsilon] \text{ and } \frac{X_i}{c_n} \ge \frac{u^\Psi_{c,\alpha}\left(\frac{i}{n}\right)}{u^\Psi_{0,\alpha}\left(\frac{i}{n}\right)} w_{n-i}\right) \right. \tag{4.26}$$
$$\left. \text{or } \left(i < n-[n\varepsilon] \text{ and } \frac{X_i}{\widehat{a}_n} \ge u^\Psi_{c,\alpha}\left(\frac{i}{n}\right)\right)\right\}$$

is an asymptotically optimal sequence of stopping times.

Proof of Theorems 4.3 – 4.5: We verify the assumptions of the Approximation Theorem 3.2. Point process convergence is shown in Theorem 4.2, (D) is satisfied in all three cases and the separation condition holds in each case since f is constant. Further, with $M_n = M_{n,1,n} = \max\{X_1, \dots, X_n\}$ and $M_n - \widehat{b}_n \le L_n - b_n$ in the Λ case and $\frac{M_n^+}{\widehat{a}_n} \le \frac{L_n^+}{a_n}$ in the Φ_α, Ψ_α case imply that (G) holds.

a) Proof of Theorem 4.3: By the independence properties of N we have for $t \in [0,1], q, r, \omega \in \mathbb{R}$

$$P(T \le t, M_{0,T-} \le q, Y_{K^T} \le r, M_{T+,1} \le w) \tag{4.27}$$
$$= \int_{o\vee u^{-1}(r)}^{(o\vee u^{-1}(r))\vee t} P(N(\{(x,y);\ 0 \le x < s, y > u(x) \wedge q\}) = 0)$$
$$\cdot P(\exists \tau_k; \tau_k \in ds, u(s) \le Y_k \le r) P(N((s,1] \times (w,\infty)) = 0)$$
$$= \int_{o\vee u^{-1}(r)}^{(o\vee u^{-1}(r))\vee t} e^{-\mu(\{(x,y);\ 0\le x<s, y>u(x)\wedge q\})}\left(e^{-u(s)}u(s)^{-c} - e^{-r}u(s)^{-c}\right) e^{-\mu((s,1]\times(w,\infty))}\,ds$$

if $u^{-1}(r) \le t$ and zero else.

The differential equation for the optimal stopping curve of N (see (2.14) in Theorem 2.5) is given by the differential equation with separated variables

$$\begin{cases} u'(t) &= -\int\limits_{u(t)}^{\infty} e^{-x}t^c \,\mathrm{d}x = -t^c e^{-u(t)}, \quad t<1 \\ u(1) &= -\infty. \end{cases} \tag{4.28}$$

$F(x) = -\int\limits_1^x t^c \,\mathrm{d}t = \frac{1-x^{1+c}}{1+c}$ and $G(y) = \int\limits_{-\infty}^{y} \frac{1}{e^{-t}}\,\mathrm{d}t = e^y$ exists and so by Proposition 2.6, (4.28) has a unique continuous solution and $G(u(t)) = e^{u(t)} = F(t) = \frac{1-t^{1+c}}{1+c}$, i.e.

$$u(t) = \log\frac{1-t^{1+c}}{1+c}. \tag{4.29}$$

To verify the lower curve condition (L) we next prove that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} E\left(\left(\frac{X_{T_n'}-\widehat{b}_n}{a_n}\right)^{-} 1_{\{T_n'>n-\{n\varepsilon\}\}}\right) = 0. \tag{4.30}$$

Since

$$\begin{aligned} & E\left(\frac{X_{T_n'}-\widehat{b}_n}{a_n}\right)^{-} 1_{\{T_n'>n-[n\varepsilon]\}} \\ &= \sum_{i=n-[n\varepsilon+1]}^{n} E\left(\frac{X_i-\widehat{b}_n}{a_n} 1_{\left\{0\geq \frac{X_i-\widehat{b}_n}{a_n}\geq u_c^\Lambda\left(\frac{i}{n}\right)-u_0^\Lambda\left(\frac{i}{n}\right)+\frac{w_{n-i}-b_n}{a_n}\right\}}\right) \\ &\quad \cdot \prod_{j=1}^{i-1} P\left(\frac{X_j-\widehat{b}_n}{a_n} < u_c^\Lambda\left(\frac{j}{n}\right) - u_0^\Lambda\left(\frac{j}{n}\right) + \frac{w_{n-j}-b_n}{a_n}\right). \end{aligned}$$

