On upper and lower prices in discrete time models

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Abstract

A simple convex ordering argument in the class of equivalent martingale measures is used to determine the upper and lower price of a convex claim in a general discrete time model (N-period model) with bounded components. Under an approximation condition the upper price is given by the price in a related Cox-Ross-Rubinstein model. As application we discuss a discrete time stochastic volatility model.

Keywords: upper price, Cox-Ross-Rubinstein model

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1 Introduction

Consider a discrete time model for a risky asset defined on a probability space $(\Omega, \mathcal{A}, P)$,

$$X_n = X_0 \prod_{k=1}^{n} Y_k, \quad 1 \le n \le N, \tag{1.1}$$

where $(Y_k, \mathcal{A}_k)$ is a stochastic sequence and Y_k is bounded, $0 \le a_k \le Y_k \le b_k$, and let B_n denote a bond with constant interest rates $r_i \ge 0$, $a_i \le 1 + r_i \le b_i$, given by

$$B_n = B_0 \prod_{k=1}^{n} (1 + r_k). \tag{1.2}$$

The value process of a self-financing portfolio $\pi = (\beta, \gamma)$, where $\beta = (\beta_n)$, $\gamma = (\gamma_n)$ are predictable sequences, is denoted by $X_n^\pi = \beta_n B_n + \gamma_n X_n = X_0^\pi + \sum_{k=1}^n (\beta_k \Delta B_k + \gamma_k \Delta X_k)$. Finally consider a European call $f_N = f(X_1, \dots, X_N) \ge 0$, where f is a convex function of $X_1, \dots, X_N$.

Then the upper price of the claim f_N is given by

$$C^*(f_N, P) = \inf\{x : \exists \pi \text{ with } X_0^\pi = x \text{ and } X_N^\pi \ge f_N\}, \tag{1.3}$$

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the lower price is given by

$$C_*(f_N, P) = \sup\{x : \exists \pi \text{ with } X_0^\pi = x \text{ and } X_N^\pi \le f_N\}. \tag{1.4}$$

These prices are induced by the class of upper hedging strategies $H^*(x, f_N, P)$ resp. by the class of lower hedging strategies $H_*(x, f_N, P)$. In an arbitrage free market the prices have the representation

$$C^*(f_N, P) = \sup_{\widetilde{P} \in \mathcal{P}(P)} B_0 E_{\widetilde{P}} \frac{f_N}{B_N}, \tag{1.5}$$

$$C_*(f_N, P) = \inf_{\widetilde{P} \in \mathcal{P}(P)} B_0 E_{\widetilde{P}} \frac{f_N}{B_N}, \tag{1.6}$$

where $\mathcal{P}(P)$ is the set of all martingale measures $\widetilde{P}$ for the discounted process (S_n/B_n) equivalent to P, which by assumption is nonempty (see Shiryaev (1999, p. 514)).

In a single step model, $N = 1$, $B_1 = B_0(1+r)$, $X_1 = S_0(1+\varrho)$ it is shown in Shiryaev (1999, p. 404/5) that under an approximation assumption on P the upper price is given by the corresponding Cox-Ross-Rubinstein model (with $\varrho \in \{d, u\}$) while the lower price is attained at the masure P_* where $Y_1 = 1 + \varrho$ is constant, $Y_1 = 1 + r (= 1 + r_1)$.

For the proof of the essential upper bound by the Cox-Ross-Rubinstein formula

$$C^*(f_N, P) \le \mathrm{CRR}(f_N) \tag{1.7}$$

it is shown in Shiryaev (1999) that the optimal hedging portfolio in the CRR-model induces an upper hedge in the single step model. Equality in (1.7) then follows from (1.5) by a simple approximation argument.

The argument used in the one-period model can be extended to a general N-period model. Again the upper bound is given by a Cox-Ross-Rubinstein model while the lower bound corresponds to constant $Y_i = 1 + r_i$. This argument based on the super-hedging portfolios has been given in detail independently in Schulz (2001) and Wrede (2000) in the case where $Y_1, \dots, Y_N$ are independent, $f_N = f(X_N)$, f convex, $a_k = 1 + d, b_k = 1 + u, r_k = r, d < r < u$. In this case under an appropriate approximation assumption the upper price is given by

$$C^*(f_N, P) = \frac{1}{(1+r)^N} \sum_{k=0}^N \binom{n}{k} \left(\frac{r-d}{u-d}\right)^{N-k} \left(\frac{u-r}{u-d}\right)^k f\left(X_0(1+u)^{N-k}(1+d)^k\right). \tag{1.8}$$

The lower price is given by

$$C_*(f_N, P) = \frac{f\left(X_0(1+r)^N\right)}{(1+r)^N}. \tag{1.9}$$

In the following section we use some natural ordering properties in the family $\mathcal{P}(P)$ of equivalent martingale measures to determine the upper and lower prices in a general discrete time model.

