Worst case portfolio vectors and diversification effects

Ludger Rüschendorf
University of Freiburg

Abstract

We consider the problem of identifying the worst case dependence structure of a portfolio $X_1, \dots, X_n$ of d-dimensional risks, which yields the largest risk of the joint portfolio. Based on a recent characterization result of law invariant convex risk measures the worst case portfolio structure is identified as a μ-comonotone risk vector for some worst case scenario measure μ. It turns out that *typically* there will be a diversification effect even in worst case situations. The only exeptions arise when risks are measured by translated max correlation risk measures. We determine the worst case portfolio structure and the worst case diversification effect in several classes of examples as for example in elliptical, in Euclidean spherical and in Archimedian type distribution classes.

Key words: dependent risks, risk bounds, diversification, comonotone vectors

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1 Introduction

In this paper we consider the problem to determine sharp risk bounds for a portfolio $X = (X_1, \dots, X_n)$ of n risk vectors $X_i \in \mathbb{R}^d$. In particular we describe in a general way the dependence structure yielding the highest risk (worst case portfolio structure) and we determine corresponding diversification effects. Here the risk distributions P_i of X_i are fixed and the risk of the joint portfolio X is measured by some risk measure $\varrho = \varrho(X)$. In this paper we concentrate essentially on risks depending on the *joint portfolio* $\varrho = \varrho(\sum_{i=1}^n X_i)$. Our developments can be extended in principle to some further aggregation risk functionals like to the maximal risk $\varrho = \varrho(\max_{i \le n} X_i)$. Some general class of multivariate risk functions of this type have been introduced in [BRü06] and [Rü06].

In the one dimensional case $d = 1$ there has been a lot of work on the problem of getting bounds for the risk of X induced by dependence between the components X_i. Here the notion of comonotonicity describes in a general way the worst case dependence structure. Let $X^+ = (F_1^{-1}(U), \dots, F_n^{-1}(U))$ denote a comonotone vector, where F_i are the distribution functions of P_i, F_i^{-1} are the generalized inverses of F_i and U is uniformly distributed on $(0,1)$, $U \sim U(0,1)$. Then by a classical result of Meilijson and Nadas (1979) for $X_i \in L^1$

$$\sum_{i=1}^n X_i \leq_{\mathrm{cx}} \sum_{i=1}^n F_i^{-1}(U), \tag{1.1}$$

where $\leq_{\mathrm{cx}}$ denotes the convex order. As consequence one obtains that

$$\varrho\Big(\sum_{i=1}^n X_i\Big) \leq \varrho\Big(\sum_{i=1}^n F_i^{-1}(U)\Big), \tag{1.2}$$

for all law invariant convex risk measures ϱ on L^1. In this sense the comonotone dependence structure is the worst case dependence structure in the one dimensional case for the joint portfolio $\sum_{i=1}^n X_i$ uniformly over all law invariant convex risk measures. In other words,

$$\sup_{\tilde{X}_i \sim P_i} \varrho\Big(\sum_{i=1}^n \tilde{X}_i\Big) = \varrho\Big(\sum_{i=1}^n F_i^{-1}(U)\Big), \tag{1.3}$$

where the sup in the left-hand side of (1.3) is over all random variables $\tilde{X}_i$ with distribution P_i. If we consider the maximal risk $\max_{i \leq n} X_i$ however, the comonotone vector is not maximal but even is of minimal risk

$$\varrho(\max_{i\leq n} F_i^{-1}(U)) = \inf_{\tilde{X}_i \sim P_i} \varrho(\max_{i \leq n} \tilde{X}_i). \tag{1.4}$$

Even in dimension 1 there are however several unsolved extremal problems of similar type of interest. For example the problem of determining sharp upper bounds for the value at risk at level α, Var_α, i.e. to determine

$$\sup_{\tilde{X}_i \sim P_i} \mathrm{Var}_\alpha \Big(\sum_{i=1}^n \tilde{X}_i\Big) \tag{1.5}$$

has been solved only for $n = 2$ and for some examples for $n \geq 3$ (see the survey [Rü05]). The comonotone dependence structure is not the worst case dependence for this risk functional. In some recent work Embrechts and Puccetti (2006a,b) have established good approximations for (1.5) in some generality.

In this paper we consider the problem of worst case dependence corresponding to (1.3) for $d \geq 1$. There is no general notion of comonotonicity available in $d \geq 1$

as in $d = 1$ and even the identical pair (X_1, X_1), for $X_1 \in \mathbb{R}^d$ may be not a worst case dependence structure due to negative dependence between the components (see [Rü04] and [PS10] for a discussion). But a suitable notion of *μ-comonotonicity* has been introduced in a recent paper by Ekeland, Galichon, and Henry (2009). Based on a recent representation result for convex law invariant risk measures we will show that a worst case dependence structure of a portfolio $(X_1, \dots, X_n)$ can be characterized by μ-comonotonicity of $X_1, \dots, X_n$ with respect to a worst case scenario measure μ. Section two deals with some basic representation properties of convex law invariant risk measures on $L^p_d = L^p_d(P)$. In Section three we then discuss the worst case dependence structure and consider several examples in Section 4. In particular we determine worst case portfolios for ellipical distributions, for discrete distributions, for distributions of spherical type and for Archimedian type distributions. It turns out that even under worst case conditions there *typically* arise diversification effects.

Our results are related to some recent developments in Ekeland, Galichon, and Henry (2009). In their paper the notion of a *strongly coherent* risk measure is introduced. A law invariant coherent lsc risk measure is called strongly coherent if for all risk vectors X_i holds

$$\sup_{\tilde{X}_i \sim X_i} \varrho\Big(\sum_{i=1}^n \tilde{X}_i\Big) = \sum_{i=1}^n \varrho(X_i). \tag{1.6}$$

This notion is motivated by the wish to *prevent giving an unnecessary premium to conglomerates and to avoid imposing an overconservative rule to the banks.* In general, the left-hand side in (1.6) is smaller than or equal to the right-hand side. Thus on the first view the notion of strong coherence of a risk measure seems to be quite intuitive and well motivated. The authors establish as a main result that strong coherence of a coherent risk measure is equivalent to ϱ being a max correlation risk measure. Thus only max correlation coherent risk measures avoid *extra premiums.* This result is considered as a particular indication for the relevance of max correlation risk measures. It can be viewed upon as a multivariate version of Kusuoka's (second) Theorem on coherent risk measures in $d = 1$ which states that max correlation risk measures for $d = 1$ (equivalently *spectral risk measures*) are the only comonotone additive risk measures. The proof given in [EGH09] of this result is direct and quite involved.

Max-correlation risk measures were introduced in [Rü06]. In that paper an extension of the (first) Kusuoka Theorem was established giving a representation of law invariant convex risk measures in $d \geq 1$ which is based on max correlation risk measures. This representation result allows us in this paper to describe in *explicit* form not only the worst case risk but also to characterize the worst case dependence structure. In particular based on this representation result we obtain a simple proof of the characterization result of strongly coherent risk measures of [EGH09], even extended to convex (not necessarily coherent) risk measures under some additional continuity assumption. As consequence typically – even under worst case dependence – a diversification effect takes place and we obtain a means to describe this effect.

