Test of Association Between Multivariate Stable Vectors*

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Abstract

Based on observations of d-dimensional random vectors in the domain of attraction of a stable distribution with (multi-)index $\alpha = (\alpha_1, \cdots, \alpha_d)$ an estimator for the dependence function of the α_i-stable variables is constructed. The estimator utilizes the α-tail-estimator and an estimator of the spectral measure of the α-stable law. This estimator gives rise to a test of association of the stable components and various quantitative measures of association.

Keywords: Asset-return distribution, dependence function, financial modeling, multivariate stable distribution

*The research of S. Mittnik was supported by the *Deutsche Forschungsgemeinschaft.* That of S.T. Rachev was partially supported by *Alexander-von-Humboldt* award during his visits at the University of Freiburg and the University of Kiel.

1 Introduction

Since the influential work of Mandelbrot(1963) and Fama (1965) it has been widely accepted that returns on financial assets do not follow a normal distribution. Empirical return distributions tend to be fat-tailed and, relative to the normal distribution, more peaked around the center. Mandelbrot proposed the stable distribution as a more realistic distributional candidate. Whereas the early work on stable modeling in finance focused on univariate questions, the problem of modeling returns on portfolios of financial assets, which are governed by multivariate stable distributions, has received a lot of attention in recent years (see Press, 1972; Mittnik and Rachev, 1989, 1993 and references therein). Typically, the assumptions for estimation have been reduced to just assuming that the observations on the vector of returns are in the domain of attraction of a multivariate α-stable law. While this is not restrictive, it was always assumed that the individual components in the vector of returns have *identical indexes of stability*.[1] Estimators of the spectral measure and the index of stability were found in this case, leading to a test for association of assets in a portfolio and a tail-estimator for the portfolio risk (cf. Rachev and Xin, 1993; Cheng and Rachev, 1995; de Haan and Resnick, 1993).

Here, we extend results of the single-stable-index portfolio to the multi-stable-index portfolio. For the latter there is neither a closed theory of stability available nor an obvious notion of portfolio risk. We define an appropriate estimator for the dependence function which gives rise to a test of association for the components in the vector of returns. Applying limiting results for the dependence function we analyze the asymptotic behavior of various quantitative measures of association.

2 Test of Association for Random Vectors in the Domain of Attraction of $(\alpha_1, \cdots, \alpha_d)$–stable laws

2.1 Association and Spectral Measure of Stable Vectors

Suppose that for a given sequence $\{Z_n = (Z_n^{(1)}, \cdots, Z_n^{(d)})\}_{n \geq i}$ of independent and identically distributed (iid) random vectors there exist $a_n \in \mathbf{R}_+^d$and$b_n \in \mathbf{R}^d$ such that

$$\left(\frac{1}{a_n^{(1)}} \sum_{j=1}^{n} Z_j^{(1)}, \cdots, \frac{1}{a_n^{(d)}} \sum_{j=1}^{d} Z_j^{(d)} \right) - b_n \xrightarrow{d} X. \tag{2.1}$$

Then, (the distribution of) X is said to be $(\alpha_1, \cdots, \alpha_d)$–stable, and $Z \stackrel{d}{=} Z_1$ is said

[1] We refer to such a portfolio as the single-stable-index portfolio.

to be in the domain of attraction of (the distribution of) X.[2] For a detailed study of these laws we refer to Resnick and Greenwood (1979).

Typically, we view $\{Z_n\}$ as a sequence of daily (weekly or monthly) returns of several assets possibly together with the corresponding differences in their trading volumes. Given the importance of understanding the tail behavior and the interest in modeling returns and trading volumes jointly we shall only assume that vectors $\{Z_n\}$ are in the domain of a random vector X having stable components with indexes $\alpha_1, \cdots, \alpha_d$, $0 < \alpha_i < 2$.

For returns with heavy-tailed distribution the notion of association seems to be a natural measure of dependence (see the survey in Lee, Rachev and Samorodnitsky, 1993). A vector $X = (X^{(1)}, \cdots, X^{(d)})$ is said to have *(positively) associated components* if $\mathrm{Cov}(f(X), g(X)) \geq 0$ for all componentwise nondecreasing functions f, g: $\mathbf{R}^d \longrightarrow \mathbf{R}$ such that the covariance exists. On the other hand X has *negatively associated components* if for any $k < d$ and nondecreasing f: $\mathbf{R}^k \longrightarrow \mathbf{R}$, g: $\mathbf{R}^{d-k} \to \mathbf{R}$, $\mathrm{Cov}(f(X'), g(X'')) \leq 0$, where X' and X'' are any k and $n-k$ dimensional subvectors of X.

The next lemma shows that we can test whether the components of $Z \stackrel{d}{=} Z_n$ (cf. (2.1)) are associated provided we have an estimator for the *spectral measure,* ν , in the characteristic function (ch.f.) of X. Note that from (2.1) it follows that X is an infinitely divisible vector and, thus, its ch.f. is given by

$$\begin{aligned} \log \mathrm{E} e^{i<\vartheta,x>} &= i < a, \vartheta > + \int_{\{|s|>1\}} \left(e^{i<\vartheta,s>} - 1\right) \nu(ds) \\ &\quad + \int_{\{0<|s|<1\}} \left(e^{i<\vartheta,s>} - 1 - i < \vartheta, s >\right) \nu(ds), \end{aligned} \tag{2.2}$$

where a is the shift vector and ν is the spectral (Lévy) measure of X. Setting $\tau(x) = (x_1^{<1/\alpha_1>}, \cdots, x_d^{<1/\alpha_d>})$, $y^{<\alpha>} := |y|^\alpha$ sign y, the *scaled spectral measure* $\tilde{\nu} = \nu \circ \tau$ satisfies the self-similarity property (see Resnick and Greenwood, 1979) for any positive a and $A \in \mathcal{B}(\mathbf{R}^d \setminus \{0\})$[3]

$$a\tilde{\nu}(A) = \tilde{\nu}(a^{-1}A). \tag{2.3}$$

We rewrite A in terms of polar coordinates: $A = \{x = (|x|,\ \vartheta(x)); |x| > r,\ \vartheta(x) \in H\}$ for some $r > 0$ and $H \in \mathcal{B}((0,\pi)^{d-2} \times [0, 2\pi))$. We next set $M(r, H) := \tilde{\nu}(A)$. Then we factorize M by mean of a spectral measure (S) on the unit sphere:

$$M(r, H) = r^{-1} S(H), \tag{2.4}$$

[2] Here, $\stackrel{d}{=}$ means equality in distribution.

[3] $\mathcal{B}(\mathbf{R}^d \setminus \{0\})$ stands for the σ-field of Borel subsets of $R^d \setminus \{0\}$.

where $S(H) = M(1, H)$ is a finite measure on $\Omega_d := [0, \pi)^{d-2} \times [0, 2\pi)$. To facilitate proofs we assume throughout that the marginal d.f.'s of Z are continuous. All the proofs are given in Section 3.