Observe that $X_i-\widehat{b}_n = Y_i - b_n + c_i - c_n$, $c_i \leq c_{i+1}$ and $\lim_{n\to\infty}\frac{c_{n-[n\varepsilon]}-c_n}{a_n} = \log(1-\varepsilon)$. Therefore, $\frac{c_i-c_n}{a_n}$ is for $i\in[n-[n\varepsilon],n]$ bounded by some constant d_1' for $n\geq n_0$ and, therefore, $\frac{X_i-\widehat{b}_n}{a_n}\geq\frac{Y_i-b_n}{a_n}-d_1'$. Also $u_c^\Lambda\left(\frac{i}{n}\right)-u_0^\Lambda\left(\frac{i}{n}\right)$ is bounded and so it is sufficient to prove that for some $m\in\mathbb{N}$

$$\begin{aligned} \lim_{\varepsilon\to 0}\lim_{n\to\infty}\sum_{i=n-[n\varepsilon]+1}^{n-m} & E\left(\frac{Y_i-b_n}{a_n}1_{\{0\geq\frac{Y_i-b_n}{a_n}\geq\frac{w_{n-i}-b_n}{a_n}-d_1\}}\right) \\ & \prod_{j=1}^{i-1} P\left(\frac{Y_j-b_n}{a_n} < d_2 + \frac{w_{n-i}-b_n}{a_n}\right) \quad = \quad 0 \end{aligned}$$

for some constants $d_1, d_2 \geq 0$ and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} E\left(\frac{X_{T_n'}-\widehat{b}_n}{a_n}1_{\{T_n'>n-m\}}\right) = 0.$$

Since $\frac{a_{[nt]}}{a_n} \to 1$, $t \in (0,1]$ it is enough to prove in fact

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sum_{i=n-[n\varepsilon]+1}^{n-m} E\left(\frac{Y_i - b_n}{a_n}1_{\{0\geq Y_i \geq w'_{n-i}\}}\right)\prod_{j=1}^{i-1} P(Y_j < d_2 + w''_{n-i}) = 0,$$

where $w'_n := w_n - d_1 a_n$ and $w''_n := w_n + d_2 a_n$. By Kennedy and Kertz (1991, section 3) it holds that $\lim_{n\to\infty} nE\left(\frac{Y_1-b_n}{a_n}1_{\{Y_1>s_n\}}\right)$ exists for all sequences s_n such that $\lim_n \frac{s_n-b_n}{a_n}$ exists. Since $\lim_n \frac{E\left(\frac{Y_1-b_n}{a_n}1_{\{Y_1>w'_n\}}\right)}{E\left(\frac{Y_1-b_n}{a_n}1_{\{Y_1>w''_n\}}\right)}$ exists it is enough to prove that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sum_{i=n-[n\varepsilon]+1}^{n-m} E\left(\frac{Y_1 - b_n}{a_n}1_{\{Y_1>w''_n\}}\right)\prod_{j=1}^{i-1} P(Y_j < w''_{n-j}) = 0$$

for some m and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} E\left(\left(\frac{Y_{T'_n} - b_n}{a_n}\right)^{-} 1_{\{T'_n>n-m\}}\right) = 0. \tag{4.31}$$

This is proved to be true in Kennedy and Kertz (1992, 3.3(i)) and finishes the proof of (4.30).

From (4.30) we conclude that

$$-\infty < \liminf_{n\to\infty} E\frac{X_{T'_n} - \widehat{b}_n}{a_n}1_{\{T'_n>n-[n\varepsilon]\}}$$

and so condition (L) holds. Theorem 3.2 implies convergence of stopping times and stopping values.

To prove asymptotic optimality of T'_n we prove convergence of the thresholds. By (4.20)

$$\lim\frac{w_{n-[nt]}-b_n}{a_n} = \left(\lim\frac{w_{n-[nt]} - b_{n-[nt]}}{a_{n-[nt]}}\frac{a_{n-[nt]}}{a_n} + \frac{b_{n-[nt]} - b_n}{a_n}\right) = \log(1-t). \tag{4.32}$$

Therefore, the thresholds of T'_n converge to the threshold u of the optimal stopping time in the limiting process. Therefore, by Proposition 2.4

$$\left(\frac{T'_n}{n}, \frac{X_{T'_n} - b_n}{a_n}\right) \to (T, Y_{K^T}). \tag{4.33}$$