The point of view towards max correlation risk measures in this paper is different from that in [EGH09]. [EGH09] point out – based on their characterization result of strongly coherent risk measure – the particular relevance of max correlation risk measures. Clearly the above mentioned aspect of *strong coherence* is one interesting property of risk measures. It is however not the only relevant property of measuring risk and should not prevent a risk manager to measure further important aspects of risks which are not realised when restricting to strongly coherent risk measures. We consider the main importance of max correlation risk measures coming essentially from the Kusuoka type representation result of law invariant risk measures – so mainly consider them as an important tool. Max correlation risk measures are intuitively well motivated but there are an abundance of well motivated further risk measures. If e.g. based on historical observations it is known that there are typically two different scenario directions μ_1, μ_2 of relevance then it seems natural to consider both of them and report the maximum of risks in both directions which is a genuine convex risk measure. There are a lot of further portfolio risk measures besides max correlation risk measures with relevance and of importance for the description of risks. The main focus in this paper is to describe that even under worst case dependence there will arise a diversification effect for *general* convex risk measures, to identify this diversification effect and also to identify the corresponding worst case portfolio structure.

2 Law invariant risk measures for portfolio vectors

There is a large and detailed literature on convex risk measures for risks in L^∞ and dimension $d = 1$. We refer in particular to the presentations in [D02], [FöS02, FöS04], [FrR02], and [JST06]. To describe the contribution of dependence in a portfolio on the risk of the portfolio it is of interest to extend risk measures to the multivariate case.

In this section we present and extend some recent representation results for risk measures to the multivariate case $d \geq 1$ and to unbounded risks. Let $(\Omega, \mathcal{A}, P)$ be a nonatomic probability space and let $L_d^p = L_d^p(P)$ be the set of all random vectors $X = (X_1, \dots, X_d)$ with p-fold integrable components, $X_i \in L^p(P)$, $1 \leq p \leq \infty$. $\varrho : L_d^p \to (-\infty, \infty]$ is called a *convex risk measure* on L_d^p if ϱ is monotone, convex, and cash invariant. Here monotonicity is with respect to the inverse order, $X_1 \leq X_2$ implies that $\varrho(X_1) \geq \varrho(X_2)$. We also assume the normalization condition that $\varrho(-me_i) = m$, $1 \leq i \leq d$. To avoid problems with the sign we typically will switch to the *insurance version* Ψ of a risk measure $\Psi(X) = \varrho(-X)$ which is monotone in the usual order. Alternatively one could also consider the utility version $u(X) = -\varrho(X)$.

Based on the Fenchel–Moreau Theorem the following representation result extends the case $p = 1$ and L^∞ to unbounded risks in L^p and $d \geq 1$ (see [BRü06, RS06, FiS07, CL09] and [KaRü09]). Let M_d be the set of $Q = (Q_1, \dots, Q_d)$ where Q_i are probability measures on $(\mathbb{R}^1, \mathcal{B}^1)$.

Theorem 2.1 (Representation result) *ϱ is a proper convex, lower semicontinuous risk measure on L^p_d, if and only if*

$$\varrho(X) = \sup_{Q \in \mathcal{Q}_{d,p}(P)} (E_Q(-X) - \alpha(Q)), \tag{2.1}$$

where the penalty α can be chosen as Legendre–Fenchel conjugate

$$\alpha(Q) = \sup_{X \in L^p_d} (E_Q(-X) - \varrho(X)),$$

$\mathcal{Q}_{d,p} = \mathcal{M}^q_d = \{Q = (Q_1, \dots, Q_d) \in M_d : \frac{dQ_i}{dP} \in L^q\}$, $1 \leq p < \infty$ is the set of d tuples of probability measures with q-integrable P-density, where $\frac{1}{q} + \frac{1}{p} = 1$, while for $p = \infty$ $\mathcal{Q}_{d,\infty} = ba_d(P)$ is the set of d tuples of P-continuous normed finitely additive measures.

There are several continuity results and results on the attainment of the sup in the representation theorem (2.1) in the literature (see [FiS07, CL09, KaRü09, FiS09]). We state a version of these results for finite risk measures on L^p_d.

Theorem 2.2 *ϱ is a finite convex risk measure on L^p_d, $1 \leq p \leq \infty$, if and only if*

$$\varrho(X) = \max_{Q \in \mathcal{Q}} \{E_Q(-X) - \varrho^*(\mathcal{Q})\} \tag{2.2}$$

for some representation set $\mathcal{Q} \subset \mathcal{Q}_{d,p}$ such that $\mathcal{D} = \{\frac{dQ_i}{dP}, 1 \leq i \leq d, Q \in \mathcal{Q}\}$ is weakly closed in L^q.

We call a convex risk measure *strongly continuous*, when the representation set $\mathcal{Q}$ in (2.2) can be chosen weakly compact in L^q.

Remark 2.3 *a) Any finite coherent risk measure is strongly continuous (see [I03], [KaRü09]). For finite convex risk measures one obtains (by the arguments in Proposition 2.10 in [KaRü09]) strong continuity of ϱ if the convex conjugate ϱ^* of ϱ is bounded above on its support. Strong continuity of general finite convex risk measures ϱ on L^p_d, as stated in Theorem 2.11 of [KaRü09], does however not seem to hold true.*

b) [RS06] resp. [CL09, Theorem 4.3] establish that nonemptyness of the algebraic interior, core $(\operatorname{dom} \varrho) \neq \emptyset$ *implies finiteness of ϱ.*

A convex risk measure ϱ on L_d^p is called *law invariant* if $X \stackrel{d}{=} Y$ (equality in distribution) implies that $\varrho(X) = \varrho(Y)$, i.e. $\varrho(X) = \varrho(P^X)$ depends only on the distribution P^X of X with respect to P. Let

$$Y \in D = \{(Y_1, \dots, Y_d) : Y_i \geq 0 \; P\text{-a.s.}, Y_i \in L^q, E_P Y_i = 1, 1 \leq i \leq d\} \subset L_d^q \tag{2.3}$$

be the set of d-tuples of P-densities and let

$$\Psi_Y(X) := EX \cdot Y \tag{2.4}$$

denote the correlation coefficient of X and Y (up to normalization). Then with $\mu = P^Y$ the distribution of Y, we define the *maximal correlation risk measure* in direction Y (resp. μ) by

$$\begin{aligned} \hat{\Psi}_Y(X) &= \sup_{\tilde{X} \sim X} E\tilde{X} \cdot Y \\ &= \sup_{\tilde{Y} \sim \mu} EX \cdot \tilde{Y} = \Psi_\mu(X) \end{aligned} \tag{2.5}$$

(see [Rü06]). $\Psi_\mu = \hat{\Psi}_Y$ is in fact a law invariant coherent risk measure.

Remark 2.4 *In dimension $d = 1$ it holds by the classical Hoeffding-Fréchet bounds (see e.g. [FöS04, Rü05])*

$$\hat{\Psi}_Y(X) = \Psi_\mu(X) = \int_0^1 F_X^{-1}(u) F_Y^{-1}(u) du. \tag{2.6}$$

By partial integration Ψ_μ can be written as a weighted average value at risk. *In dimension $d \geq 1$*

$$\Psi_\mu(X) = \hat{\Psi}(X, Y) = \sup \Big\{ \int x \cdot y \; d\tau(x, y) : \tau \in M(P^X, P^Y) \Big\} \tag{2.7}$$

can be seen as an instance of the classical optimal mass transportation problem (see [RaRü98]).