Lemma 2.1. *Let Z be in the domain of attraction of X and suppose that Z is either (i) positively associated or (ii) negatively associated. Then, in case (i) the spectral measure S satisfies $S(\Omega_d^-) = 0$, where $\Omega_d^- = \{\vartheta \in \Omega_d;\ \vartheta_i\vartheta_j < 0$, for some $i \neq j\}$, while in case (ii) $S(\Omega_d^+) = 0$, where $\Omega_d^+ := \{\vartheta \in \Omega_d : \exists\, i \neq j, \vartheta_i\vartheta_j > 0\}$.*

With Lemma 2.1 in hand the problem of testing association of Z is reduced to the problem of choosing an appropriate estimator for $\tilde{\nu}-$ the spectral measure generated by S (see (2.3)). Moreover, it is sufficient to obtain a "good"estimator for the normalized dependence measure, μ, induced by $\tilde{\nu}$. To define μ, we set for $i = 1, \cdots, d$,

$$p_i = \lim P(Z^{(i)} > x)/P(|Z^{(i)}| > x) \quad \text{and} \quad q_i = 1 - p_i \tag{2.5}$$

and define μ on each orthant of $\mathbf{R}^d \backslash \{0\}$ as follows:
For $J \subset \{1, \cdots, d\}$ and $x = (x_J, x_{J^c}) \in \mathbf{R}^d$ with $x_j > 0$ for $j \in J$ and $x_j < 0$ for $j \in J^c$; define

$$\begin{aligned} m(x) &:= \mu((0, x_J] \times [x_{J^c}, 0)) \\ &:= \tilde{\nu}\left(\prod_{j \in J} [\frac{p_j}{x_j}, \infty) \times \prod_{j \in J^c} (-\infty, \frac{q_j}{x_j}]\right). \end{aligned} \tag{2.6}$$

The function m is an analogue to the so-called dependence function in the tail (see Xin, 1992). We list some properties of m. Let $\Omega_{d,J}$ denote the orthant with $x_j \geq 0$ for $j \in J$ and $x_j < 0$ for $j \in J^c$.

Lemma 2.2.

a) Representation of m through the spectral df S: For X with $\vartheta(x)$ in the orthant $\Omega_{d,J}$

$$m(x) = \frac{1}{d-1} \int_{\Omega_{d,J}} \left(\overset{\min}{j \in J} \left(\frac{\tau_j(\vartheta) x_i}{p_j} \right)^{d-1} - \overset{\max}{j \in J^c} \left(\frac{\tau_j(\vartheta) X_j}{q_g} \right)^{d-1} \right) dS(\vartheta), \tag{2.7}$$

where

$$\tau_j(\vartheta) := \begin{cases} \prod_{i=1}^{d-1} \sin \vartheta_i, & \textit{if} \quad j = 1 \\ \prod_{i=1}^{j-1} \sin \vartheta_i \cos \vartheta_j & \textit{if} \quad j \geq 2; \end{cases}$$

b) The one-dimensional marginal distributions of m *are given by*

$$m(\infty,\ldots,x_j,\ldots,-\infty)=|x_j|, \quad \textit{for all} \qquad x_j\in\mathbf{R}^1\backslash\{0\}, j=1,\ldots,d; \quad (2.8)$$

c) Homogeneity property:

$$m(ux)=um(x), \quad \textit{for all}\, u>0,\ x\in\mathbf{R}^d\backslash\{0\}. \qquad (2.9)$$

In the case $d=2$ *(2.7) simplifies to*

$$\begin{aligned}
m(x_1,x_2) &= \int\limits_0^{\pi/2} \tfrac{x_1\cos\vartheta}{p_1} \wedge \tfrac{x_2\sin\vartheta}{p_2} S(d\vartheta),\ \textit{for}\, x_1,x_2>0;\\
m(x_1,x_2) &= \int\limits_{\pi/2}^{\pi} \tfrac{x_1\cos\vartheta}{q_1} \wedge \tfrac{x_2\sin\vartheta}{p_2} S(d\vartheta),\ \textit{for}\, x_1<0,x_2>0;\\
m(x_1,x_2) &= \int\limits_{\pi}^{3\pi/2} \tfrac{x_1\cos\vartheta}{q_1} \wedge \tfrac{x_2\sin\vartheta}{q_2} S(d\vartheta)\ \textit{for}\, x_1<0,x_2>C0;
\end{aligned}$$

and

$$m(x_1,x_2) = \int\limits_{3\pi/2}^{2\pi} \tfrac{x_1\cos\vartheta}{p_1} \wedge \tfrac{x_2\sin\vartheta}{q_2} S(d\vartheta),\ \textit{for}\, x_1>0,x_2>0.$$

2.2 Estimation of the Dependence Function

The domain-of-attraction condition (2.1) can be re-written in terms of the normalized Lévy-measure $\tilde{\nu}$ in the following way. For all $\tilde{\nu}$-continuity sets $A\in\mathcal{B}(\mathbf{R}^2\backslash\{0\})$ with $\tilde{\nu}(A)<\infty$,

$$\lim_{n\to\infty} nP\left((U_1(Z^{(1)}),\ldots,U_d(Z^{(d)}))\in nA\right)=\tilde{\nu}(A), \qquad (2.10)$$

where the U_i are monotone functions equalizing the tails of the $Z^{(i)}$ by defining

$$U_i(x)=\frac{p_i}{P(Z^{(i)}>x)}I_{\{x>0\}}-\frac{q_i}{P(Z^{(i)}\le x)}I_{\{x\le 0\}}, \qquad (2.11)$$

(see Resnick and Greenwood, 1979; and the proof of Lemma 2.1 in Section 3).

From Lemmas 2.1 and 2.2 we have that under the hypothesis H_o, that the components of the vector $(Z^{(1)},\cdots,Z^{(d)})$ are positively associated m and μ are concentrated on the orthants without sign chance of the components.

To define a tail estimator for μ or, equivalently, m, let $Z_1, \cdots, Z_n$ be a sample from the distribution of Z and $(Z^{(1)}_{k:n}, \cdots, Z^{(d)}_{k:n})$, $k = 1, \cdots, n$ the corresponding order statistics. Then, for $x_J \geq 0$ and $x_{J^c} < 0$,

$$\hat{m}(x) \quad = \quad \frac{1}{k} \sum_{i=1}^{n} I \left\{ Z_i^{(j)} > Z^{(j)}_{n-[kx_j]+1:n}, j \in J; Z_i^{(j)} \leq Z^{(j)}_{[-kx_j]:n}, j \in J^c \right\} \tag{2.12}$$

is a tail estimator of m. The estimator of μ, $\hat{\mu}$, is defined analogously.

The estimator $\hat{m}$ can be viewed as a tail version of the multivariate rank order process in Rüschendorf (1976) or, alternatively, as the empirical dependence function in Deheuvels (1979). Variants of this estimator have been used in Xin (1992), Rachev and Xin (1993) and Cheng and Rachev (1995). The fact that the estimator uses only tail information is a consequence of the distributional assumption, which is only concerned with the tail behavior. If an explicit parametric model is assumed, it would be more natural and efficient to use the rank-order process or the dependence function itself.

Regularity conditions: In order to obtain asymptotic normality for $\hat{m}$, we need some second-order conditions on the tail of the Z–laws. Assuming that the components of Z have continuous $df's$ $F_i, 1 \leq i \geq d$, let us rewrite the domain of attraction condition as follows. For all $x \in \mathbf{R}^d \backslash \{0\}$,

$$m(x) = \lim_{t \downarrow 0} \frac{1}{t} m^{(t)}(X), \tag{2.13}$$

where

$$m^{(t)}(x) \quad := \quad P\left(\bar{F}_J(Z^{(J)}) \leq t x_J, F_{J^c}(Z^{(J^c)}\right) \leq -t x_J) \tag{2.14}$$

with $\bar{F}_i := 1 - F_i$, $x_J = (x_j)_{j \in J} \geq 0$, $x_{J^c} < 0$ and $\bar{F}_J = (\bar{F}_j)_{j \in J}$.

The proof of (2.13) is given in Section 3, see (3.4).

For the dependence function in the tail of the Z-distribution, the following regularity conditions are assumed:

RC1 $m(x)$ has continuous partial derivatives of first order $m_i' = \frac{\partial m}{\partial x_i}, 1 \leq i \leq d$,

and

RC2 as $t \downarrow 0$, uniformly on the unit sphere $\{x : ||x|| = 1\}$,
$m^t(x) = t\, m(x) + O(t^\delta)$ for some $\delta > 1$.