2 Comparison of the N-period model to the Cox-Ross-Rubinstein model

A well known technique to increase the risk of a distribution is given by the notion of mean preserving spread resp. balayage. A probability measure Q_2 is a mean preserving spread of Q_1 in some interval $[a,b]$ if on this interval a mass is shifted to the borders of the interval while the mean is preserved. In terms of random variables: Z is a mean preserving spread of Y on the interval $[a,b]$ if

$$\text{a)} \qquad P^Z(B) = P^Y(B) \quad \text{for } B \cap [a,b] = \phi \tag{2.1}$$

$$\text{b)} \qquad P^{Y1_{[a,b]}(Y)} \preccurlyeq_{cx} P^{Z1_{[a,b]}(Z)}, \tag{2.2}$$

where $\preccurlyeq_{cx}$ is the convex ordering. (2.1) and (2.2) imply that

$$EZ1_{[a,b]}(Z) = EY1_{[a,b]}(Y).$$

Further, for any convex function f such that $Ef(Z)$ exists, holds

$$Ef(Y) \le Ef(Z); \tag{2.3}$$

so Y is smaller than Z in the convex ordering, $Y \le_{cx} Z$.

A particular construction of this type of mean preserving spread is the balayage $Z = Y_a^b$ of Y with distribution given by

$$P(Y_a^b = a) = \frac{1}{b-a} E1_{[a,b]}(Y)(b-Y) \tag{2.4}$$

and

$$P(Y_a^b = b) = \frac{1}{b-a} E1_{[a,b]}(Y)(Y-a). \tag{2.5}$$

All mass of Y in $[a,b]$ is sweaped out to the borders a, b of the interval while retaining the mean. This balayage has been discussed in the context of comparison of the expectation of products with independent rates in Schmitz and Wrede (2000). In the following we extend this ordering argument to the comparison of convex calls in general discrete time models.

Let $(X_n, \mathcal{A}_n)$ be a nonnegative stochastic sequence, let $X_n = \prod_{i=1}^n Y_i$, where $a_i \le Y_i \le b_i$, $a_i \le 1 \le b_i$ and let $f(x_1, \dots x_N)$ be a convex function. Obviously, any discrete time sequence with bounded returns is of this form. Then the value of the convex claim $f_N = f(X_1, \dots, X_N)$ can be compared to the value in a corresponding Cox-Ross-Rubinstein model.

Proposition 1 (Comparison lemma) *Let $Q \in \mathcal{P}(P)$ be an equivalent martingale measure for (X_n), then*

$$E_Q f(X_1, \dots, X_N) \le E_Q f(\widetilde{X}_1, \dots, \widetilde{X}_N)$$

where $\widetilde{X}_n = \prod_{i=1}^n \widetilde{Y}_i$, $(\widetilde{Y}_i)$ *are independent and* $Q^{\widetilde{Y}_i} = Q^{(Y_i)_{a_i}^{b_i}}$. $\widetilde{X}_1, \dots, \widetilde{X}_N$ *is a Cox-Ross-Rubinstein model and*

$$E_Q f(\widetilde{X}_1, \dots \widetilde{X}_N) = \sum_{k=0}^N \sum_{\substack{T\subset\{1,\dots,N\}\\|T|=k}} \prod_{k\in T} q_k^* \prod_{k\notin T} p_k^* \, f\left(\prod_{k\in T} a_k \prod_{k\notin T} b_k\right)$$

where $p_k^* = \frac{1-a_k}{b_k-a_k}$, $q_k^* = \frac{b_k-1}{b_k-a_k}$.

Proof: Consider the conditional expectation

$$E_Q(f(X_1,\dots,X_N)|X_{(N-1)}) = E_Q(f(X_1,\dots,X_{N-1},X_{N-1}Y_N|Y_1,\dots,Y_{N-1}) \tag{2.6}$$

where $X_{(N-1)} = (X_1,\dots,X_{N-1})$. Then from the martingale assumption we obtain $E_Q(Y_N|Y_1,\dots,Y_{N-1}) = 1$. Therefore, from (2.3) applied to the balayage w.r.t. the conditional distribution and from convexity of f in the last variable follows

$$\begin{aligned} E_Q(f(X_1,\dots,X_N)|X_{(N-1)}) &\le E_Q\left(f(X_1,\dots,X_{N-1},X_{N-1}(Y_N)_{a_N}^{b_N}|X_{(N-1)}\right) \\ &= E_Q\left(f(X_1,\dots,X_{N-1},X_{N-1}\widetilde{Y}_N)|X_{(N-1)}\right) =: H_N. \end{aligned} \tag{2.7}$$