The following is an extension of the Kusuoka representation result for convex law invariant risk measures as given for $d = 1$ and L^∞ in [FrR05] and for $d \geq 1$ and L^∞ in [Rü06].

Theorem 2.5 (Law invariant convex risk measures) *Let Ψ be a finite convex law invariant risk measure on L_d^p, $1 \leq p \leq \infty$. Then*

$$\Psi(X) = \max_{\mu \in A} (\Psi_\mu(X) - \alpha(\mu)), \tag{2.8}$$

where Ψ_μ is the maximal correlation risk measure in direction μ, $\alpha(\mu)$ is some law invariant penalty function, and $A \subset \mathcal{Q}_{d,p}$ is weakly closed (i.e. $\mathcal{D}_d^q = \{\frac{dQ_i}{dP} : 1 \leq i \leq d, Q \in A\}$ is weakly closed in L^q).

Proof: The representation result follows for $p < \infty$ as in [Rü06] where it is stated in the case $d = 1, p = \infty$. The attainment of the sup follows from [CL09, Theorem 4.3]. In the case $p = \infty$ the analogous representation result holds true as follows from [Rü06] and the Fatou continuity result due to [JST06]. $\square$

As consequence of Theorem 2.5 and Remark 2.3 finite law invariant coherent risk measures on L_d^p have a representation of the form

$$\Psi(X) = \max_{\mu \in A} \Psi_\mu(X) \tag{2.9}$$

for some weakly compact set $A \subset \mathcal{Q}_{d,p}$. Thus the maximal correlation risk measures are the building blocks of the class of all law invariant risk measures. The statement of Theorem 2.5 implies in particular that the *max* in the representation of Ψ is attained. A scenario measure $\mu \in \mathcal{Q}$ is called *worst case scenario* measure (for the risk X), if the max is attained in μ, i.e.

$$\Psi(X) = \Psi_\mu(X) - \alpha(\mu).$$

Remark 2.6 *A convex risk measure Ψ on L_d^p, $1 \leq p \leq \infty$, is law invariant if and only if it is consistent w.r.t. the increasing convex order ($\leq_{\mathrm{icx}}$) resp. the convex order $\leq_{\mathrm{cx}}$ on L_d^p, i.e.*

$$X \leq_{\mathrm{cx}} Y \textit{ (resp. } X \leq_{\mathrm{icx}} Y\textit{) implies } \Psi(X) \leq \Psi(Y) \tag{2.10}$$

(see [BRü06] for the case $p = \infty$). Thus an increase in convex order implies an increase in risk for all law invariant convex risk measures Ψ. This property however is not useful in order to identify worst case dependence structures. If X, Y have identical marginal distributions, i.e. $X_i \sim Y_i$ (where $X_i \sim Y_i$ means equality in distribution) and if $X \leq_{\mathrm{cx}} Y$ then by a well-known result in stochastic ordering already $X \sim Y$ holds. As consequence of this observation we obtain that there is no worst case dependence structure uniformly for all convex law invariant risk measures. A worst case dependence structure however can be given and described (even uniformly) for certain aggregation functionals as in (1.2)*.*

3 Worst case joint portfolios and diversification

Let $M^1(\mathbb{R}^d, \mathcal{B}^d)$ denote the class of probability measures on $(\mathbb{R}^d, \mathcal{B}^d)$. Let $X_i \sim P_i \in M^1(\mathbb{R}^d, \mathcal{B}^d)$, $1 \leq i \leq n$, $X_i \in \mathcal{L}_d^p$ be a portfolio of risks with finite p-th moments and let Ψ be a risk measure on the set of joint portfolios. Then $X = (X_1, \dots, X_n)$ is a worst case portfolio w.r.t. Ψ if it maximizes the risk

$$\Psi(X_1, \dots, X_n) = \sup_{\tilde{X}_i \sim X_i} \Psi(\tilde{X}_1, \dots, \tilde{X}_n). \tag{3.1}$$

In dimension $d = 1$ the comonotone dependence structure represents in some general sense a worst case dependence structure *independent* of the convex law invariant risk measure Ψ. See the survey on results of this type in [Rü05]. In dimension $d \geq 1$ there does not exist an analogous notion of comonotonicity. For some types of risk measures it is however possible to construct *explicitly* worst case dependence structures. Consider for example a risk measure Ψ of the series form

$$\Psi(X) = \Psi_1(X_1, X_2) + \Psi_2(X_2, X_3) + \cdots + \Psi_{n-1}(X_{n-1}, X_n). \tag{3.2}$$

By an iterative coupling argument we obtain

$$\sup_{\tilde{X}_i \sim X_i} \Psi(\tilde{X}_1, \dots, \tilde{X}_n) = \sum_{i=1}^{n-1} \sup_{\substack{\tilde{X}_i \sim X_i \\ \tilde{X}_{i+1} \sim X_{i+1}}} \Psi_i(\tilde{X}_i, \tilde{X}_{i+1}), \tag{3.3}$$

i.e. the worst case pair-wise dependence structures can be combined iteratively to yield a worst case joint dependence structure. But this possibility of recursive constructions is not typical.

We restrict in the following to risk measures of the joint portfolio of the form $\Psi(\sum_{i=1}^n X_i)$, where Ψ is a finite, convex, law invariant risk measure on L_d^p. A portfolio $X = (X_1, \dots, X_n)$, $X_i \in L_d^p$, is called a *worst case portfolio* with respect to Ψ if

$$\Psi\Big(\frac{1}{n} \sum_{i=1}^n X_i\Big) = \sup_{\tilde{X}_i \sim X_i} \Psi\Big(\frac{1}{n} \sum_{i=1}^n \tilde{X}_i\Big). \tag{3.4}$$

For convex risk measures Ψ we consider (in some contrast with (3.1)) the worst case risk of the average portfolio. In the case of coherent risk measures Ψ we can equivalently consider the risk $\Psi(\sum_{i=1}^n X_i)$ of the joint portfolio (as in (3.1)).

In the first case we assume that $\Psi = \Psi_\mu$ is a max correlation risk measure and thus a coherent risk measure, for some scenario measure (measure of direction) $\mu \in \mathcal{M}_d^q$.

A portfolio $X = (X_1, \dots, X_n)$, $X_i \sim P_i \in \mathcal{M}_d^p$ is called *μ-comonotone* (see [EGH09]) if for some density vector $Y \sim \mu$, $Y \in D_d^q$, holds that all X_i are optimally coupled to Y,

$$X_i \sim_{\mathrm{oc}} Y, \quad 1 \leq i \leq n. \tag{3.5}$$

Here optimally coupled is defined in the sense of optimal transportation, i.e.

$$\Psi_\mu(X_i) = \sup_{\tilde{Y} \sim Y} EX_i \cdot \tilde{Y} = EX_i \cdot Y, \quad 1 \leq i \leq n. \tag{3.6}$$

From results on optimal transportation there exist optimal couplings $X_i \sim_{\mathrm{oc}} Y$ of P_i, μ where $Y \in D_d^q$, $Y \sim \mu$ can be chosen independent of i (see [RüRa90, RaRü98]). By the characterization of optimal couplings in [RüRa90]

$$X_i \in \partial f_i(Y) \text{ a.s. for some } f_i \text{ convex, lsc,}$$

it follows, that

$$\sum_{i=1}^{n} X_i \in \partial \textstyle\sum_{i=1}^{n} f_i\,(Y) \text{ and } \sum_{i=1}^{n} f_i \text{ is convex and lsc.}$$

Thus

$$\sum_{i=1}^{n} X_i \sim_{\mathrm{oc}} Y, \tag{3.7}$$

the sum of X_i is optimally coupled to Y. This also follows from a simple direct argument. For this observation see e.g. [RüU97] or [EGH09, Theorem 1]. As consequence one obtains the following characterization of worst case dependence for max correlation risk measures (see [EGH09, Theorem 1] in case $p = q = 2$).

