Variance minimization and random variables with constant sum

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Abstract

Motivated by the problem of variance minimization and the method of antithetic variates we consider the problem of construction of random variables with given marginals and constant sum. In the case of one dimensional symmetric, unimodal distributions we give a simple general construction. An alternative more complicated construction had been given previously by Knott and Smith (1998). In the multivariate case we consider the corresponding problem for affine transforms of products, elliptically contoured distributions, α-symmetric distributions and α-Cauchy distributions.

1 Introduction

A well known problem coming from the method of antithetic variates in Monte Carlo simulation is to construct real random variables $X_1, \ldots, X_n$ with given distributions $P_1, \ldots, P_n$ such that the variance of the sum is minimal

$$\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \min! \tag{1.1}$$

w.r.t. all similar constructions. For $n = 2$ the solution is given by the antithetic variates $X_1 = F_1^{-1}(U), X_2 = F_2^{-1}(1-U)$, where F_i are the distribution functions of P_i and U is uniform on $(0,1)$. For some cases like $P_i = \mathcal{B}(1, \vartheta)$ or $P_i = U(0,1)$ the uniform distribution on $(0,1)$ or $P_i = U(\{1, \ldots, m\})$ the uniform distribution on $\{1, \ldots, m\}$ and some other distributions, solutions have been constructed for general n (see Rachev and Rüschendorf (1998) for references). The idea of the construction is to try to concentrate the sum $S_n = \sum_{j=1}^{n} X_j$ at the expectation as much as possible. Obviously, for an optimal solution $X_1, \ldots, X_n$ any of the variables X_j has to be optimally coupled to $T_i = \sum_{j \neq i} X_j$ in the sense of antithetic variates but

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this necessary condition is not sufficient for optimality in general (see Rüschendorf and Uckelmann (1997)). For the dual problem of variance maximization it has been proved in a recent paper in Rüschendorf and Uckelmann (1998) that an idea of Knott and Smith (1994) to reduce the n-coupling problem to two coupling problems by optimal monotone coupling to the sum can be justified in general under some technical condition. This leads to some examples with explicit results.

A general characterization of optimal random variables for variance minimization has been given in Rüschendorf and Uckelmann (1997, 1998) where the problem is reduced to several two coupling problems – but for some more complicated coupling functionals. It is however not easy to solve these two coupling problems in general.

In this note we consider also the multivariate extension of the variance minimization problem

$$E\left\|\sum_{i=1}^{n} X_i\right\|^2 = \inf \tag{1.2}$$

where the inf is on all random variables X_i with distributions P_i on $\mathbb{R}^d$ given. An obvious solution of (1.2) is obtained if $X_i \stackrel{d}{=} P_i$ are constructed such that

$$\sum_{j=1}^{n} X_j = c. \tag{1.3}$$

Here $X_i \stackrel{d}{=} P_i$ denotes, that X_i has distribution P_i.

Knott and Smith (1998) gave a construction of a solution of (1.3) in dimension $d = 1, n = 3, c = 0$ if $P_1 = P_2 = P_3$ is absolutely continuous, symmetric and unimodal.
In the first part of this note we give a simplified construction of solutions of (1.3) in $d = 1$ and then in the second part provide some extensions to the multivariate case.

2 Symmetric unimodal one dimensional distributions

For the variance minimization problem (1.2) with $n = 3, P_1 = P_2 = P_3 = P$ a probability with a symmetric, unimodal density f Knott and Smith (1998) constructed a solution $X \stackrel{d}{=} Y \stackrel{d}{=} Z \stackrel{d}{=} P$ of (1.3) of the following type

$$X = R\cos U, Y = R\cos\left(U + \frac{2}{3}\pi\right), Z = R\cos\left(U - \frac{2}{3}\pi\right) \tag{2.1}$$

where U is uniformly distributed on $(0, 2\pi)$ independent of R. Then by the addition theorem for the cosine function

$$X + Y + Z = 0. \tag{2.2}$$

By means of the Mellintransform one gets the density of R in the form

$$\tilde{f}_R(t) = -t\frac{d}{dt}\int_1^\infty f(ut)\frac{2}{u\sqrt{u^2-1}}du \tag{2.3}$$

(see Knott and Smith (1998), formula (10)).