The limiting process B. Let W be a continuous zero-mean Gaussian field on $\mathbf{R}^d$ such that the restrictions of W on each orthant are independent and, such that the covariance function on each orthant is given by

$$\mathrm{E}\, W(x)W(y) = m((x_j \wedge y_j)_{j \in J}, (x_j \vee y_i)_{j \in J^c}), \tag{2.15}$$

where

$$x_j, y_j \geq 0, \quad j \in J$$

and

$$x_j, y_j < 0, \quad j \in J^c.$$

The process W generates a signed-random measure $\underline{W}$ by

$$W(x) = \underline{W}((0, x_J] \times [x_{J^c}, 0)), \quad \text{for } x_J \geq 0, x_{J^c} < 0.$$

For the next lemma we define the Gaussian field

$$B(x) = W(x) - \sum_{i=1}^{d} (\text{sign} x_i)\, m_i(x) W(\infty, \cdots, x_i, \cdots, \infty), x \in \mathrm{R}^d, \tag{2.16}$$

and write $\xrightarrow{D_\delta}$ to denote the convergence of random elements in the Skorohod space $D(\mathbf{R}^d \backslash \{0\})$ under uniform convergence on compacts with metric $\delta(m, m^*) = \sum_{k \geq 1} 2^{-k} \frac{\delta_k(m,m^*)}{1+\delta_k(m,m^*)}$, where $\delta_k(m, m^*) := sup_{|x_i| \leq k} |m(x) - m^*(x)|$, (cf. Pollard, 1984, Sec. V.5.).

Lemma 2.3. *Suppose $RC1$, $RC2$ and*

RC3 $k = o(n^{(2\delta-2)/(2\delta-1)})$

hold. Then, as $n \to \infty$,

$$\sqrt{k}(\hat{m} - m) \xrightarrow{D_\delta} B. \tag{2.17}$$

Consistency of $\hat{m}$: While regularity conditions $RC1$, $RC2$ and $RC3$ imply the asymptotic normality of $\hat{m}$, the asymptotic consistency of $\hat{m}$ requires weaker assumptions.

Lemma 2.4. *Suppose that the domain-of-attraction condition (2.13) holds and let $k = k(n) \to \infty$, $k \leq n$, $\frac{k}{n} \to 0$. Then,*[4]

$$sup_{|x_i| \leq T} |\hat{m}(x) - m(x)| \xrightarrow{P} 0.$$

[4]Here, $\xrightarrow{P}$ stands for convergence in probability. I.e., as $n \to \infty$, $x_n \xrightarrow{P} x$, if $\mathrm{P}(|x_n - x| > \varepsilon) \to 0$,
for all $\varepsilon > 0$.

2.3 Testing the Homogeneity of $\hat{m}$ and Testing for Association

The dependence function m satisfies the homogeneity condition $m(ux) = um(x)$, for all $x \in \mathbf{R}^d$ and $u > 0$. Note that although $\hat{m}$ is not a homogeneous function it should be homogeneous in the limit.

It is desirable to test homogeneity of $\hat{m}$ prior to the test for association. In fact, if $\hat{m}(t\tau(\vartheta)) \approx t\hat{m}(\tau(\vartheta))$ and if $(Z^{(1)}, \cdots, Z^{(d)})$ are positively associated (see Lemma 2.1), then $\hat{m}(t\tau(\vartheta)) \approx 0$ for $\vartheta \in \Omega_d^-, t > 0$. As test statistic we suggest the maximum difference of $\frac{1}{t}\hat{m}(t\tau(\vartheta))$ and $\hat{m}(\tau(\vartheta))$. I.e., for $0 < T_1 < T_2$ define the test statistic

$$h_n(T_1, T_2) := \sup_{\substack{T_1 \le t \le T_2 \\ \vartheta \in \Omega_d}} \left| \frac{1}{t}\hat{m}(t\tau(\vartheta)) - \hat{m}(\tau(\vartheta)) \right|,$$

Alternatively, one may consider the test statistic

$$h_n(T_1, T_2) := \sup_{\substack{T_1 \le t \le T_2 \\ \vartheta \in \Omega_d}} \left| \frac{\frac{1}{t}\hat{m}(t\tau(\vartheta))}{\hat{m}(t\tau(\vartheta))} - 1 \right|.$$

However, this statistic will lead to some difficulties due to possible singularities of the quotient.

Lemma 2.5. *Suppose $(RC1), (RC2)$ and $(RC3)$ hold. Then, as $n \to \infty$,*

$$\sqrt{k}h_n(T_1, T_2) \xrightarrow{w} h(T_1, T_2), \textit{ where } h(T_1, T_2) = sup_{\substack{T_1 \le t \le T_2 \\ \vartheta \in \Omega_d}} \left| \frac{1}{t}B(t\tau(\vartheta)) - B(\tau(\vartheta)) \right|, \tag{2.18}$$

and the Gaussian process B is defined by (2.17).

Accepting homogeneity of $\hat{m}$, means

$$\hat{m}(t\tau(\vartheta)) \approx t\hat{m}(\tau(\vartheta)) \tag{2.19}$$

If m is concentrated on the orthants that have no sign changes, we expect that $\hat{m}$ is almost zero on Ω_d^-. Due to (2.19), this is equivalent to $\hat{m}(t\tau(\vartheta)) \approx 0$, for $t > 0$. This leads to the following test for association.

Theorem 2.1. *Suppose $Z = (Z^{(1)}, \cdots, Z^{(d)})$ is in the domain of attraction of a $(\alpha_1, \cdots, \alpha_d)$-stable r.v. X with c.f. (2.2) and suppose that either Z is positively or negatively associated. Moreover, assume that regularity conditions $RC1, RC2$ and $RC3$ hold. Then, in case of positive association*

$$\sqrt{k}\; sup_{\vartheta \in \Omega_d^-} |\hat{m}(\tau(\vartheta))| \xrightarrow{\omega} sup_{\vartheta \in \Omega_d^-} |B(\tau(\vartheta))|. \tag{2.20}$$

In case of negative association, (2.20) holds with Ω_d^- being replaced by Ω_d^+.

If Z is positively associated, then $m(\tau(\vartheta)) = 0$, for $\vartheta \in \Omega_d^-$, and so $m(t\tau(\vartheta)) = 0$, for all $t > 0$ and $\vartheta \in \Omega_d^-$. Cov $B(\tau(\vartheta)) = W(\tau(\vartheta)), \vartheta \in \Omega_d^-$. Applying this to Theorem 2.1 leads to the simplified covariance function of W instead of B. Consequently, the simulation of $\sup_{\vartheta \in \Omega_d^-} |B(\tau(\vartheta))|$ can be simplified.

2.4 Asymptotic Confidence Bound and the Bootstrap Method

To estimate $\mathrm{P}(h(T_1, T_2) > \epsilon)$ and $\mathrm{P}(\sup_{\vartheta \in \Omega_d^-} |B(\tau(\vartheta))| > \epsilon)$ in the right-hand- side of the limit relations (2.18) and (2.20) we use two approaches: one consists of construct an universal upper bound; the other generates bootstrap estimates of the confidence intervals for $h(T_1, T_2)$ and $\sup_{\vartheta \in \Omega_d^-} |B(\tau(\vartheta))|$.