Proposition 3.1 *Let $\Psi = \Psi_\mu$ be a max correlation risk measure on L_d^p with scenario measure $\mu \in \mathcal{M}_d^q$. Then for $X_i \in L_d^p$ with distributions P_i, $X_i \sim P_i$, $1 \le i \le n$ holds: $(X_1, \dots, X_n)$ is a worst case dependence structure for Ψ_μ if and only if $X_1, \dots, X_n$ are μ-comonotone.*

Further in this case there is no worst case diversification effect:

$$\sup_{\widetilde{X}_i \sim X_i} \Psi_\mu\Big(\sum_{i=1}^{n} \widetilde{X}_i\Big) = \Psi_\mu\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} \Psi_\mu(X_i). \tag{3.8}$$

As mentioned in the introduction there are many relevant convex risk measures besides max correlation risk measures. We will see in the following that a diversification effect is present for all of them even under worst case portfolios. Let Ψ be now a finite, convex, law invariant risk measure on L_d^p. By Theorem 2.5 Ψ has a representation of the form

$$\Psi(X) = \max_{\mu \in A}(\Psi_\mu(X) - \alpha(\mu)), \tag{3.9}$$

where $A \subset \mathcal{M}_d^q$ is a weakly closed set of scenario measures. For a portfolio $X_i \sim P_i$, $1 \le i \le n$ and a risk measure Ψ as in (3.9) we define the *average risk* functional

$$F(\mu) := \frac{1}{n} \sum_{i=1} \Psi_\mu(X_i) - \alpha(\mu). \tag{3.10}$$

A scenario measure $\mu_0 \in A$ is called a *worst case scenario measure* if it maximizes the average risk functional F, i.e.

$$F(\mu_0) = \sup_{\mu \in A} F(\mu). \tag{3.11}$$

The following theorem determines the worst case risk and identifies the worst case joint portfolio as comonotone portfolio with respect to a worst case scenario measure. It also allows to determine the worst case diversification effect.

Theorem 3.2 (Worst case joint portfolio) *Let $X_i \sim P_i$, $1 \le i \le n$ be a portfolio and consider a finite, convex, law invariant risk measure Ψ as in* (3.9)*.*

a) The worst case risk is given by

$$\sup_{\tilde{X}_i \sim X_i} \Psi\Big(\frac{1}{n}\sum_{i=1}^n \tilde{X}_i\Big) = \sup_{\mu \in A} F(\mu) \tag{3.12}$$

b) If μ_0 is a worst case scenario measure and if $X_i^* \sim P_i$ are μ_0 comonotone, then $X_1^*, \dots, X_n^*$ is a worst case portfolio.

c) If Ψ is strongly continuous then there exists a worst case scenario measure $\mu_0 \in A$

$$F(\mu_0) = \sup_{\mu \in A} F(\mu). \tag{3.13}$$

Proof:

a) Generally for law invariant convex risk measures Ψ holds the inequality

$$\Psi\Big(\frac{1}{n}\sum_{i=1}^n \tilde{X}_i\Big) \le \frac{1}{n}\sum_{i=1}^n \Psi(\tilde{X}_i) = \frac{1}{n}\sum_{i=1}^n \Psi(X_i). \tag{3.14}$$

Furthermore, by Proposition 3.1, formula (3.8), and by the representation in (3.9) holds

$$\begin{aligned}
\sup_{\tilde{X}_i \sim X_i} \Psi\Big(\frac{1}{n}\sum_{i=1}^n \tilde{X}_i\Big) &= \sup_{\tilde{X}_i \sim X_i} \sup_{\mu \in A} \Big(\Psi_\mu\Big(\frac{1}{n}\sum_{i=1}^n \tilde{X}_i\Big) - \alpha(\mu)\Big) \\
&= \sup_{\mu \in A} \Big(\sup_{\tilde{X}_i \sim X_i} \Psi_\mu\Big(\frac{1}{n}\sum_{i=1}^n \tilde{X}_i\Big) - \alpha(\mu)\Big) \\
&= \sup_{\mu \in A} \Big(\frac{1}{n}\sum_{i=1}^n \Psi_\mu(X_i) - \alpha(\mu)\Big) \\
&= \sup_{\mu \in A} F(\mu).
\end{aligned} \tag{3.15}$$

b) If μ_0 is a worst case scenario measure and if $X_i^* \sim X_i$ are μ_0-comonotone, i.e. $X_i^* \sim_{\mathrm{oc}} Y$, $1 \le i \le n$, for some $Y \sim \mu_0$, then $\frac{1}{n}\sum_{i=1}^n X_i^* \sim_{\mathrm{oc}} Y$ and

$$\Psi\Big(\frac{1}{n}\sum_{i=1}^n X_i^*\Big) = F(\mu_0).$$

Therefore, (X_i^*) is a worst case portfolio.

c) If Ψ is strongly continuous, then the scenario set A is weakly compact in $\mathcal{M}_d^q$. Since the function $\mu \to \Psi_\mu(X) = \sup\{EX \cdot \tilde{Y}; \tilde{Y} \sim \mu\}$ is usc in the weak topology on L_d^q it follows that the sup in a) is attained at some $\mu_0 \in A$. Thus a worst case scenario measure $\mu_0 \in A$ exists. □

In the coherent case the result can also be formulated in terms of the risk of the joint portfolio.

Corollary 3.3 (Coherent case) *Any finite coherent law invariant risk measure Ψ on L_d^p has a representation of the form*

$$\Psi(X) = \max_{\mu \in A} \Psi_\mu(X) \tag{3.16}$$

with some weakly compact subset $A \subset \mathcal{M}_d^q$. Furthermore,

$$\sup_{\tilde{X}_i \sim X_i} \Psi\Big(\sum_{i=1}^n \tilde{X}_i\Big) = \sup_{\mu \in A} F_c(\mu) = F_c(\mu_0) \tag{3.17}$$

for some worst case scenario measure $\mu_0 \in A$, with respect to total risk $F_c(\mu) := \sum_{i=1}^n \Psi_\mu(X_i)$. *If $X_i^* \sim P_i$ are μ_0-comonotone, then (X_i^*) is a worst case portfolio, i.e.*

$$\sup_{\tilde{X}_i \sim X_i} \Psi\Big(\sum_{i=1}^n \tilde{X}_i\Big) = \Psi\Big(\sum_{i=1}^n X_i^*\Big).$$

Remark 3.4 *a) By definition the average risk functional*

$$F(\mu) = \frac{1}{n} \sum_{i=1}^n \Psi_\mu(X_i) - \alpha(\mu)$$

involves only the marginal distributions P_i of X_i but does not involve the joint distribution of $X_1, \dots, X_n$. To apply Theorem 3.2 we have to analyse the convex average risk functional F and have to determine a worst case *scenario measure $\mu_0 \in A$. Then the worst case dependence structure is given by the comonotone vector $(X_1^*, \dots, X_n^*)$ w.r.t. the worst case scenario measure μ_0.*

b) Also the converse of Theorem 3.2 and Corollary 3.3 holds true. If $X_1^*, \dots, X_n^*$ is a worst case portfolio, then (X_i^*) are μ-comonotone w.r.t. any worst case scenario $\mu \in A$.