Note that by construction of the balayage

$$Q\left((Y_N)_{a_N}^{b_N} = b_N|X_{(N-1)}\right) = \frac{1-a_N}{b_N-a_N} =: p_N^* \tag{2.8}$$

and

$$Q\left((Y_N)_{a_N}^{b_N} = a_N|X_{(N-1)}\right) = \frac{b_N-1}{b_N-a_N} =: q_N^* \tag{2.9}$$

are independent of the conditions. Therefore, $Q^{(Y_N)_{a_N}^{b_N}|X_{(N-1)}} = Q^{\widetilde{Y}_N}$ and $(Y_N)_{a_N}^{b_N}$ is independent from $X_{(N-1)} = (X_1,\dots,X_{N-1})$. From (2.7) we obtain

$$\begin{aligned} E_Q f(X_1,\dots,X_N) &\le E_Q H_N \\ &= p_N^* E_Q f(X_1,\dots,X_{N-1},X_{N-1}\cdot b_N) \\ &\quad + q_N^* E_Q f(X_1,\dots,X_{N-1},X_{N-1}\cdot a_N). \end{aligned} \tag{2.10}$$

Again the conditional distribution can be estimated from above in the convex order

$$\begin{aligned} Q^{X_{N-2}Y_{N-1}}|X_{(N-2)} &\preccurlyeq_{cx} Q^{X_{N-2}(Y_{N-1})_{a_{N-1}}^{b_{N-1}}|X_{(N-2)}} \\ &= Q^{X_{N-2}\widetilde{Y}_{N-1}|X_{(N-2)}} \end{aligned} \tag{2.11}$$

and $f(X_1,\dots,X_{N-2}Y_{N-1},X_{N-2}Y_{N-1}c_N)$, $c_N \in \{a_N,b_N\}$ is convex in Y_{N-1}. So we obtain

$$\begin{aligned} E_Q f(X_1,\dots,X_{N-1},X_{N-1}c_N) &\le E_Q f(X_1,\dots,X_{N-2}\widetilde{Y}_{N-1},X_{N-2}\widetilde{Y}_{N-1}c_N) \\ &= p_{N-1}^* E_Q f(X_1,\dots,X_{N-2}b_{N-1},X_{N-2}b_{N-1}c_N) \\ &\quad + q_{N-1}^* E_Q f(X_1,\dots,X_{N-2}a_{N-1},X_{N-2}a_{N-1}c_N). \end{aligned}$$

The result now follows by induction. $\square$

Remark 2 *We remark that for the proof of Proposition 1 we need that*

$$g(y_1,\dots,y_N) := f(x_1,\dots,x_N), \quad x_k = \prod_{i=1}^k y_i$$

is componentwise convex in y_i. This is fulfilled in particular under the condition that f is convex, as in Proposition 1, but it is a weaker condition.

Proposition 1 allows to compare the martingale $X_1,\dots,X_N$ with the Cox-Ross-Rubinstein martingale $\widetilde{X}_1,\dots,\widetilde{X}_N$ which produces higher costs of the claim $f_N = f(X_1,\dots,X_N)$ by indroducing higher risk.

Consider now the discrete time model as introduced in section 1, (1.1), (1.2), with $0 \le a_k \le 1 + r_k \le b_k$. Assume the approximation condition:

(A*): there exists a sequence $(Q_n) \subset \mathcal{P}(P)$ such that $Q_n^{(X_1,\dots,X_N)}$ converges weakly to the Cox-Ross-Rubinstein measure $Q^* = \otimes_{i=1}^N Q_i^*$ with

$$Q_k^*\{b_k\} = p_k^* = \frac{1 + r_k - a_k}{b_k - a_k}, \qquad Q_k^*(\{a_k\}) = q_k^* = \frac{b_k - 1 - r_k}{b_k - a_k}. \tag{2.12}$$

Condition (A*) is fulfilled if for all $\varepsilon > 0$ and $T \subset \{1,\dots,N\}$

$$P\left(\prod_{k\in T}[a_k, a_k+\varepsilon] \times \prod_{k\notin T}[b_k - \varepsilon, b_k]\right) > 0.$$

Then we conclude

Theorem 3 (Upper and lower price formula) *Under condition (A*) the upper and lower price for the discrete time N-component model* (1.1), (1.2) *are given by*

$$C^*(f_N, P) = B_0 E_{Q^*} \frac{f_N}{B_N}$$

and

$$C_*(f_N, P) = B_0 E_{Q_*} \frac{f_N}{B_N}.$$

Here Q^* is the Cox-Ross-Rubinstein martingale measure (see (2.12)*) while $Q_* = \otimes_{i=1}^N (Q_i)_*$, where $(Q_i)_*(\{1 + r_i\}) = 1$, is the one-point martingale measure for $\left(\frac{X_n}{B_n}\right)$.*

Note that the ordering argument for the lower bound is quite obvious.