Lemma 2.6. *For any $\epsilon > 0$ and $0 < T_1 < T_2$, the following bounds for the tail probabilities hold:*

$$P(h(T_1, T_2) > \epsilon) \;\leq\; 2^d \phi\left(\frac{T_1 \epsilon}{(d+1)\sqrt{T_2}}\right) \;+\; 2^d \phi\left(\frac{\epsilon}{(d+1)T_2}\right), \tag{2.21}$$

and

$$P(sup_{\vartheta \in \Omega_d^-} |B(\tau(\vartheta))| > \epsilon) \leq 2^d \left(\phi\left(\frac{\epsilon}{2d}\right) + \phi\left(\frac{\epsilon}{2}\right)\right), \tag{2.22}$$

where $\phi(\epsilon) = \frac{2}{\sqrt{2\pi}} \int_\epsilon^\infty e^{-x2/2} dx$.

To obtain slight improvements for these bounds,note that the covariance function, K, of B is of the form

$$K(x,y) = m(x \wedge y) \;+\; \sum_{i,j} m_i'(x) m_j'(y) m(\infty, \cdots, x_i, \cdots y_j, \cdots, \infty)$$

$$-\sum_{i=1}^{d} \left(m_i'(x) m(y_1, \cdots, x_i \wedge y_i, \cdots, y_d) \;+\; m_i'(y) m(x_1, \cdots, x_i \wedge y_i, \cdots, x_d)\right). \tag{2.23}$$

Defining

$$\begin{aligned} \mathrm{p}(u) &= \max\left\{[\mathrm{E}(B(s) - B(t))^2]^{\frac{1}{2}}; s, t \in \mathbf{R}^d, ||s-t||_\infty \leq u\sqrt{d}\right\} \\ &= \max_{||t-s||_\infty \leq u\sqrt{d}} (K(t,t) + K(s,s) - 2K(s,t))^{1/2}, \end{aligned}$$

where $||t||_\infty = \max |t_i|$, one obtains from Garsia (1970, Theorem 1) a sharper bound for the suprema in Lemma 2.6. Assuming $\int_o^T (\log \frac{1}{u})^{1/2} dp(u) < \infty$, there exists a version X of B such that

$$\sup_{||t-s|| \leq r} |X(t) - X(s)| \;\leq\; 16(\ln Z)^{1/2} p(r) \;+\; 16(2k)^{1/2} \int_o^r \left(\ln \frac{1}{u}\right)^{1/2} dp(u), \tag{2.24}$$

where $Z \geq 0$ is a random variable with E $Z \leq \sqrt{8}$. This leads by use of Tschebycheff's inequality to a sharpened bound in (2.21) and (2.22). We will not discuss this further because the bootstrap technique will lead to bounds for the tail probabilities in (2.21) and (2.22) which are considerably better.

The bounds in Lemma 2.6 are indeed too conservative. To get a bootstrap estimator for the limiting process B in (2.18) on the positive orthant, we construct a bootstrap version of $\hat{m}$. First, we outline the procedure in the case $d = 2$ (see Xin, 1992, for detail).

Let $\rho(\theta) = 1/\mu((0, \cos\theta) \times (0, \infty) \cup (0, \infty) \times (0, \sin\theta))$ for $0 < \theta < \frac{\pi}{2}$, where μ is defined by (2.6). Invoking estimator $\hat{m}$ (see (2.12)) define

$$\hat{\rho}(\theta) = \left\{ \frac{[K\cos\theta]}{K} + \frac{[K\sin\theta]}{K} - \hat{m}(\cos\theta, \sin\theta) \right\}. \tag{2.25}$$

Next, define the trimmed version of $\hat{\rho}$ by

$$\hat{\rho}(\theta) = \ \min\left(\hat{\rho}(\theta), \frac{1}{\cos\theta}, \frac{1}{\sin\theta}\right);$$

and let $(\tilde{\rho}(\theta), \theta)_{0 \le \theta \le \pi_2}$ be the unique concave curve satisfying $\tilde{\rho}(\theta) \ge \hat{\rho}_1(\theta),\ 0 \le \theta \le \frac{\pi}{2}$. We set $\tilde{\rho}(\theta)$ to be the minimal concave curve with this property.

To define the bootstrap version of $\hat{m}$ in the first quadrant, set for $(x, y) \in \mathbf{R}^2_+$,

$$\tilde{m}(x, y) = x + y + \frac{\sqrt{x^2 + y^2}}{\tilde{\rho}(\mathrm{arctg}\ \frac{y}{x})}. \tag{2.26}$$

The attractive feature of $\tilde{m}$ is that the trimmed version of $\tilde{m}$,

$$\tilde{m}_n(x, y) = \tilde{m}\left(x \wedge \frac{1}{\tilde{m}(1,1)},\ \ y \wedge \frac{1}{\tilde{m}(1,1)}\right),$$

is a distribution function on $\mathbf{R}^2_+$.

We now describe the bootstrap procedure for the estimator $\hat{m}$ in (2.12). We choose k replications $(Z^*_j = (Z^{*(1)}_j, Z^{*(2)}_j))_{1 \le j \le k}$ of Z^* with d.f. $\tilde{m}_n$. Under some regularity conditions, the bootstrap estimator

$$m^*(x, y) = \frac{1}{h} \sum_{j=1}^{k} \mathbf{1}_{\left\{\mathbf{Z}^{*(1)}_{\mathbf{j}} \le \mathbf{Z}^{*(1)}_{[\mathbf{hx}]:\mathbf{k}}, \mathbf{Z}^{*(2)}_{\mathbf{j}} \le \mathbf{Z}^{*(2)}_{[\mathbf{hy}]:\mathbf{k}}\right\}} \tag{2.27}$$

is asymptotically normal. More precisely, on $D([0; T]^2)$

$$\sqrt{h}(m^* - \hat{m}) \xrightarrow{w} B. \tag{2.28}$$

In fact, the following limiting result holds (cf. Xin, 1992, Theorem 5).

Lemma 2.7. *Let $0 < h \le k$ and, as $n \to \infty$, $h \to \infty$, $\frac{h^{3/2}}{k} \to 0$ and $\frac{k^{\alpha+1}}{h^{1/\alpha} n^{\alpha}} \to 0$. Moreover, assume that for each fixed $M > 0$*

(i) $\frac{k}{\sqrt{h}} \left\{ \frac{h}{k} M - m(\frac{h}{k} M, 1) \right\} \to 0$; and

(ii) $\frac{k}{\sqrt{h}} \left\{ \frac{h}{k} M - m(1, \frac{h}{k} M) \right\} \to 0$.

Then, for each $\epsilon > 0$, there exist $N_\epsilon > 0$ and a set $A_n(\epsilon)$ in the Borel σ-field generated by $D([0, T]^2)$ with $P(A_n(\epsilon)) \ge 1 - \epsilon$, such that, as $n \to \infty$,

$$\sqrt{h}\,(m^*(\cdot, w_n) - \hat{m}(\cdot, w_n)) \xrightarrow{w} B \tag{2.29}$$

on $D([0, T]^2)$, for all $w_n \in A_n(\epsilon)$.

Unfortunately, it seems difficult to extend this bootstrap approach for higher dimensions, i.e. $d \geq 3$. An alternative bootstrap procedure, working for any dimension, can be obtained by following the construction of de Haan and Resnick (1993). It rests on the following result for Poisson processes (cf. deHaan and Resnick, 1993, Lemma 5.2).