c) Theorem 3.2 also implies a description of the worst case portfolio and the diversification effect for the joint portfolio in the case of general convex risk measures Ψ. This follows from the rewriting $\Psi(\sum_{i=1}^n X_i) = \Psi(\frac{1}{n} \sum_{i=1}^n nX_i)$ and thus

$$\sup_{\tilde{X}_i \sim X_i} \Psi\Big(\sum_{i=1}^n \tilde{X}_i\Big) = \sup_{\mu \in A} \; F_n(\mu) \tag{3.18}$$

with $F_n(\mu) = \frac{1}{n} \sum_{i=1}^n \Psi_\mu(nX_i) - \alpha(\mu) = \sum_{i=1}^n \Psi_\mu(X_i) - \alpha(\mu)$.

For general convex risk measures Ψ on L_d^p we have the inequality

$$\sup_{\tilde{X}_i \sim X_i} \Psi\Big(\frac{1}{n}\sum_{i=1}^n \tilde{X}_i\Big) \le \frac{1}{n}\sum_{i=1}^n \Psi(X_i). \tag{3.19}$$

The difference

$$\frac{1}{n}\sum_{i=1}^n \Psi(X_i) - \Psi\Big(\frac{1}{n}\sum_{i=1}^n X_i\Big) \quad \text{resp.} \quad \sum_{i=1}^n \Psi(X_i) - \Psi\Big(\sum_{i=1}^n X_i\Big)$$

is called the *diversification effect* of portfolio (X_i). Equality holds in (3.19) by Proposition 3.1 for max correlation risk measures. We interpret this result by saying that for max correlation risk measures there is no *diversification effect* in the worst case dependence situation. A diversification effect however arises quite generally for convex risk measures even under worst case dependence.

Let $\Psi(X) = \max_{\mu \in A}(\Psi_\mu(X) - \alpha(\mu))$ be a finite convex, law invariant risk measure on L_d^p. The key comparison argument describing the worst case diversification effect can be demonstrated most clearly in the case $n = 2$.

Proposition 3.5 *Assume that for some X, $Y \in L_d^p$, μ_1, $\mu_2 \in A$, $\mu_1 \neq \mu_2$ are two different unique worst case scenarios, i.e.*

$$\begin{aligned} &\Psi_{\mu_1}(X) - \alpha(\mu_1) > \Psi_\mu(X) - \alpha(\mu), \quad \forall \mu \in A, \mu \neq \mu_1 \\ &\textit{and} \\ &\Psi_{\mu_2}(Y) - \alpha(\mu_2) > \Psi_\mu(Y) - \alpha(\mu), \quad \forall \mu \in A, \mu \neq \mu_2. \end{aligned} \tag{3.20}$$

Then

$$\sup_{\tilde{X} \sim X, \tilde{Y} \sim Y} \Psi\Big(\frac{1}{2}(\tilde{X} + \tilde{Y})\Big) < \frac{1}{2}\big(\Psi(X) + \Psi(Y)\big). \tag{3.21}$$

Proof: By Theorem 3.2 and assumption (3.20) it follows with the worst case scenario measure $\mu_0 \in A$ for the portfolio (X, Y)

$$\begin{aligned} \sup_{\tilde{X} \sim X, \tilde{Y} \sim Y} \Psi\Big(\frac{1}{2}(\tilde{X} + \tilde{Y})\Big) &= \sup_{\mu \in A} \Big(\frac{1}{2}(\Psi_\mu(X) + \Psi_\mu(Y)) - \alpha(\mu)\Big) \\ &= \frac{1}{2}\big(\Psi_{\mu_0}(X) - \alpha(\mu_0) + \Psi_{\mu_0}(Y) - \alpha(\mu_0)\big) \\ &< \frac{1}{2}(\Psi_{\mu_1}(X) - \alpha(\mu_1) + \Psi_{\mu_2}(Y) - \alpha(\mu_2)) \\ &= \frac{1}{2}\big(\Psi(X) + \Psi(Y)\big). \end{aligned}$$

□

Remark 3.6 *Proposition 3.5 shows clearly when a diversification effect can be expected. A similar result holds obviously also for a portfolio with* $n \geq 2$ *components. The exposure condition* (3.20) *implies a diversification effect. No diversification effect arises only in case that all risks* X_i *admit the same worst case scenario measure* $\mu_0 \in A$. *If* $|A| \geq 2$, *then by the classical Bishop–Phelps theorem it follows that there exist some (strongly) exposed points* μ_1, $\mu_2 \in A$ *satisfying the exposure condition* (3.20). *This type of argument has also been used essentially in the recent paper of [EGH09] who consider the coherent case.*

[EGH09] have characterized the max correlation risk measures in the class of coherent risk measures as those which do not allow worst case diversification effects, in their language as the only *structural neutral* coherent risk measures. Their argument is given in a direct way and is quite involved. In particular, it uses the Bishop–Phelps theorem (compare also Remark 3.6). In the following we extend this result to the case of strongly continuous convex risk measures and give a simplified proof. Our proof is based essentially on the representation result for law invariant convex risk measures (Theorem 2.5) and uses an idea from the classical Kusuoka proof in the one dimensional case $d = 1$.

Let Ψ be a strongly continuous, law invariant convex risk measure on L_d^p. By Theorem 2.5 Ψ has a representation of the form

$$\Psi(X) = \max_{\mu \in A}(\Psi_\mu(X) - \alpha(\mu)),$$

with weakly compact scenario set $A \subset \mathcal{M}_d^q$ and penalty $\alpha(\mu)$. If Ψ is a translated max correlation risk measure then Ψ does not allow a worst case diversification effect for any portfolio. The following theorem states the converse, that the translated max correlation risk measures are the only strongly continuous convex, law invariant risk measures with no worst case diversification effect.

Theorem 3.7 (Worst case diversification effect) *Let* Ψ *be a strongly continuous, convex law invariant risk measure on* L_d^p. *Then it holds:*
Ψ *has no worst case diversification effect, i.e., for all portfolios* (X_i) *holds*

$$\sup_{\tilde{X}_i \sim X_i} \Psi\Big(\frac{1}{n}\sum_{i=1}^n \tilde{X}_i\Big) = \frac{1}{n}\sum_{i=1}^n \Psi(X_i) \tag{3.22}$$

if and only if Ψ *is a translated max correlation risk measure,*

$$\Psi = \Psi_\mu - \alpha(\mu)$$

for some scenario measure $\mu \in \mathcal{M}_d^q$ *and* $\alpha(\mu) \in \mathbb{R}^1$.

Proof: By Proposition 3.1 any translated max correlation risk measure has no worst case diversification effect.