Formula (2.3) is not easy to evaluate in general. A simplified construction for random variables with constant sum is obtained as follows.

Consider at first the case of uniform distributions on $(-1,1)$. Define for $n=3$.

$$\begin{aligned} U_1 &:= U \stackrel{d}{=} U(-1,1) \\ U_2 &:= \begin{cases} -2U-1 & \text{if} \quad -1 \le U \le 0 \\ 1-2U & \text{if} \quad 0 < U \le 1 \end{cases} \\ U_3 &:= \begin{cases} U+1 & \text{if} \quad -1 \le U \le 0 \\ U-1 & \text{if} \quad 0 < U \le 1 \end{cases} \end{aligned} \tag{2.4}$$

then $U_i \stackrel{d}{=} U(-1,1)$ are uniformly distributed on $(-1,1)$ and (for $n=3$)

$$\sum_{i=1}^{3} U_i = 0. \tag{2.5}$$

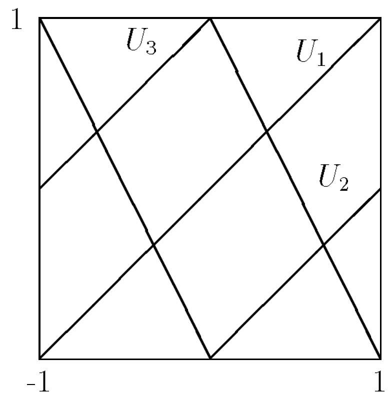


Figure 1: Construction of U_i for $n=3$

With the antithetic construction $\tilde{U}_1 = U, \tilde{U}_2 = -U$ one obtains for the case $n=2$

$$\tilde{U}_1 + \tilde{U}_2 = 0. \tag{2.6}$$

By combination of (2.5), (2.6) one obtains for any n

$$U_i \stackrel{d}{=} U(-1,1) \text{ with } \sum_{i=1}^{n} U_i = 0. \tag{2.7}$$

(see also Gaffke and Rüschendorf (1981), Rüschendorf and Uckelmann (1997)).

We next give a construction of random variables with constant sum and fixed symmetric unimodal distribution of the marginals for general n.

Theorem 2.1 (Random variables with constant sum) *Let P be a symmetric unimodal distribution on $(\mathbb{R}^1, \mathcal{B}^1)$ with a.s. differentiable Lebesgue density f. Then $f_R(x) := -xf'(x)$ is a Lebesgue density. If R is a random variable with density f_R, then $(X_1, \ldots, X_n)$, where $X_i = RU_i, 1 \le i \le n$ and (U_i) as in (2.7), is a solution of the variance minimization problem and $\sum_{j=1}^n X_j = 0$.*

Proof: Obviously by (2.7), $\sum_{j=1}^n X_j = R\sum_{j=1}^n U_j = 0$ so we have only to prove that f_R is a density and that $RU \stackrel{d}{=} P$.

By unimodality and symmetry of f it holds $f_R(x) = -xf'(x) \ge 0$ and

$$\begin{aligned}\int_{-\infty}^{\infty} f_R(x)dx &= 2\int_0^\infty -xf'(x)dx \\ &= 2\lim_{y\to\infty} -xf(x)\Big|_0^y + 2\int_0^\infty f(x)dx = 1;\end{aligned}$$

so f_R is a density.
For the characteristic function of the product RU we have

$$\begin{aligned}\varphi_{RU}(t) &= \int \varphi_U(xt)f_R(x)dx \\ &= -\int \tfrac{\sin(tx)}{t} f'(x)dx.\end{aligned}$$