A similar result holds for stopping times $\geq nt$ resp. $\geq t$. Since

$$E\frac{X_{T'_n} - \widehat{b}_n}{a_n} = E\frac{X_{T'_n} - \widehat{b}_n}{a_n}1_{\{T'_n\leq n-[n\varepsilon]\}} + E\frac{X_{T'_n} - \widehat{b}_n}{a_n}1_{\{T'_n>n-[n\varepsilon]\}}$$

it follows from (4.33)

$$\begin{cases} E\frac{X_{T'_n}-\widehat{b}_n}{a_n}1_{\{T'_n\leq n-[n\varepsilon]\}} & \to \quad EY_{K^T}1_{\{T\leq 1-\varepsilon\}} \\ \text{and} & \\ EY_{K^T}1_{\{T\geq 1-\varepsilon\}} & \underset{\varepsilon\to 0}{\to} \quad EY_{K^T}. \end{cases} \tag{4.34}$$

From (4.30), (4.33) and Fatou it follows that $EY_{K^{T\geq t}} > -\infty$, for any $t < 1$. Therefore, by Theorem 2.5 e) u is the optimal stopping curve of N. Thus, $E\frac{X_{T_n'} - \widehat{b}_n}{a_n} \to EY_{K^T} = u(0)$.

From (4.27) we obtain

$$\begin{aligned} P(Y_{K^T} \leq x) &= -\int\limits_{o \vee u^{-1}(x)}^{1} e^{-\int\limits_0^s e^{-(u(y)-c\log y)\,\mathrm{d}y}} \left(e^{-u(s)}s^c - e^{-x}s^c\right)\mathrm{d}s \qquad (4.35)\\ &= \int\limits_{(1-e^x(1+c))_+^{1/(1+c)}}^{1} (1-s^{1+c})s^c\left(\frac{1+c}{1-s^{1+c}} - e^{-x}\right)\mathrm{d}s \\ &= \int\limits_{(1-e^x(1+c))_+^{1/(1+c)}}^{1} \frac{1}{1+c}\left(1+c-\frac{e^{-x}(1-\Psi)}{1+c}\right)\mathrm{d}\Psi \\ &= \left(\Psi - \frac{e^{-x}\left(\Psi - \frac{1}{2}\Psi^2\right)}{1+c}\right)\Bigg|_{(1-e^x(1+c))_+}^{1} \\ &= \begin{cases} 1-\dfrac{1}{2}\dfrac{e^{-x}}{1+c} & \textit{for} \;\; x \geq -\log(1+c) \\ \dfrac{1}{2}e^x(1+c) & \textit{for} \;\; x < -\log(1+c). \end{cases} \end{aligned}$$

b) Proof of Theorem 4.4 The differential equation for the optimal stopping curve u in the limiting process is given by an equation with separate variables

$$\begin{cases} u'(t) &= -\int\limits_{u(t)}^{\infty} x^{-\alpha}t^{c\alpha}\,\mathrm{d}x \;=\; -t^{c\alpha}\dfrac{u(t)^{1-\alpha}}{\alpha-1}, \quad t<1 \\ u(1) &= 0. \end{cases} \qquad (4.36)$$

With $F(x) = -\int\limits_1^x t^{c\alpha}\,\mathrm{d}t = \frac{1-x^{1+c\alpha}}{1+c\alpha}$, $G(y) = \int\limits_0^y t^{\alpha-1}(\alpha-1)\,\mathrm{d}t = (\alpha-1)\frac{y^\alpha}{\alpha}$, u is the unique solution of $G(u(t)) = F(t)$ i.e.

$$u(t) = \left(\alpha\frac{1-t^{1+c\alpha}}{(1+c\alpha)(\alpha-1)}\right)^{1/\alpha}. \qquad (4.37)$$

Condition (L) is trivially fulfilled in this case while the other part is handled similarly to that in the proof of Theorem 4.3.

c) Proof of Theorem 4.5 In this case u is the unique solution of

$$\begin{cases} u'(t) &= -t^{c\alpha}\dfrac{(-u(t))^{1+\alpha}}{1+\alpha} \\ u(1) &= -\infty \end{cases} \qquad (4.38)$$

which is given by

$$u(t) = -\left(\frac{1-t^{1+c\alpha}}{1+c\alpha}\,\frac{\alpha}{1+\alpha}\right)^{-1/\alpha}.$$

The conditions of Theorem 3.2 are verified as in the proof of Theorem 4.3.

In the case $c = 0$ the results simplify and yield in particular the iid case as derived in Kennedy and Kertz (1991). In this iid case the asymptotic properties of the optimal stopping times and values could be established directly. This direct method however will not work in the examples with discount and observation costs considered in this paper.

As in Kennedy and Kertz (1991) for the iid case we obtain also the following relations between optimal stopping value and expected maxima.