For the converse direction assume that Ψ has no worst case diversification effect. Define

$$\mathcal{M}(\Psi, X) = \big\{\mu \in A : \Psi_\mu(X) - \alpha(\mu) = \Psi(X) = \sup_{\tilde\mu \in A}(\Psi_{\tilde\mu}(X) - \alpha(\tilde\mu))\big\} = \mathcal{M}(\Psi, P^X). \tag{3.23}$$

Since for any $X \in L_d^p$ the mapping $\tilde\mu \to \Psi_{\tilde\mu}(X) - \alpha(\tilde\mu)$ is usc with respect to the weak topology on $\mathcal{M}_d^q$ and since $A \subset \mathcal{M}_d^q$ is weakly compact it follows that $\mathcal{M}(\Psi, X) \neq \emptyset$ is a nonempty closed subset of A.

For $X_1, \dots, X_n \in L_d^p$ with distributions $P_1, \dots, P_n \in \mathcal{M}_d^p$ let $\mu_0 \in A$ be a worst case scenario measure for the portfolio (X_i). Thus from our assumption (3.22) we obtain

$$\begin{aligned} \sup_{\tilde X_i \sim X_i} \Psi\Big(\frac{1}{n}\sum_{i=1}^n \tilde X_i\Big) &= F(\mu_0) = \frac{1}{n}\sum_{i=1}^n (\Psi_{\mu_0}(X_i) - \alpha(\mu_0)) \\ &= \frac{1}{n}\sum_{i=1}^n \Psi(X_i). \end{aligned} \tag{3.24}$$

This implies that $\Psi(X_i) = \Psi_{\mu_0}(X_i) - \alpha(\mu_0)$ for $1 \le i \le n$ and thus

$$\mu_0 \in \bigcap_{i=1}^n \mathcal{M}(\Psi, X_i) = \bigcap_{i=1}^n \mathcal{M}(\Psi, P_i),$$

i.e. finite intersections of $\mathcal{M}(\Psi, P_i)$, $P_i \in \mathcal{M}_d^p$, $1 \le i \le n$ are nonempty. By weak compactness of A this implies

$$\bigcap_{P \in \mathcal{M}_d^p} \mathcal{M}(\Psi, P) \neq \emptyset.$$

Thus there exists some $\mu \in A$ such that

$$\Psi_\mu(X) - \alpha(\mu) = \sup_{\tilde\mu \in A}(\Psi_{\tilde\mu}(X) - \alpha(\tilde\mu)) = \Psi(X)$$

i.e., Ψ is a translated max correlation risk measure. □

4 Remarks on optimal couplings and some classes of examples

By Theorem 3.2 the worst case portfolio (dependence) structure for portfolio distributions $P_1, \dots, P_n$ and a risk measure Ψ is given by comonotone random vectors $X_1, \dots, X_n$, $X_i \sim P_i$ with respect to a worst case scenario measure $\mu_0 \in A$. By the basic characterization of optimal couplings w.r.t. L^2-distance in [RüRa90] holds: $X_i \sim P_i$, $Y \sim \mu_0$ is an optimal coupling, $X_i \sim_{\text{oc}} Y$ if and only if X_i lies a.s. in the subgradient

$$X_i \in \partial f_i(Y) \quad \text{a.s.} \tag{4.1}$$

of some lsc convex function f_i. Criterion (4.1) holds without any continuity assumptions on the scenario measure μ_0 and Y can be chosen independent from the index i. If μ_0 is absolutely continuous the subgradient reduces to the gradient a.s. and then (4.1) is equivalent to

$$X_i = \nabla f_i(Y) \quad \text{a.s.,} \tag{4.2}$$

X_i is given by the gradient of f_i applied to Y. Thus determination of the worst case portfolio structure is reduced by Theorem 3.2 to an optimal coupling problem w.r.t. the worst case scenario measure μ_0 as in (4.1), (4.2).

a) Discrete distributions and approximation

For the numerical analysis it is important that the optimal coupling problem can be approximated by optimal couplings between discrete distributions. For the case that $Q = \Sigma_i \alpha_i \varepsilon_{\{y_i\}}$ is discrete, where Q stands for some portfolio measure P_i, one can restrict in (4.1), (4.2) to convex functions f of the form

$$f(x) = \max_i(\langle x, y_i\rangle + a_i) = f_{(a_i)}(x) \tag{4.3}$$

Then with $A_i := \{x : f(x) = \langle x, y_i\rangle + a_i\}$ holds

$$y_i \in \partial f(x) \text{ if and only if } x \in A_i, \tag{4.4}$$

see [RüU00]. The optimal shifts a_i can be determined by the condition

$$P(A_i) = \alpha_i. \tag{4.5}$$

Numerically this can be done most efficiently by a gradient approach to the minimization of the convex function of the shifts (a_i).

$$f_{(a_i)}(x) - \sum_i \alpha_i a_i = \inf_{(a_i)}, \tag{4.6}$$

as was observed in a related problem on combinatorial Voronoi type partitioning in [AHA98], see also [RüU00] and [EGH09]. The solutions (a_i) of the optimization problem (4.6) determine the optimal shifts (a_i) and thus by (4.1) resp. (4.2) they determine the optimal couplings $X \sim Q$, $Y \sim \mu$ by the rule

$$X = y_i \quad \text{implies} \quad Y \in A_i. \tag{4.7}$$

If μ is absolutely continuous, then (4.7) determines X uniquely as a function of Y: a.s. holds

$$X = y_i \quad \text{if and only if} \quad Y \in A_i. \tag{4.8}$$

If μ is not continuous, then X has to be chosen on the boundaries of A_i such that additionally (4.5) holds (which is typically an easy task).

The procedure above allows a numerical solution of the optimal coupling problem and has been applied successfully in a series of examples (see [RüU00, EGH09]). Iteratively applying this procedure to all pairs (P_i, μ) we obtain as a result approximatively a worst case portfolio $X_1^*, \dots, X_n^*$.

b) Location – scale families, elliptical distributions.
For a random vector $X \in \mathbb{R}^d$ with distribution Q, $X \sim Q$ consider the generated location scale family

$$\mathcal{Q} := \{Q_{a,B} : B \in \mathcal{A}, a \in \mathbb{R}^d\} \tag{4.9}$$

where $Q_{a,B} \sim X_{a,B} := BX + a$ and where $\mathcal{A}$ is some set of $d \times d$ scaling matrices. Consider the scenario measure $\mu = Q \equiv Q_{0,I}$, $X \sim Q$ and assume that the portfolio distributions $P_i = Q_{a_i,B_i} \in \mathcal{Q}$, $1 \le i \le n$ are in the generated scale family $\mathcal{Q}$.

b$_1$) If $\mathcal{A} \subset NN(d)$ lies in the class of positive semidefinite matrices then by the optimal coupling criterion (1.1) holds

$$X_i := X_{a_i,B_i} \sim_{\mathrm{oc}} X, \quad 1 \le i \le n \tag{4.10}$$

and

$$X_1, \dots, X_n \text{ are } \mu\text{-comonotone.} \tag{4.11}$$

Further in this case the worst case risk of the portfolio $P_1, \dots, P_n$ w.r.t. Ψ_μ is given by

$$\sup_{\tilde{X}_i \sim X_i} \Psi_\mu\Big(\sum_{i=1}^n \tilde{X}_i\Big) = \Psi_\mu\Big(\sum_{i=1}^n X_i\Big) = \operatorname{tr}\Big(\Big(\sum_{i=1}^n B_i\Big)\Sigma\Big) \tag{4.12}$$

with $\Sigma = \operatorname{Cov} X$ the covariance matrix of $X \sim Q$ and tr the trace operator.

b$_2$) Assume that the basic measure Q in b$_1$) is invariant under orthogonal transformations like e.g. the normal distribution $N(0, I)$ or the uniform distribution on a ball around 0. Then we can extend in b$_1$) to general affine linear transformations $Q_{a,B} \sim a + BX$, $X \sim Q$, $B \in A = M(d, \mathbb{R})$. By the polar factorization theorem holds

$$B = PO \tag{4.13}$$

where P is positive semidefinite and O is orthogonal. Therefore,

$$BX \sim POX \sim PY, \tag{4.14}$$

where $Y := OX$, $X \sim Y$. Thus the optimal coupling problem in this case is reduced to the optimal coupling in the positive semidefinite case.