Lemma 2.8. *Let $(N_n), n \geq 1$, be Poisson processes with intensity measures $(\mu_n)_{n\geq 1}$ and dependence functions $(m_n)_{n\geq 1}$ (cf. (2.6)). Suppose that the sequence $h(n)$ satisfies:*
(i) $h(n) \to \infty$,
(ii) $\frac{h(n)}{n} \to 0$, and
(iii) $\frac{m_n - hm}{\sqrt{h}}$ converges weakly to 0 in $D(\mathbf{R}^d/\{0\})$, where m is the dependence function generated by μ.
Then, the empirical dependence function $\hat{m}_n$ corresponding to N_n is asymptotically normal

$$\frac{1}{\sqrt{h}}(\hat{m}_n - hm)$$

converges weakly to a Gaussian process Y with covariance function

$$Cov(Y(x), Y(y)) = m((x_{\wedge J} y_J), (x_{J^c} \vee y_{J^c})),$$

for x, y in the orthant Ω_J with $x_J \geq 0$, and

$$Cov(Y(x), Y(y)) = 0,$$

for x, y in any other orthant.

For the bootstrap procedure we choose $h = o(k)$, where $k = k(n)$ is specified in Lemma 2.3 and generate iid copies, $N_n^{(i)}$, of a Poisson process with intensity measures $\hat{\nu}_n = h\hat{\mu}_n$, where $\hat{\mu}_n$ is the measure corresponding to the empirical dependence function $\hat{m}_n$ as defined in (2.12). Then, by Lemma 2.3,

$$\frac{1}{\sqrt{n}}(h\hat{m}_n - hm) = \sqrt{h}(\hat{m}_n - m) = \sqrt{\frac{h}{k}}\,\sqrt{k}(\hat{m}_n - m)$$

converges weakly to 0. Because $h\hat{m}_n$ is the dependence function of $\hat{\nu}_n$, Lemma 2.8 implies that, as $n \to \infty$, $\frac{1}{\sqrt{h}}(\hat{m}_n^{(i)} - hm)$ converges weakly to a Gaussian process $B^{(i)}$ with covariance structure given in (2.17). From the replications one obtains approximations of the quantiles of the supremum of the Gaussian random field B by taking the corresponding empirical quantile. As a consequence of the approximation result in Lemma 2.3 we obtain

$$\sqrt{k}\Big(\int f d\hat{\mu}_n - \int f d\mu\Big) \to \int f dB, \quad \text{as}\, n \longrightarrow \infty,$$

for all bounded continuous functions f, where $\hat{\mu}_n$ is the measure corresponding to the empirical dependence function $\hat{m}_n$, and μ is generated by m.
This allows us to obtain estimates for measures of association and (bootstrap) confidence intervals. In particular, one could consider functionals of the form

$$F(\mu) \;=\; \frac{1}{r_1}\int_{\Omega_{1,+}} ||\vartheta - \vartheta_1||^2 d\mu(\vartheta) \;+\; \frac{1}{r_2}\int_{\Omega_{1,-}} ||\vartheta - \vartheta_2||^2 d\mu(\vartheta) \tag{2.30}$$

where $\Omega_{1,+}$ and $\Omega_{1,-}$ are the intersections of the sphere with the positive and negative orthants respectively, and $r_1 = \mu(\Omega_{1,+}), r_2 = \mu(\Omega_{1,-})$. The related empirical version defined by

$$F(\hat{\mu}_n) \;:=\; \frac{1}{\hat{r}_1}\int_{\Omega_{1,+}} ||\vartheta - \hat{\vartheta}_1||^2 d\hat{\mu}_n(\vartheta) \;+\; \frac{1}{\hat{r}_2}\int_{\Omega_{1,-}} ||\vartheta - \hat{\vartheta}_2||^2 d\hat{\mu}_n(\vartheta) \tag{2.31}$$

with $\hat{\vartheta}_1 = \int_{\Omega_{1,+}} \vartheta\; d\hat{\mu}_n(\vartheta), \quad \hat{\vartheta}_2 = \int_{\Omega_{1,-}} \vartheta\; d\hat{\mu}_n(\vartheta)$. It is strongly consistent under the conditions of Lemma 2.4. Moreover, from Lemma 2.3, $\sqrt{k}(F(\hat{\mu}_n) - F(\mu))$ converges weakly to the corresponding functional of the Gaussian measure B.

If $F(\mu)$ is zero, then μ has support on ϑ_i and, therefore, the $Z^{(i)}$ are positively associated. Hence, $F(\hat{\mu})$ can be used as an empirical measure of positive association.

2.5 Measuring Portfolio Risk

In contrast to single-stable-index portfolios there is no accepted notion of portfolio risk in multiple-stable-index portfolios. Here, we present a practical approach to express portfolio risk.

Let $X = (X_1, \cdots, X_d)'$ denote the return vector of a d-asset portfolio and $c = (c_1, \cdots, c_d)$, $\sum_{i=1}^d c_i = 1$, the vector of portfolio weights. We assume there is no short-selling, i.e., $c_i \geq 0$, $i = 1, \cdots,$ d. We define the portfolio risk associated with weight vector c by

$$r(c) := E|c \cdot X - c \cdot EX|,$$

where ^{T} denotes the transpose of a vector or matrix. In fact, $r(c)$ is the L_1-*variation* (assuming that all $\alpha_i \geq 1$) or the L_1-*risk.* A portfolio c^* is then *risk minimal,* if $r(c^*) \leq r(c)$ for all portfolios c on the simplex $\{c_i \geq 0, \sum c_i = 1\}$.

In order to estimate c^* and $r(c^*)$ we need an empirical version of $r(c)$, denoted by $\hat{r}_n(c)$. To this end, we use the empirical distribution of the return vector X. It can be obtained from the empirical versions of the dependence function (or the spectral measure). To construct the spectral measure ν of the distribution (2.2), we have to estimate the transformation $\tau(x)$, and, therefore, indexes $\alpha_1, \cdots, \alpha_d$.

This can be achieved by employing the one-dimensional Hill tail-estimator for each α_i or alternative estimators given in Mittnik and Rachev (1995). Next, we need to numerically invert the empirical c.f.. The final step is to construct confidence intervals. For this we use the dual representation for the distance between c.f.'s and distances between d.f.'s, see Rachev (1991).

3 Proofs

3.1 Proof of Lemma 2.1

Recall that for $i = 1, \cdots, d$,

$$U_i(x) := \begin{cases} \left(\frac{p_i}{P(Z^{(i)} > x)}\right) & \text{if } x > 0, \\ \left(\frac{-q_i}{P(Z^{(i)} \leq x)}\right) & \text{if } x < 0, \end{cases} \tag{3.1}$$

where $q_i = 1 - p_i$ and

$$p_i := \lim_{x \longrightarrow \infty} P(Z^{(i)} > x)/P(|Z^{(i)}| > x). \tag{3.2}$$

Then, $U = (U_1(Z^{(1)}), \cdots, U_d(Z^{(d)}))$ belongs to the domain of attraction of a $(1, \cdots, 1)-$ stable law. We now set $(|U|, \vartheta(U))$ to be the polar coordinates of U and, thus, rewrite the domain of attraction condition (2.1) as

$$\lim_{x \to \infty} \frac{P(|U| > xr,\ \vartheta(U) \leq \vartheta_1)}{P(|U| > xr,\ \vartheta(U) \leq \vartheta_2)} = r^{-1} \frac{S(\vartheta_1)}{S(\vartheta_2)} \tag{3.3}$$

for all $\vartheta_1, \vartheta_2 \in \Omega_d$, (see Resnick and Greenwood, 1979). In other words, U is in the domain of attraction of a 1-stable vector Y with spectral *df* S, i.e.

$$\begin{aligned} &\log Ee^{i<y,t>} = \\ &\quad -|t| \textstyle\int_{\Omega_d} |\cos < t, \vartheta > |(1 + i \text{ sign } (\cos < t, \vartheta >) \frac{2}{\pi}\ ln(\rho| \cos < t, \vartheta > |) dS(\vartheta), \end{aligned} \tag{3.4}$$

for $t = (\rho \sin\varphi_1 \cdots \sin\varphi_{d-1}, \cdots, \rho\cos\varphi_1)$ and $\cos(t, \vartheta) := (\prod_{i=1}^d \sin\varphi_i \sin\vartheta_i) + \cdots + \cos\varphi_1 \cos\vartheta_1$.