Corollary 4.6 *a) For $F \in D(\Lambda)$ we obtain under the conditions of Theorem 4.3*

$$\lim_{n\to\infty} \frac{EM_n - EX_{T_n}}{a_n} = \gamma \tag{4.39}$$

where $\gamma = 0,5772\dots$ is the Euler constant.

b) For $F \in D(\Phi_\alpha)$, $\alpha > 1$ and under the conditions of Theorem 4.4 holds

$$\begin{aligned} \lim_{n\to\infty} \frac{EM_n - EX_{T_n}}{a_n} &= (1+c\alpha)^{-1/\alpha}\left(\Gamma\left(1-\frac{1}{2}\right) - \left(\frac{\alpha-1}{\alpha}\right)^{1/\alpha}\right) \\ \lim \frac{EM_n}{EX_{T_n}} &= \left(\frac{\alpha-1}{\alpha}\right)^{-1/\alpha}\Gamma\left(1-\frac{1}{\alpha}\right). \end{aligned} \tag{4.40}$$

c) For $F \in D(\Psi_\alpha)$, $\alpha > 0$ and under the conditions of Theorem 4.5 holds

$$\begin{aligned} \lim \frac{EM_n - EX_{T_n}}{a_n} &= (1+c\alpha)^{1/\alpha}\left(\Gamma\left(1+\frac{1}{\alpha}\right) - \left(\frac{\alpha-1}{\alpha}\right)^{1-\frac{1}{\alpha}}\right) \\ \lim \frac{EM_n}{EX_{T_n}} &= \left(\frac{\alpha-1}{\alpha}\right)^{-1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right). \end{aligned} \tag{4.41}$$

Proof: a) By condition (G) it follows that $\left\{\frac{(M_n-b_n)^+}{a_n}\right\}_{n\in\mathbb{N}}$ is uniformly integrable. Also by (4.30) $\left\{\frac{(M_n-b_n)^-}{a_n}\right\}_{n\in\mathbb{N}}$ and, therefore, $\left\{\frac{M_n-b_n}{a_n}\right\}_{n\in\mathbb{N}}$ is uniformly integrable, and we obtain from the convergence in distribution $E\frac{M_n-b_n}{a_n} \to EM$ where

$$\begin{aligned} P(M \le x) &= P(N([0,1]\times(x,\infty)) = 0) \\ &= e^{-\mu([0,1]\times(x,\infty))} = e^{-\frac{1}{1+c}e^{-x}} \\ &= e^{-e^{-x-\log(1+c)}}. \end{aligned}$$

Therefore, $P(M + \log(1+c) \le x) = e^{-e^{-x}}$ and $EM = \gamma - \log(1+c)$. By Theorem 4.3 (4.39) follows.

b), c) The proof of b), c) is analogous. $\square$

Finally we state some asymptotic independence properties as in Kennedy and Kertz (1990) for the iid case. Let $M_{1,t} = \max_{1 \le k \le t} X_k$, $M_{t,n} = \max_{t \le k \le n} X_k$ be the pre resp. past t maximum.

Corollary 4.7 *a) If $F \in D(\Lambda)$ and under the conditions of Theorem 4.3 the random variables*

$$\left(\frac{T_n}{n}, \frac{M_{1,T_n-1} - u_{n,T_n}}{a_n}\right), \quad \frac{X_{T_n} - u_{n,T_n}}{a_n}, \quad \frac{M_{T_n+1,n-u_{n,T_n}}}{a_n} \tag{4.42}$$

are asymptotically independent.

b) If $F \in D(\Phi_\alpha)$, $\alpha > 1$ or $F \in D(\Psi_\alpha)$, $\alpha > 0$ then under the conditions of Theorem 4.4 resp. Theorem 4.5 the random variables

$$\left(\frac{T_n}{n}, \frac{M_{1,t_n-1}}{u_{n,T_n}}\right), \quad \frac{X_{T_n}}{u_{n,T_n}}, \quad \frac{M_{T_n+1,n}}{u_{n,T_n}} \tag{4.43}$$

are asymptotically independent.

Proof: Note that $u_{n,[nt]} \to u(t)$ in each case. From Theorems 4.3, 4.4, 4.5 it is enough to prove independence in the limiting Poisson process. But this can be seen from (4.27) and the related formulas in b). $\square$

Remark 4.8 Our method of proof of approximation of stopping problems is based on the approximation of the normalized imbedded point processes by the Poisson point process. This point process convergence is closely related to convergence of maxima to extreme value distributions. We therefore expect that the rate of approximation in the convergence of the stopping values is of the same order as the rate of convergence for moments of maxima in the extreme value distributions (see Resnick (1987, chapter 2)).

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