Interesting examples of b_1), b_2) are multivariate normal distributions $N(\mu, \Sigma)$, uniform distributions on ellipses and general elliptical distributions. The optimal coupling results available for multivariate normal distributions (see [RüRa90, RaRü98]) extend in the same form to these scale families. Thus for a scenario measure $\mu \sim Q$ and $P_i \in \mathcal{Q}$ we explicitly obtain the worst case dependence structure. In terms of the covariances Σ_i of P_i, Σ_0 of Q and a scenario vector $T \sim Q$ the worst case portfolio for location scale families is given by

$$X_i = S_i T \tag{4.15}$$

where $S_i = \Sigma_i^{1/2}(\Sigma_i^{1/2}\Sigma_0\Sigma_i^{1/2})^{-1/2}\Sigma_i^{1/2}$ (see [RüRa90]).

If the class of scenario measures A is a subclass of $\mathcal{Q}$, then the determination of the worst case scenario measure reduces to a standard optimization problem of the form

$$\operatorname{tr}\Big[\Big(\sum_{i=1}^n S_i^\top\Big) B\Sigma_0\Big] = \sup_{B\in A} \tag{4.16}$$

where $\Sigma_0 = \mathrm{Cov}(T)$ is the covariance matrix of Q.

c) Coupling to the sum

In some cases even if the explicit representation of the convex risk measures is not known explicitly it is possible to determine the worst case dependence structure and the worst case scenario measure. We consider e.g. the variation risk measure

$$\Psi(X) = (E\|X\|^2)^{1/2} \tag{4.17}$$

where $\|X\|$ is the usual Euclidean norm of X. Then it has been shown in [RüU02] that the property of worst case dependence of a portfolio $X_i \sim P_i$, $1 \le i \le n$, is closely related to the fact, that all X_i are optimally coupled to their sum $T = \sum_{i=1}^n X_i$. More precisely, optimal coupling to the sum is a necessary condition and together with a regularity condition on the support of T also a sufficient condition (see [RüU02]) for a worst case portfolio. In our context this means that the worst case scenario measure μ_0 is given by the distribution of the (worst case) sum T. The worst case dependence structure is given by μ_0-comonotone vectors X_i. Since the variation risk measure is not monotone we leave in this example formally the framework of the previous sections and have to allow also non positive directions as scenarios.

In the case of normal distributions $X_i \sim P_i$, $P_i = N(0, \Sigma_i)$, $1 \le i \le n$, Σ_i positive definite covariance matrices, the worst case scenario measure μ_0 is given by

$$\mu_0 = N(0, \Sigma_0), \tag{4.18}$$

where Σ_0 is a (unique) positive definite solution of the matrix equation

$$\sum_{i=1}^n (\Sigma_0^{1/2}\Sigma_i\Sigma_0^{1/2})^{1/2} = \Sigma_0. \tag{4.19}$$

The worst case dependence structure is given by

$$X_1 \sim N(0,\Sigma_1), \quad X_i = S_i S_1^{-1} X_1, \quad i = 2,\dots,n, \tag{4.20}$$

where $S_i = \Sigma_i^{1/2}(\Sigma_i^{1/2}\Sigma_0\Sigma_i^{1/2})^{1/2}\Sigma_i^{1/2}$. (For details see [RüU02].) Since the optimal coupling property is a property of the couplings (mappings) and not of the underlying distributions it follows that this determination of the worst case dependence structure extends in the same way to location scale families, in particular to elliptical distributions as in b_1), b_2).

d) Distributions of spherical type

Let U be a random vector on the unit sphere in $\mathbb{R}^d$ w.r.t. Euclidean distance and let $X = RU$ with some scaling real random variable $R > 0$ independent of U. Then we call X of Euclidean spherical type. Special cases of distributions of Euclidean spherical type are *spherical invariant* distributions which are invariant under orthogonal transformations. In this case U is uniformly distributed on the Euclidean unit sphere. Examples are uniform distributions on spheres or on balls and normal distributions $N(0,\sigma^2 I_d)$ which have exponential tails. But also interesting unsymmetric distributions are of Euclidean spherical type. If R has polynomial tails like for positive stable distributions then also $X = RU$ has polynomial tails and forms a class of distributions of interest in extreme value theory.

Assume that a portfolio is given by $X_i = R_i \cdot U_i$, $1 \le i \le n$ where $U_1 \stackrel{d}{=} U_2 \stackrel{d}{=} \dots \stackrel{d}{=} U_n \stackrel{d}{=} U$, $R_i \ge 0$ are independent of U_i with distribution functions F_i. Denote by P_i the portfolio distribution of X_i.

Proposition 4.1 (Euclidean spherical type portfolio) *Let $X_i = R_i \cdot U_i$, $1 \le i \le n$ be a portfolio of Euclidean spherical type with*

$$U_i \stackrel{d}{=} U, \quad 1 \le i \le n.$$

Define $X_i^* = F_i^{-1}(V) \cdot U$, $1 \le i \le n$, for some uniformly on $(0,1)$ distributed random variable V independent from U. Then $X_1^*,\dots,X_n^*$ is a worst case portfolio structure with respect to the variation risk measure $\Psi_2(X) = (E\|X\|_2^2)^{1/2}$ – $\|\ \|_2$ the Euclidean norm – and the worst case risk is given by

$$\sup_{\widetilde{X}_i \sim X_i} \Psi_2\Big(\sum_{i=1}^n \widetilde{X}_i\Big) = \Psi_2\Big(\sum_{i=1}^n X_i^*\Big) = \Big(E\Big(\sum_{i=1}^n F_i^{-1}(V)\Big)^2\Big)^{1/2}. \tag{4.21}$$

Proof: By definition $\|X_i\| = R_i$, $i \le i \le n$, and with $R_i^* := F_i^{-1}(V)$ it holds $\|X_i^*\| = R_i^*$. Thus we obtain for $i \ne j$,

$$\|X_i - X_j\|_2 \ge \big|\|X_i\|_2 - \|X_j\|_2\big| = |R_i - R_j|\,.$$

This implies by a well known one dimensional coupling result

$$E\|X_i - X_j\|_2^2 \geq E|R_i - R_j|^2 \geq E|R_i^* - R_j^*|^2 = E\|X_i^* - X_j^*\|_2^2. \tag{4.22}$$

In consequence all pairs X_i^*, X_j^* are optimally coupled, $X_i^* \sim_{\mathrm{oc}} X_j^*$. This implies directly that $X_1^*, \dots, X_n^*$ is a worst case portfolio with respect to Ψ_2 and (4.21) follows from (4.22). $\square$