By assumption (i) on Z the vector U has positively associated components and, because U is in the domain of attraction of Y, it follows that Y is also positively associated. To complete the proof, we apply the criterion for association of $\alpha-$stable vectors (cf. Lee, Rachev and Samorodnitsky, 1993) and conclude that $S(\Omega_d^-) = 0$. The proof of the case of negative association is analogous. $\Box$.

3.2 Proof of Lemma 2.2

.

The proof of (2.7) follows the arguments of de Haan and Resnick (1977). For x with $\vartheta(x)$ in the orthant $\Omega_{d,J}$ we have[5]

$$m(x) = \tilde{\nu}\left(\prod_{j\in J}[\frac{p_j}{x_j},\infty)\times(\prod_{j\in J^c}(-\infty,\frac{q_j}{x_j}]\right) = \int \frac{1}{r^d}\,dr\,dS(\vartheta)$$

where the integral is taken over the region

$$\left\{(r,\vartheta): r\tau_j(\vartheta)\geq\frac{p_j}{x_j},\ j\in J, r\tau_j(\vartheta)\leq\frac{q_j}{x_j}, j\in J^c\right\}.$$

Thus, $m(x)=\int\ r^{-d}\ dr\ dS(\vartheta)$, where the integration is now over the region

$$\left\{(r,\vartheta): r\geq\max\left(\frac{p_j}{\tau_j(\vartheta)x_j},\ j\in J\right), r\geq\min\left(\frac{q_j}{\tau_j(\vartheta)x_j},\ j\in J\right)\right\}$$

This implies that

$$m(x) = \int_{\theta\in\Omega_{d,J}}\left(\int \mathbf{1}_{\left\{\max_{j\in J}\left(\frac{p_j}{\tau_j(\vartheta)x_j}\right)\leq r\leq\min_{j\in J^c}\left(\frac{q_j}{\tau_j(\theta)x_j}\right)\right\}} d(\frac{r^{-d+1}}{-d+1})\right) dS(\vartheta) \quad (3.5)$$

$$= \frac{1}{d-1}\int_{\Omega_{d,J}}\left(\min(\frac{\tau_j(\vartheta)x_j}{p_j})_J^{d-1} - \max\left(\frac{\tau_j(\vartheta)x_j}{q_j}\right)_{J^c}^{d-1}\right) dS(\vartheta). \quad (3.6)$$

To show (2.8), we use another representation for $m(x)$. If the $Z^{(i)}$'s have continuous *df's*, F_i, the domain of attraction condition (2.10) can be written as:

$$m(x) = \lim_{t\downarrow 0}\frac{1}{t}m^{(t)}(x), \quad (3.7)$$

for all $x\in\mathbf{R}^d/\{0\}$. Setting $\bar{F}_i = 1-F_i$ and defining $F_i^{\leftarrow} and \bar{F}_i^{\leftarrow}$ to be the inverses of F_i and $\bar{F}_i$ respectively, as well as analogously $\bar{F}_J^{\leftarrow} = (\bar{F}_j^{\leftarrow})_{j\in J}, F_{J^c}^{\leftarrow} := (F_j^{\leftarrow})j\in J$, then $m^{(t)}(x)$, in (3.4), has the form:

$$m^{(t)}(x) \ := \ P(Z^{(J)}\geq\bar{F}_J^{\leftarrow}(tx_J), Z^{(J^c)}\leq F_{J^c}^{\leftarrow}(-tx_{J^c})),$$

for $x_J\geq 0, x_{J^c}<0$.

[5]-Recall that $x_j\geq 0,\ j\in J$ and $x_j<0,\ j\in J^c$ in this orthant

To prove (3.34) note that by definition

$$\begin{aligned}
m(x) &= \lim_{n\to\infty} nP\left(U_j(Z^{(j)})/n \geq \tfrac{p_j}{x_j},\ j\in J,\ U_j(Z^{(j)})/n \leq \tfrac{q_j}{x_j},\ j\in J^c\right)\\
&= \lim_{n\to\infty} nP\left(\tfrac{p_j}{n\bar F_j(Z^{(j)})I_{\{Z^{(j)}>0\}}} \geq \tfrac{p_j}{x_j},\ j\in J,\right.\\
&\qquad \left.\tfrac{q_j}{n\bar F_j Z^{(j)}} I_{\{Z^{(j)}<0\}} \leq \tfrac{q_j}{x_j},\ j\in J^c\right)\\
&= \lim_{n\to\infty} nP\left(n\bar F_{j,+}(Z^{(j)}) \leq x_j,\ j\in J, n\bar F_{j,+}(Z^{(j))} \leq x_j,\ j\in J^c\right)\\
&\quad \left(\text{with } \bar F_{j,+}(x) = P(Z^{(j)}>x)I_{\{x>0\}}\right)\\
&= \lim_{n\to\infty} nP\left(n\bar F_j(Z^{(j)}) \leq x_j, Z^{(j)}>0,\ j\in J,\right.\\
&\quad \left. n\bar F_j(Z^{(j)}) \geq x_j, Z^{(j)}<0,\ j\in J^c\right)\\
&= \lim_{n\to\infty} nP\left(Z^{(j)} \geq \bar F_j^{\leftarrow}(\tfrac{x_j}{n}), Z^{(j)}>0,\ j\in J,\right.\\
&\quad \left. Z^{(j)} \leq \bar F_j^{\leftarrow}(\tfrac{x_j}{n}), Z^{(j)}<0,\ j\in J^c\right)\\
&= \lim_{n\to\infty} \tfrac{1}{t}P\left(Z^{(j)} \geq (\bar F_j^{\leftarrow}(tx_j))_+,\ j\in J,\right.\\
&\quad \left. Z^{(j)} \leq (\bar F_j^{\leftarrow}(tx_j))(+x_j)_j\ j\in J^c\right)
\end{aligned}$$

Where $(a)_+ := max(0,a)$ and $(a)_- := min(0,a)$
Then,

$$m(x) = \lim_{n\to\infty} \frac{1}{t} P\left(Z^{(j)} \geq \bar F_j^{\leftarrow}(tx_j),\ j\in J, Z^{(j)} \leq \bar F_j^{\leftarrow}(tx_j),\ j\in J^c\right),$$

since as $t\to 0, x>0, \bar F_j^{\leftarrow}(tx_j)\to\infty,\ j\in J$ and $\bar F_j^{\leftarrow}(tx_j)\to -\infty,\ j\in J^c$.