With $h(x) := \frac{1}{t}\sin(tx)$ we obtain using symmetry of P

$$\begin{aligned}(\varphi_P - \varphi_{RU})(t) &= \int \cos(tx)f(x)dx + \int \tfrac{\sin(tx)}{t} f'(x)dx \\ &= \int h'(x)f(x)dx + \int h(x)f'(x)dx \\ &= \lim_{y\to\infty} h(x)f(x)\Big|_{-y}^y = \lim_{y\to\infty} \tfrac{1}{t}\sin(tx)f(x)\Big|_{-y}^y = 0.\end{aligned}$$

Therefore, $RU \stackrel{d}{=} P$ and $(X_1, \ldots, X_n)$ is a solution of the variance minimization problem. □

Remark 2.2 *The factorization in Theorem 2.1 is closely related to the wellknown factorization of symmetric unimodal distributions by Khinchin. The coupling distribution f_R in Theorem 2.1 can be calculated easily in examples.*

Example 2.3

a) For $P = N(0,\sigma^2)$, $f_P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$, one obtains $f_R(x) = \frac{1}{\sqrt{2\pi}\sigma^3} x^2 \exp\left(-\frac{1}{2\sigma^2}x^2\right)$ the density of a double-Maxwell distribution with parameter $\alpha = \frac{1}{\sigma^2}$

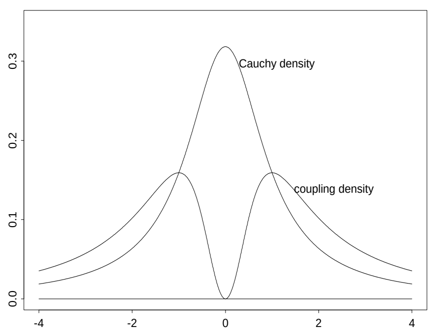
b) For P the Cauchy distribution with density

$$\begin{aligned} f_C(x) &= \frac{1}{\pi}\frac{1}{1+(\beta x)^2} \quad \text{one obtains} \\ f_R(x) &= \frac{2\beta}{\pi}\left(\frac{x}{1+(\beta x)^2}\right)^2 \end{aligned}$$

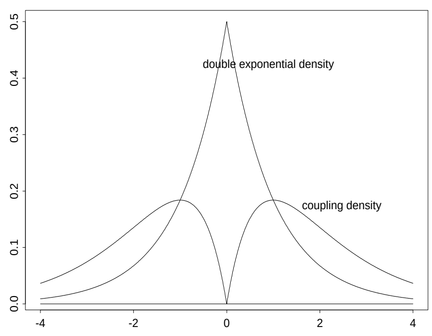
c) For P the Laplace-distribution with density

$$\begin{aligned} f(x) &= \frac{1}{2}\alpha \exp(-\alpha|x|) \quad \text{one obtains} \\ f_R(x) &= \frac{1}{2}\alpha^2 |x| \exp(-\alpha|x|). \end{aligned}$$

In this example the coupling distribution in Knott and Smith's construction (2.3) cannot be calculated in explicit form.



Cauchy distribution



Laplace distribution

Figure 2: Coupling densities

3 Extensions to multivariate distributions

In the multivariate case there is not a general construction principle as in Theorem 2.1. The idea is to transfer solutions in standard cases to more general situations. We consider some standard situations.

a) Affine transforms of products
Let $Q = \bigotimes_{i=1}^m Q_i$, where Q_i are symmetric, unimodal distributions with a.s. differentiable densities f_{Q_i}, then $Q_i \stackrel{d}{=} R_i U_i^j = X_i^j$, where $f_{R_i}(t) = -t f'_{Q_i}(t)$ and $(R_1, U_1^j), \ldots, (R_m, U_m^j)$ are independent, $U_i^j \stackrel{d}{=} U(-1,1)$ with $\sum_{j=1}^n U_i^j = 0$ as in (2.7). Let $P = Q^A$ be the image of Q under a linear mapping A and let $X^j = (X_i^j)_{1\le i\le m}, 1 \le j \le n$, then we have