Remark 4.2 *Obviously in the Euclidean spherical portfolio above all X_i^* are optimal coupled to the spherical part U of the distribution and also are optimally couped to the sum $\sum_{i=1}^n X_i^*$. Thus μ the distribution of U is a worst case scenario measure in this situation. From (4.21) we see that typically a worst case diversification effect arises in this class of distributions. We remark that a similar coupling result has also been discussed in [CRT93].*

The argument for the Euclidean spherical type portfolio also extends to distributions of spherical type with respect to other norms on $\mathbb{R}^d$. Consider for example the one norm $\|x\|_1 = \sum_{i=1}^n |x_i|$. Let U be distributed on the unit one-sphere and let $X_i = R_i U_i$, $1 \leq i \leq n$ be a portfolio of spherical type (w.r.t. the one-sphere) with $U_i \sim U$. In case that U_i are uniformly distributed on the one-sphere $\{x \in \mathbb{R}^d : \|x\|_1 = 1\}$ we obtain exactly the class of Archimedian distributions, i.e. those distributions which have Archimedian copulas. Therefore, we call this class of spherical type distributions distributions od *Archimedian type. Archimedian type* distributions have been used a lot in recent dependence modelling. Consider the risk measure

$$\Psi_1(X) := E\|X\|_1 \tag{4.23}$$

defined by the L^1-norm.

Proposition 4.3 (Archimedian type portfolio) *Let $X_i = R_i U_i$, $1 \leq i \leq n$, be an Archimedian type portfolio, with R_i independent of U_i and $U_i \stackrel{d}{=} U$, $1 \leq i \leq n$, are distributed as U. Let (R_i) be independent of U. Then $X_i^* := R_i U$, $1 \leq i \leq n$, is a worst case portfolio w.r.t. Ψ_1 and the worst case risk is given by*

$$\sup_{\widetilde{X}_i \sim X_i} \Psi_1\Big(\sum_{i=1}^n \widetilde{X}_i\Big) = \Psi_1\Big(\sum_{i=1}^n X_i^*\Big) = E\sum_{i=1}^n R_i = \sum_{i=1}^n \Psi_1(X_i). \tag{4.24}$$

There is no worst case diversification effect.

Proof: For the proof note that $X_i^* = R_i U$ have the correct portfolio distribution, $X_i^* \sim P_i$. Furthermore, we obtain for any portfolio $X_i \sim P_i$ by Minkowski's inequality

$$\begin{aligned}\Psi_1\Big(\sum_{i=1}^n X_i\Big) &= E\Big\|\sum_{i=12}^n X_i\Big\|_1 \\ &\le E\sum_{i=1}^n R_i = E\Big\|\sum_{i=1}^n R_i U\Big\|_1 = \Psi_1\Big(\sum_{i=1}^n X_i^*\Big).\end{aligned} \tag{4.25}$$

Thus (X_i^*) is a worst case portfolio. Since $X_i^* = R_i U$ we obtain further

$$\Psi_1\Big(\sum_{i=1}^n X_i^*\Big) = \sum_{i=1}^n ER_i = \sum_{i=1}^n \Psi_1(X_i),$$

i.e. there is no worst case diversification effect. □

For the L^1-norm risk we obtain by a classical result, that $\Psi_1(X)$ is identical to a max-correlation risk measure with worst case scenario given by the sign of X. This explains the disappearance of the worst case diversification effect in (4.24). Note that in this case again we have to allow negative scenarios.

In fact the arguments of Propositon 4.3 can be generalized to the following general spherical equivalent models. Let $\|\ \|$ be any norm on $\mathbb{R}^d$ and let $X = R \cdot U$ be the polar representation of X with radial part $R = \|X\|$ and spherical part $U = {}^{X}\!/\!_{\|X\|}$. We assume that the radial part R is independent of the spherical part U. A random vector Y is called *spherically equivalent* to X if the spherical part of Y is identically distributed, ${}^{Y}\!/\!_{\|Y\|} \overset{d}{=} {}^{X}\!/\!_{\|X\|}$ and the radial part of Y is independent of the spherical part.

Let f be a convex nondecreasing function $f : [0,\infty) \to [0,\infty)$ and g be a nondecreasing function $g : [0,\infty) \to [0,\infty)$ and consider a risk measure Ψ of the form

$$\Psi(X) = g(Ef(\|X\|)). \tag{4.26}$$

Examples are the p-norms, $p \ge 1$, i.e. $\Psi(X) = \|X\|_p$ and with norms of the form $\|x\| = x^\top Ax$, we get as examples in particular models of elliptical type distributions as in 4.b).

Theorem 4.4 (Spherically equivalent portfolios) *Let $X_i = R_i U_i$ be a portfolio of spherical type $X = RU$ and let Ψ be a risk measure as in* (4.26)*. Let V be independent of U uniformly distributed on* $(0,1)$ *and define*

$$R_i^* := F_i^{-1}(V), \quad X_i^* := R_i^* U, \quad 1 \le i \le n. \tag{4.27}$$

Then $X_1^*, \dots, X_n^*$ is a worst case portfolio structure with respect to Ψ and the worst *case risk is given by*

$$\sup_{\tilde{X}_i \sim X_i} \Psi\Big(\sum_{i=1}^n \tilde{X}_i) = \Psi\Big(\sum_{i=1}^n X_i^*\Big) \tag{4.28}$$
$$= g\Big(Ef\Big(\sum_{i=1}^n R_i^*\Big)\Big).$$

Proof: All elements X_i^* have the same spherical part U and by construction X_i^* have the correct marginal distributions, $X_i^* \sim P_i$. Furthermore, by Minkowski's inequalitiy we have for any portfolio (X_i) with marginals (P_i)

$$\Big\|\sum_{i=1}^n X_i\Big\| \leq \sum_{i=1}^n \|X_i\| = \sum_{i=1}^n R_i,$$

while for the portfolio (X_i^*) holds $\|\sum_{i=1}^n X_i^*\| = \sum_{i=1}^n R_i^*$. By our construction $R_i^* \overset{d}{\simeq} R_i$. Thus we can apply a classical optimal coupling result for real random variables (see (1.1)) which implies

$$Ef\Big(\Big\|\sum_{i=1}^n X_i\Big\|\Big) \leq Ef\Big(\sum_{i=1}^n R_i\Big)$$
$$\leq Ef\Big(\sum_{i=1}^n R_i^*\Big) = Ef\Big(\Big\|\sum_{i=1}^n X_i^*\Big\|\Big).$$

As consequence we obtain that (X_i^*) is a worst case dependence structure

$$\sup_{\tilde{X}_i \sim X_i} \Psi\Big(\sum_{i=1}^n \tilde{X}_i\Big) = \Psi\Big(\sum_{i=1}^n X_i^*\Big). \qquad \square$$

Remark 4.5 *a) In typical cases Theorem 4.4 implies a worst case diversification effect. An exception is the situation of an Archimedian type portfolio with one-norm as in Proposition 4.3. Theorem 4.4 gives a tool to calculate worst case portfolios in many examples and to determine the corresponding worst case diversification effect.*

b) As proposed by a reviewer it will be of interest to develop more examples of worst case portfolios for further multivariate risk measures with a natural appeal to financial and insurance risks.

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Ludger Rüschendorf
Department of Mathematical Stochastics
University of Freiburg
Eckerstraße 1
79104 Freiburg
Germany
ruschen@stochastik.uni-freiburg.de