3.3 Proof of Lemma 2.3.

On the first orthant $\Omega_{d,+} := \{x; x\geq 0\}$ the assertion of Lemma 2.3 reads as follows. *Suppose that for some* $\delta>1$

$$sup_{||x||=1}[\bar F(\bar F_1^{\leftarrow}(tx_1),\cdots,\bar F_d^{\leftarrow}(tx_d)) - tm(x)] = O(t^\delta),\ as\, t\to 0, \tag{3.8}$$

where $\bar F(x) = P(Z>x)$*, and that* m *has continuous partial derivatives* m_i'*. Letting* $k = o(n^{\frac{2\delta-2}{2\delta-1}})$*, then, on* $\Omega_{d,+}$*,*

$$\sqrt{k}(\hat m - m) \xrightarrow{D_\delta} B, \tag{3.9}$$

where $B(x) = W(x) - \sum_{i=1}^d m_i'(x)W(\infty,\cdots,x_i,\cdots,\infty)$*; and* W *is a continuous zero mean Gaussian process with covariance* $EW(x)W(y) = m(x_1\wedge y_1,\cdots,x_d\wedge y_d)$*.*

To show (3.9), we first uniformize the marginals. Let

$$\begin{aligned}
&U_j^{(i)} := \bar F_i(Z_j^{(i)}), \qquad i=1,\cdots,d,\\
&U \stackrel{d}{=} U_1 = (U_1^{(1)},\cdots,U_1^{(d)}),\\
&\tilde U(x) = \bar F(\bar F_1^{\leftarrow}(x_1),\cdots,\bar F_d^{\leftarrow}(x_d)) = P(U^{(1)}\leq x_1,\cdots,U^{(d)}\leq x_d),
\end{aligned}$$

and let $E_n(x)$ be the empirical counterpart of $\frac{n}{k}\,\tilde U(\frac{k}{n}x)$, i.e.,

$$E_n(x) := \frac{1}{k}\sum_{j=1}^n \mathbf{1}(U_j^{(1)} \leq \frac{k}{n}x_1,\cdots,U_j^{(d)} \leq \frac{k}{n}x_d).$$

Claim 1. On the Skorohod space $D(\Omega_{d,+})$ the tail process $W_n(x) := \sqrt{k}(E_n(x) - \frac{n}{k}\tilde{U}(\frac{k}{n}x))$ converges in D_ρ to $W(x)$.

Proof of Claim 1. The convergence of finite dimensional distributions can be checked as in Rachev and Xin (1993), applying the Cramr-Wold device and Lyapunov's Theorem. To establish the D_ρ-convergence of W_n to W, we need to show that for each $K > 0$ and $\epsilon > 0$ there is a grid $I_i = \prod_{j=1}^{d}[\frac{k}{m}i_j, \frac{k}{m}(i_j+1)]$, $i = (i_1, \cdots, i_d)$, $0 \le i_j \le m-1$, such that the $\rho-$tightness condition holds. The latter has the form

$$\lim\ sup\ P\left(\max_{0\le i_j \le m-1} sup_{x\in I_i}|W_n(x) - W(\frac{k}{m}i)| > \epsilon\right) < \epsilon.$$

We apply the following inequality due to Einmahl (1987). Let $(Z_i)_{i=1,\cdots,n}$ be a random sample from a probability law Q on $\mathbf{R}^d$ and let Q_n be the corresponding empirical measure of Q. Let $\mathcal{A}$ be the semiring of rectangles in $\mathbf{R}^d$; and $A \in \mathcal{A}$ with $0 < Q(A) < \frac{1}{2}$ and $0 < \delta < 1$. Then, there exists a function $\varphi(\lambda) > 0$ with $\varphi(\lambda) \uparrow 1$ as $\lambda \downarrow 0$, such that for all $\lambda > 0$,

$$P\left(\sup_{\tilde{A}\subset A}|\sqrt{n}(Q_n(\tilde{A}) - Q(\tilde{A}))| \ge \lambda\right) \;\le\; K(\delta)\exp\left(\frac{-(1-\delta)}{2Q(A)}\lambda^2\varphi\left(\frac{\lambda}{n^{1/2}Q(A)}\right)\right),$$

where $K(\delta)$ is an absolute positive constant. Taking $K = 1$ and applying Einmahl's inequality we get

$$\begin{aligned}
&P\left(\max_{0\le i_j\le m-1}(\sup_{I_i}|W_n(x) - W_n(\tfrac{1}{m}i)| > \epsilon)\right)\\
\le\;& \sum_{j=0}^{d-1}\sum_{0\le i_j\le m-1} P(\sup_{I_i}|W_n(x_{i,j}) - W_n(x_{i,j+1})| > \tfrac{\epsilon}{d})\\
&\left(\text{where } x_{i,j} = (x_1, \cdots, x_{d-j}, \tfrac{i_{d-j+1}}{m}, \cdots, \tfrac{i_d}{m})\right)\\
\le\;& \sum_{j=0}^{d-1}\sum_{i_j\le m-1} P(\sup_{I_i}\sqrt{n}|\tfrac{1}{n}\textstyle\sum_{l=1}^{n}\{I(U_l^{(1)} \le \tfrac{k}{n}x_1, \cdots, \tfrac{k}{n}\tfrac{i_d}{m} \le U_l^{(d)} \le \tfrac{k}{n}\tfrac{i_d+1}{m})\\
-\;& P(U^{(1)} \le \tfrac{k}{n}x_1, \cdots, \tfrac{k}{n}\tfrac{i_d}{m} \le U^{(d)} \le \tfrac{k}{n}\tfrac{i_d+1}{m})\}| \ge \tfrac{k}{\sqrt{n}}\epsilon)\\
\le\;& \sum_{j=0}^{d-1}\sum_{0\le i_j\le m-1}[K(\tfrac{1}{2})\exp(\tfrac{-\epsilon^2\frac{k^2}{n}}{8P(G_{l,j})}\varphi(\tfrac{\epsilon\frac{k}{\sqrt{n}}}{2\sqrt{n}P(G_{l,j})})]\\
&(\text{with } G_{i,j} = (\tfrac{k}{n}\tfrac{i_j}{m} \le U^{(j)} \le \tfrac{k}{n}\tfrac{i_j+1}{m}))\\
\le\;& d\sum_{0\le i_j\le m-1, 1\le j\le d} K(\tfrac{1}{2})\exp\left\{\tfrac{-\epsilon^2 mn}{8}\varphi(\tfrac{\epsilon mn}{2})\right\}.
\end{aligned}$$

This proves the $\rho-$tightness condition and, thus, completes the proof of the claim.

From Claim 1 we have for the first marginal of W_n

$$\begin{aligned}
W_n(x_1, \infty, \cdots, \infty) &= \sqrt{k}(E_n(x_1, \infty, \cdots, \infty) - \tfrac{n}{k}\tilde{U}(\tfrac{k}{n}x_1, \infty, \cdots, \infty))\\
&= \sqrt{k}\left(\tfrac{1}{k}\textstyle\sum_{j=1}^{n} I(U_j^{(1)} \le \tfrac{k}{n}x_1) - \tfrac{n}{k}P(U^{(1)} \le \tfrac{k}{n}x_1)\right)\\
&= \sqrt{k}\left(\tfrac{1}{k}\textstyle\sum_{j=1}^{n} I(U_j^{(1)} \le \tfrac{k}{n}x_1) - x_1\right) \xrightarrow{D_\rho} W(x_1, \infty, \cdots, \infty).
\end{aligned}$$

Inverting $\frac{1}{k}\sum_{j=1}^{n} I(U_j^{(1)} \leq \frac{k}{n}x)$ and applying Lemma 1 in Vervaat (1972) we get

$$\sqrt{k}\left(\frac{n}{k}U^{(1)}_{[kx_1]:n} - x_1\right) \xrightarrow{D_\rho} -W(x_1, \infty, \cdots, \infty).$$

In a similar way, we obtain for the other components

$$\sqrt{k}\left(\frac{n}{k}U^{(j)}_{[kx_j]:n} - x_j\right) \xrightarrow{D_\rho} -W(\infty, \cdots, x_j, \cdots, \infty).$$

Finally, applying again the claim in combination with regularity condition (3.8) we get[6]

$$\begin{aligned}
\sqrt{k}(E_n(x) - m(x)) &= \sqrt{k}\left(E_n(x) - \frac{n}{k}\tilde{U}(\frac{k}{n}x)\right) + \sqrt{k}\left\{\frac{n}{k}\tilde{U}(\frac{k}{n}x) - m(x)\right\} \\
&= W_n(x) - \sqrt{k}\left\{\frac{n}{k}\bar{F}(\bar{F}_1^{\leftarrow}(\frac{k}{n}x_1), \cdots, \bar{F}_d^{\leftarrow}(\frac{k}{n}x_d)) - m(x)\right\} \\
&= W_n(x) - \sqrt{k}\left\{\frac{n}{k}O((\frac{k}{n})^\rho)\right\} \text{ (recall that } k = O(n^{\frac{2\delta-2}{2\delta-1}})) \\
&= W_n(x) - o(1).
\end{aligned}$$