Theorem 3.1 *Define $Y^j = AX^j, 1 \le j \le n$; then $Y^j \stackrel{d}{=} P, (Y^1, \ldots, Y^n)$ solves the multivariate variance minimization problem and $\sum_{j=1}^n Y^j = 0$.*

Proof: Since $X^j = (X_i^j)_{1\le i\le m} \stackrel{d}{=} \bigotimes_{i=1}^m Q_i$ we obtain $AX^j \stackrel{d}{=} P$. Also

$$\begin{aligned}\sum_{j=1}^n Y^j &= A \sum_{j=1}^n X^j = A\left(\sum_{j=1}^n R_i U_i^j\right)\\ &= A\left(R_i \sum_{j=1}^n U_i^j\right) = 0.\end{aligned}$$

Therefore, $X^1, \ldots, X_n$ solves the variance minimization problem. $\square$

b) α-symmetric distributions
A random vector $X = (X_1, \ldots, X_n)$ has an α-symmetric distribution on $\mathbb{R}^n$, $\alpha > 0$, if the characteristic function of X has a representation of the form

$$\varphi_X(t) = \Phi\left((|t_1|^\alpha + \cdots + |t_n|^\alpha)^{1/\alpha}\right) \tag{3.1}$$

for some real function Φ.

For $\underline{\alpha = 2}$ $\varphi_X(t) = \Phi\left(\|t\|^2\right)$ if and only if

$$\Phi(t) = \int_0^\infty \Omega(rt)\, dF(r) \tag{3.2}$$

for some distribution function F on $\mathbb{R}_+$ and

$$\Omega(t) = \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{t}\right)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(t),$$

J a Besselfunction. $\Omega(\|x\|)$ is the characteristic function of the uniform distribution on the unit sphere $S_{n-1} = \{x \in \mathbb{R}^d,\ \|x\| = 1\}$. Therefore, (3.1) and (3.2) (with $\alpha = 2$) are equivalent to a stochastic representation of the form

$$X \stackrel{d}{=} RU \tag{3.3}$$

where U is uniformly distributed on S_{n-1} and $R \geq 0$ is independent of U.

Similarly, for $\underline{\alpha = 1}$ X is 1-symmetric if and only if

$$X \stackrel{d}{=} R \left(\frac{U_1}{D_1^{1/2}}, \ldots, \frac{U_n}{D_n^{1/2}} \right)^T \tag{3.4}$$

where D is Dirichlet-distributed with parameter $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ and the density of $D' = (D_1, \ldots, D_{n-1})$ is given by

$$f_{D'}(x) = \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{1}{2}\right)^{-n} \left(\left(1 - \sum_{i=1}^{n-1} x_i\right) \prod_{i=1}^{n-1} x_i \right)^{-1/2},$$

$x_i \geq 0, \sum_{i=1}^{n-1} x_i \leq 1$ (cf. Gneiting (1998), Fang and Zhang (1990)).

Our next aim is to construct $X \stackrel{d}{=} Y \stackrel{d}{=} Z \stackrel{d}{=} P$ for 1- or 2-symmetric distributions P with constant sum $X + Y + Z = 0$. To that purpose consider the orthogonal mapping on $\mathbb{R}^{2n}$

$$S = \begin{pmatrix} S_1 & & 0 \\ & \ddots & \\ 0 & & S_n \end{pmatrix} \text{ where } S_i = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}. \tag{3.5}$$

As $S_i^2 = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$ we have

$$I_{2n} + S + S^2 = \begin{pmatrix} I_2 + S_1 + S_1^2 & & 0 \\ & \ddots & \\ 0 & & I_2 + S_n + S_n^2 \end{pmatrix} = 0. \tag{3.6}$$

Let U be uniformly distributed on the unit sphere S_{2n-1} in $\mathbb{R}^{2n}$, then by orthogonality of S $U \stackrel{d}{=} SU \stackrel{d}{=} S^2U$ and $U + SU + S^2U = 0$.
Define

$$\text{for } \alpha = 2 \quad X = RU, Y = RSU, Z = RS^2U \tag{3.7}$$

with R as in (3.3) and define

$$\text{for } \alpha = 1 \quad X = R(\frac{U_i}{\sqrt{D_i}}),\ Y = R\left(\frac{(SU)_i}{\sqrt{D_i}}\right),\ Z = \left(\frac{(S^2U)_i}{\sqrt{D_i}}\right) \tag{3.8}$$

with R as in (3.4).