3 A discrete time stochastic volatility model

The convex ordering argument for the martingale measures in section 2 based on the dual representation of the upper and lower prices is flexible enough to be applied to further models. Consider, e.g., the following stochastic volatility model which is a discrete analog to continuous time volatility models of the form

$$\frac{dS_t}{S_t} = \alpha_t dt + \sigma_t dM_t, \tag{3.1}$$

where M_t is a martingale. Let $(\mathcal{A}_n)$ be a filtration and let

$$S_n = \prod_{i=1}^n (\alpha_i + \sigma_i \varepsilon_i) \tag{3.2}$$

where $(\alpha_i, \mathcal{A}_i)$, $(\sigma_i, \mathcal{A}_i)$, $(\varepsilon_i, \mathcal{A}_i)$ are stochastic sequences, α_i, σ_i are predictable, and assume that ε_i are bounded

$$A_i \le \varepsilon_i \le B_i, \quad 1 \le i \le N, \quad A_i \le 0 \le B_i \tag{3.3}$$

and

$$E_Q(\varepsilon_i | \mathcal{A}_{i-1}) = 0 \tag{3.4}$$

for some given measure Q.

Then Q is a martingale measure for (S_n) if and only if $\alpha_i = 1, 1 \le i \le N$. Assume that σ_i is known to lie in between two deterministic bounds

$$0 \le \underline{\sigma_i} \le \sigma_i \le \overline{\sigma_i} \quad \text{where } 1 + \overline{\sigma_i} A_i \ge 0. \tag{3.5}$$

Consider again a convex claim

$$f_N = f(S_1, \dots, S_N). \tag{3.6}$$

Then with $Y_i = 1 + \sigma_i \varepsilon_i$ holds

$$0 \le 1 + A_i \overline{\sigma_i} =: a_i \le Y_i \le 1 + B_i \overline{\sigma_i} =: b_i \tag{3.7}$$

Assume as in section 2 that $0 \le a_i \le 1 + r_i \le b_i$. Then, from Theorem 3 we get an upper bound of the upper price by the corresponding CRR-model. As in Theorem 3 under an approximation assumption one can construct an approximating sequence of martingale measures such that the upper bound is attained. As a result we obtain:

Corollary 4 (Bounds for the stochastic volatility model)
Under assumption (3.5) *on the volatilites* (σ_i) *and* (3.3) *on the noise process* (ε_i) *holds:*

$$C^*(f_N, P_\sigma) \le C^*(f_N, P_{\overline{\sigma}}) = B_0 E_{Q^*_{\overline{\sigma}}} \frac{f_N}{B_N} \tag{3.8}$$

and

$$C_*(f_N, P_\sigma) \ge B_0 E_{Q_*} \frac{f_N}{B_N},$$

where P_σ *corresponds to a model with stochastic volatility* $\sigma = (\sigma_1, \dots, \sigma_N)$, $Q^*_{\overline{\sigma}}$ *is the CRR-martingale measure* $\bigotimes Q^*_{i,\overline{\sigma}}$ *with bounds* $a_i = 1 + A_i \overline{\sigma_i}$, $b_i = 1 + B_i \overline{\sigma_i}$. Q_* *corresponds to the one-point martingale measure.*

Remarks 5 *The lower bounds $\underline{\sigma_i}$ on the volatility σ_i do not enter the upper resp. lower price bounds. Shataev (1998) considered upper and lower prices in a related discrete time stochastic volatility model, where ε_i are conditionally Gaussian (and, therefore, unbounded). In this case one gets the trivial upper and lower bounds*

$$C^* = S_0, \quad C_* = (S_0 - K)^+ \tag{3.9}$$

for the European call in the model

$$S_n = S_0 e^{\sum_{i=1}^n h_i}, \quad h_n = \sigma_n \varepsilon_n,$$

where $\sigma_n > 0$ is any predictable positive volatility sequence. A result related to Corollary 4 for the continuous time stochastic volatility model with bounds on the volatility process was given in Frey and Sin (1999) and Frey (2000).

Conclusion: The comparison method for martingale measures used in this paper seems to be of potential use also for further models and types of claims. It shows that typically in discrete time models with unbounded support one can expect only trivial and too large arbitrage intervals. So hedging and arbitrage considerations are not of much use for determining useful option prices. In particular superhedging strategies typically seem not to be a useful tool in this type of non-complete models. A similar remark in the context of stochastic volatility models has been made in Frey and Sin (1999).

Acknowledgement: In a recent paper of Gushchin and Mordecki (2001) a characterization of attainability of the trivial upper and lower bounds by predictable characteristics in general semimartingale models is given. This result is based on a general statistical ordering concept in connection with comparison of option prices. In the N-component model independently a related comparison result to the Cox-Ross-Rubinstein model is given in this paper.

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