Therefore,

$$\sqrt{k}(E_n(x) - m(x)) \xrightarrow{D_\rho} W(x).$$

We now replace x in the above limit relation with $\left(\frac{n}{k}U^{(i)}_{[kx_i]:n}\right)$. Then

$$\begin{aligned}
E_n\left((\frac{n}{k}U^{(i)}_{[kx_i]:n})_{i=1,\cdots,d}\right) &= \frac{1}{k}\sum_{j=1}^{n} I(U_j^{(1)} \leq U^{(1)}_{[kx_1]:n}, \cdots, U_j^{(d)} \leq U^{(d)}_{[kx_d]:n}) \\
&= \frac{1}{k}\sum_{j=1}^{n} I\left(Z_j^{(1)} \geq Z^{(1)}_{n-[kx_1]+1:n}, \cdots, Z_j^{(d)} \geq Z^{(d)}_{n-[nx_d]+1:n}\right) \\
&= \hat{m}(x).
\end{aligned}$$

From the mean-value-theorem

$$\sqrt{k}\left(m(\frac{n}{k}U^{(1)}_{[kx_1]:n}, \cdots, \frac{n}{k}U^{(d)}_{[kx_d]:n}\right) - m(x) \xrightarrow{D_\rho} -\sum_{i=1}^{d} m_i'(x)W(\infty, \cdots, x_i, \cdots, \infty).$$

Implying finally, that

$$\sqrt{k}(\hat{m} - m) \xrightarrow{D_\rho(Q_1)} B.$$

In a similar way we show convergence on the other orthants and obtain the desired result. $\Box$.

3.4 Proof of Lemma 2.4.

The proof follows the arguments in Xin (1992, p. 27) and Rachev and Xin (1993, Lemma 1, (B)).

[6]Observe that from the homogeneity of m one can take the maximum in (3.5) over every bounded set in $\mathbf{R}_+^d$

3.5 Proof of Lemma 2.5.

The limit relation (2.19) is a direct consequence of the asymptotic normality of $\hat{m}$, see (2.18).

3.6 Proof of Theorem 2.1.

Under (i) $m(\tau(\vartheta)) = 0$, for $\vartheta \in \Omega_d^-$, so by (2.18)

$$\sup_{\vartheta\in\Omega_d^-} \sqrt{k}|\hat{m}(\tau(\vartheta))| \xrightarrow{w} \sup_{\vartheta\in\Omega_d^-} |B\tau(\vartheta)| \tag{3.10}$$

Note that for $\vartheta \in \Omega_d^-$, $m(\tau(\vartheta)) = 0$ and, therefore, by using the homogeneity of m $B(\tau(\vartheta)) = W(\tau(\vartheta))$. $\Box$.

3.7 Proof of Lemma 2.6.

Recall the definition of $B(x) = W(x) - \sum_{i=l}^d (\text{sign } x_i) m_i'(x) W(\infty, \cdots, x_i, \cdots, \infty)$. For any $\delta > 0$ and $x = (x_1, \cdots, x_d) > 0$, $m(x_1 + \delta, x_2, \cdots, x_d) - m(x) = \mu((x_1, x_1 + \delta] \times \prod_{i=2}^d (0, x_i]) \leq \delta$. Therefore, $m_1'(x) \leq 1$. In the same way we see that $|m_i'| \leq 1$. Set now $M_B(T) := \sup_{0\leq x_i\leq T} |B(x)|$ and $M_W(T) := \sup_{0\leq x_i\leq T} |W(x)|$. Then,

$$P(M_B(T) > \epsilon) \leq P\left(M_W(T) > \frac{\epsilon}{d+1}\right) + \sum_{i=1}^d P(\sup_{0\leq x_i\leq T} W(\infty, \cdots, x_i, \cdots, \infty) > \frac{\epsilon}{d+1})$$

and, as in Xin (1992, p. 48), we have

$$P(M_B(T) > \epsilon) \leq 2^d \phi\left(\frac{\epsilon}{(d+1)\sqrt{T}}\right). \tag{3.11}$$

This implies (2.21). By (3.11),

$$\begin{aligned} P(h(T_1, T_2) > \epsilon) &\leq P(\frac{1}{T_1} \sup_{\substack{T_1\leq t\leq T_2 \\ \vartheta\in\Omega_d}}, |B(\tau(\vartheta))| > \frac{\epsilon}{2}) + P(\sup_{\substack{T_1\leq t\leq T_2 \\ \vartheta\in\Omega_d}}, |B(\tau(\vartheta))| > \frac{\epsilon}{2}) \\ &\leq 2^d \phi\left(\frac{\epsilon T_1}{(d+1)\sqrt{T_2}}\right) + 2^d \phi\left(\frac{\epsilon}{(d+1)\sqrt{T_2}}\right). \end{aligned}$$

$\Box$.

Remark: A slightly better estimate is possible. By the representation

$$\begin{aligned} \left|\frac{1}{t}B(t\tau(\vartheta)) - B(\tau(\vartheta))\right| &\leq \left|\frac{1}{t}W(t\tau(\vartheta)) - W(\tau(\vartheta))\right| \\ &+ \left|\sum_{i=1}^d [\text{sign}(t\tau_i(\vartheta)) \frac{m_i'(t\tau(\vartheta))W(\infty, \cdots, t\tau_i(\vartheta), \cdots, \infty)}{t}\right. \\ &- \text{sign}(\tau_i(\vartheta)) m_i'(\tau(\vartheta)) W(\infty, \cdots, \tau_i(\vartheta), \cdots, \infty)]| . \end{aligned}$$

Observe that, because of homogeneity of m, $m_i'(t\tau(\vartheta)) = m_i'(\tau(\vartheta))$. For $\tau(\vartheta)$ in the positive orthant we have

$$E\frac{1}{t}W(t\tau(\vartheta))W(\tau(\vartheta)) = \frac{1}{t}m((t\wedge 1)\tau(\vartheta)) = \frac{t\wedge 1}{t}m(\tau((\vartheta)).$$

Therefore, as in Xin (1992), we obtain

$$P\left(\sup_{\substack{T_1\le t\le T_2\\ \tau(\vartheta))\in Q_1}}\left|\frac{1}{t}W(t\tau(\vartheta)) - W(\tau(\vartheta))\right| > \lambda\right) \le 4d\phi(\lambda).$$

For the second term observe that with A being defined by

$$A = \sum_{i=1}^{d}|m_i'(\tau(\vartheta))|\left|W(\infty,\cdots,\tau_i(\vartheta),\cdots,\infty) - \frac{W(\infty,\cdots,t\tau_i(\vartheta),\cdots,\infty)}{t}\right|,$$

we have

$$P(A(\vartheta)\ge\lambda) \le dP\left(\left|W(\infty,\cdots,\tau_i(\vartheta),\cdots,\infty) - \frac{W(\infty,\cdots,t\tau_i(\vartheta),\cdots,\infty)}{t}\right| \ge \frac{\lambda}{d}\right)$$
$$\le d\phi(\frac{\lambda}{d}).$$

Therefore, $P(h(T_1,T_2)\ge\epsilon) \le 4d\phi(\frac{\epsilon}{2}) + d\phi(\frac{\epsilon}{2d})$, which is, indeed, a better bound than (2.21).

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