Theorem 3.2 *(X, Y, Z) as defined in (3.7) for $\alpha = 2$ and in (3.8) for $\alpha = 1$ is a solution to the variance minimization problem for α-symmetric distributions, $\alpha = 1, 2$ for the case of even dimension $2n$ and $X + Y + Z = 0$.*

Remark:

1) Note that the construction above works in even dimension for three summands. For two summands we have the trivial construction with $X = RU, Y = R(-U)$ in any dimension. Therefore, we obtain by combination a construction with constant sum in even dimension for any number of summands.

2) The coupling distributions of R are given for $\alpha = 2$ by $R = \|X\|$. For $\alpha = 1$ R has the density

$$\begin{aligned} f_R(r) &= 2\Gamma^{-2}\left(\tfrac{n}{2}\right) r^{n-1} B_n^{m-1}(r^2) \\ \text{where } B_n(t) &= (-1)^{\frac{n-2}{2}} t^{\frac{n-1}{2}} \int_0^\infty \sin(u\sqrt{t})\Phi(u)du \\ \text{and } \varphi_P(t) &= \Phi(|t_1| + \cdots + |t_n|) \end{aligned}$$

(see Cambanis, Keener and Simons (1983)).

α-symmetric distributions $0 < \alpha \leq 2$

A characterization of the class $\Phi_n(\alpha)$ of functions Φ satisfying (3.1) is not known for $\alpha \neq 1, 2$. But it is known that $\Phi \in \Phi_\infty(\alpha)$ for $0 < \alpha \leq 2$ if and only if

$$\Phi(t) = \int_0^\infty e^{-rt^\alpha} dF(r) \tag{3.9}$$

and

$$\Phi_n(\alpha) = \{1\} \text{ for } n \geq 3, \quad \alpha \in (2, \infty]. \tag{3.10}$$

(For references see Gneiting (1998).)
So X is a finite segment of an infinite dimensional vector whose finite dimensional distributions are α-symmetric $0 < \alpha \leq 2$ if and only if

$$X \stackrel{d}{=} RY \tag{3.11}$$

where $R \geq 0$ is independent of Y which has independent and identically distributed symmetric stable components of index α and F is the distribution function of R. From the results of section 1 we therefore get for this subclass of α-symmetric distributions for any $n \in \mathbb{N}$ a construction of random variables $X^j, 1 \leq j \leq n$ such that $X^j \stackrel{d}{=} X$ and

$$\sum_{j=1}^n X^j = 0. \tag{3.12}$$

As consequence as in section 2 one obtains variance minimization results for affine transformations of α-symmetric distributions by (AX, AY, AZ) where $X \stackrel{d}{=} Y \stackrel{d}{=} Z \stackrel{d}{=} Q$ is α-symmetric, $P = Q^A$ and $X + Y + Z = 0$ as constructed above. $\square$

Examples 3.3

A) <u>Normal distribution</u>
Let $P = N(0, \Sigma), \Sigma > 0$, then $\Sigma = AA^T$ and we can apply both constructions (that in a) and that in b)) to this example.

1) For $Q = N(0, I)) = \bigotimes_{i=1}^m N(0,1)$ the coupling vector $R = (R_1, \ldots, R_m)$ has the density

$$\begin{aligned} f_R(x) &= \Pi_{i=1}^m \frac{1}{\sqrt{2\pi}} x_i^2 \exp\left(-\frac{1}{2}x_i^2\right) \\ &= (2\pi)^{m/2} \exp\left(-\frac{1}{2}\|x\|^2\right) h(x) \end{aligned} \tag{3.13}$$

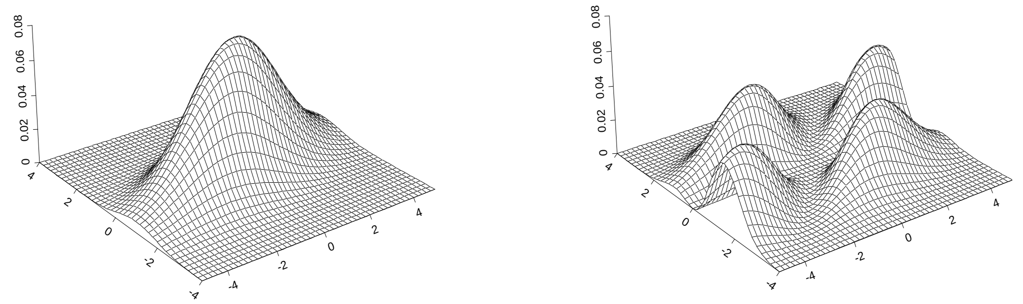
with $h(x) = \Pi_{i=1}^m x_i^2$. So AR has the density

$$f_{AR}(x) = h\left(A^{-1}x\right) f_P(x)$$

and an optimal n-tuple is given by

$$(ARU^1, ARU^2, \ldots, ARU^n) \text{ with } RU^j = (R_1U_1^j, \ldots, R_mU_m^j) \tag{3.14}$$

where $U^j = (U_1^j, \ldots, U_m^j)$ have independent uniform components in $[-1,1]$ and as in a) $\sum_{j=1}^n U_i^j = 0, \quad 1 \le i \le m$. The following figure shows the coupling density of AR for $m = 2, n = 3$, $\sum = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$



normal density f_Σ coupling density f_{AR}

Figure 3: Normal density and coupling distribution

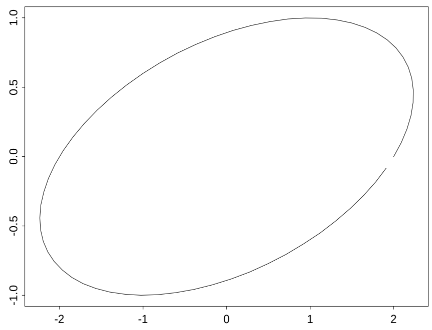
2) $N(0, I)$ is 2-symmetric, so the second construction with $U^i = S^iU$, $i = 0,1,2$ can be applied and we obtain the solution (for $n = 3$)

$$(RAU^1, RAU^2, RAU^3). \tag{3.15}$$

U^i is uniformly distributed on the unit sphere S^2. By calculating the densities of $\|X\|$ we obtain

$$f_R(t) = \Gamma\left(\frac{n}{2}\right)^{-1} t^{n-1} \exp\left(-\frac{1}{2}t^2\right), \quad t \geq 0. \tag{3.16}$$

The following figure gives the support of AU (an ellipse) and the coupling density f_R for $P = N(0, \Sigma)$ with $\Sigma = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$ as in 1).



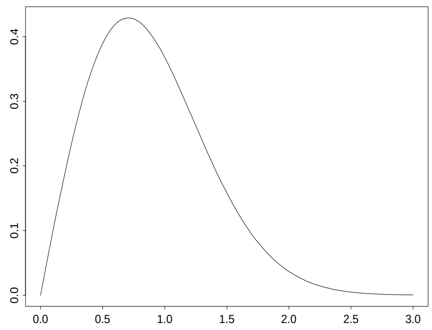


Figure 4: density f_{AU} and coupling density f_R

B) Elliptically contoured distributions
A random vector X has an elliptically contoured distribution denoted by $EC_m(0, \Sigma, \Phi)$ if the characteristic function φ_X is of the form $\varphi_X(t) = \Phi(t^T \Sigma t)$. A special case are normal distributions $N(0, \Sigma)$ where $\Phi(u) = \exp\left(-\frac{u}{2}\right)$. Then $X \stackrel{d}{=} EC_m(0, \Sigma, \Phi)$ with $rg\Sigma = k$ if and only if $X \stackrel{d}{=} RA^T U$ where U is uniformly distributed on the unit sphere in $\mathbb{R}^k$, A is a $k \times m$ matrix with $\Sigma = A^T A$, $R \geq 0$ is a real random variable independent of U with characteristic function Φ (see Cambanis, Huang and Simons (1981) and Fang and Zhang (pg. 65, 1990)). Therefore, as in the normal example in A 2) we obtain a construction of $X^j \stackrel{d}{=} EC_m(0, \Sigma, \Phi)$ with $\Sigma_{j=1}^n X^j = 0$ in even dimension.

C) Cauchy-distributions
For $X = (X_1, \ldots, X_n) \stackrel{d}{=} P$ with stochastically independent Cauchy-distributed components, $X_i \stackrel{d}{=} C(1), 1 \leq i \leq n$, we have

$$\varphi_P(t) = \Pi_{i=1}^n \varphi_{X_i}(t_i) = \exp\left(-(|t_1| + \cdots + |t_n|)\right) \tag{3.17}$$

P is 1-symmetric and the coupling distribution R in Theorem 3.2 has the density

$$f_R(t) = 2\Gamma(n)\Gamma\left(\frac{n}{2}\right)^{-2} t^{n-1} \frac{1}{(1-t^2)^n}, \quad t \geq 0 \tag{3.18}$$

(see Cambanis, Keener and Simons (1983)).

D) α-Cauchy distribution
The m-dimensional α-Cauchy distribution $C_m^{(\alpha)}$ has the density

$$c_m^{(\alpha)}(x) = \frac{2}{J_{m+1}(\alpha)} \frac{1}{\left(1 + \sum_{j=1}^m |x_j|^\alpha\right)^{(m+1)/\alpha}} \tag{3.19}$$

for any $\alpha > 0$. Let $X = (X_1, \dots, X_{m+1})$ be uniformly distributed on the unit sphere $S_{\alpha,m}$ in $\mathbb{R}^{m+1}$ w.r.t. $\| \|_\alpha$, then

$$\left(\frac{X_1}{X_{m+1}}, \dots, \frac{X_m}{X_{m+1}}\right) \stackrel{d}{=} C_m^{(\alpha)} \tag{3.20}$$

(see Szablowski (1998), Lemma 3.1).
Also it is known that if $Y_1, \dots, Y_{m+1}$ are independent, each with density

$$f_\alpha(x) = \frac{1}{2} \frac{\alpha^{1-1/2}}{\Gamma(1/2)} \exp\left(-\frac{|x|^\alpha}{\alpha}\right), \tag{3.21}$$

then $X = \frac{Y}{\|Y\|_\alpha}$ is uniformly distributed on the norm α unit sphere in $\mathbb{R}^{m+1}$ where $Y = (Y_1, \dots, Y_{m+1})$ (see Rachev and Rüschendorf (1991)). Therefore, according to our first one-dimensional construction method we find for any $n \in \mathbb{N}$ $Y_{m+1}, \left\{Y_i^j, 1 \le j \le n\right\}$, such that $Y_1^j, \dots, Y_m^j, Y_{m+1}$ are independent with density f_α and $\sum_{j=1}^n Y_i^j = 0$, $1 \le i \le m$. Then define for $1 \le j \le n$

$$Y^j = \frac{1}{Y_{m+1}} \left(Y_1^j, \dots, Y_m^j\right). \tag{3.22}$$

Since with $\widetilde{Y}^j = \left(Y_1^j, \dots, Y_m^j, Y_{m+1}\right)$, $\frac{\widetilde{Y}^j}{\|\widetilde{Y}^j\|_\alpha}$ is uniform on the unit sphere, we obtain by (3.20) that Y^j is $C_m^{(\alpha)}$-distributed and

$$\sum_{j=1}^n Y^j = 0 \tag{3.23}$$

So $Y^1, \dots, Y^n$ solve the variance minimization problem for α-Cauchy distributions.

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