On a best choice problem for discounted sequences

Robert Kühne and Ludger Rüschendorf*
Institut für Mathematische Stochastik, University of Freiburg,
Eckerstr. 1, 79104 Freiburg, Germany

Abstract

The optimal choice problem is considered for a discounted sequence of random variables in the domain of a max-stable distribution. Asymptotically optimal stopping times and the asymptotic value of the stopping problem are determined. For the proof of these results the best choice problem for the discounted sequence is related to a best choice problem in an associated Poisson process.

Keywords: best-choice problem, Poisson process, stopping problem, max-stable
AMS 1991 subject classification: 60 G40, 62 L15

1 Introduction

Gilbert and Mosteller (1966) found the solution of the best choice problem for an iid sequence $X_1, \ldots, X_n$ with known continuous distribution F. The best choice problem is to find an optimal stopping time τ_n maximizing the probability $P(X_\tau = M_n)$, where $M_n = \max\{X_1, \ldots, X_n\}$, under all stopping times $\tau \le n$. The optimal stopping rule is given by

$$\begin{aligned} T_n &= \min\{k;\ X_k = M_k, F(X_k) \ge b_{n-k}\}, \\ \text{where } b_0 &= 0, \sum_{j=1}^{i} \binom{i}{j} \left(b_i^{-1} - 1\right)^j j^{-1} = 1, \qquad i = 1, 2 \ldots \end{aligned} \tag{1.1}$$

The boundary numbers b_i satisfy $b_i \uparrow 1$, $i(1-b_i) \to b = 0.80435\ldots$, b is the solution to $\sum_{j=1}^{\infty} \frac{b^j}{j!j} = 1$. The optimal choice probability $v_n = P(X_{T_n} = M_n)$ is strictly decreasing in n, independent of F and

$$\begin{aligned} v_n \longrightarrow v_\infty &= e^{-b} + (e^b - b - 1) \int_1^\infty x^{-1} e^{-bx} dx \\ &= 0.58016\ldots \end{aligned} \tag{1.2}$$

*Corresponding author. E-mail: ruschen@stochastik.uni-freiburg.de

Gnedin (1996) proved that the limiting value v_∞ is identical to the optimal choice probability of the best choice problem for a Poisson process $N = \sum \varepsilon(\tau_k, y_k)$ on $[0,1] \times \mathbb{R}^1$ where the intensity μ is given by the product of the Lebesgue measure on $[0,1]$ with the Lebesgue measure on $(-\infty, 0]$, $\mu = \lambda_{[0,1]} \otimes \lambda_{(-\infty,0]}$. $\varepsilon_{(\tau_k, y_k)}$ is the one point measure in time point τ_k with coordinate y_k. Our aim is to stop at the time point with the highest y-value. The information at time t is given by the observation of all points (τ_k, y_k) with $\tau_k \leq t$ (see the detailed description of the problem in Gnedin (1996)). The optimal stopping time for this problem is given by stopping at the first time point where a boundary $u(t)$ is exceeded and the observed value is identical to the maximum value until this time. Gnedin (1996) also imbedded an iid exponentially distributed sequence into the Poisson process and proved in this way the approximation result $v_n \to v_\infty$ (even with order $O(n^{-1})$). Some related optimal choice problems in the iid case based on other criteria like expected rank are discussed in Bruss and Ferguson (1993) and in Saario and Sakaguchi (1995).

In this paper we consider a class of best choice problems for discounted sequences

$$X_i = c_i Y_i \tag{1.3}$$

under the assumption that (Y_i) is an iid sequence with distribution in the domain of a max-stable distribution i.e. the normalized maxima converge in distribution. By the wellknown limit theorems of Gnedenko (1943) (see Resnick (1987) pg. 9) the limiting distribution is of one of three possible types Λ, Ψ_α and Φ_α where

$$\Lambda(x) = e^{-e^{-x}}, x \in \mathbb{R}, \quad \Phi_\alpha(x) = \begin{cases} e^{-x^{-\alpha}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad \text{for some } \alpha > 0 \text{ and}$$

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha} & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \text{for some } \alpha > 0.$$

We assume that $0 < c_i$, (c_i) bounded by a constant. We also assume that (c_i) is either monotonically nonincreasing or nondecreasing and for some real c

$$\lim_{n\to\infty} \frac{c_{[nt]}}{c_n} = t^{-c}, \quad t \in [0,1]. \tag{1.4}$$

If $c_i \geq c_{i+1}$ then $c \geq 0$, if $c_i \leq c_{i+1}$ then $c \leq 0$. We deal explicitly with the case where the distribution of Y_1 is in the max-domain of Ψ_α i.e. $P^{Y_1} \in D(\Psi_\alpha)$, $\alpha > 0$. Just for normalization we assume $Y_i \leq 0$ with upper bound on the support $w_F = F^{-1}(1-) = 0$, F the distribution function of Y_i. The cases where the distribution of Y_1 is in the domain of attraction of Φ_α for some $\alpha > 1$ or Λ, $P^{Y_1} \in D(\Lambda)$ resp. $P^{Y_1} \in D(\Phi_\alpha)$, $\alpha > 1$ are dealt with analogously. For this non-iid best choice problem Gnedins imbedding technique cannot be applied since a Poisson process cannot be reduced as in his paper to reproduce the sequence (X_i). Instead we directly approximate the associated embedded point process

$$N_n = \sum_{i=1}^n \varepsilon_{(\frac{i}{n}, \frac{X_i}{\overline{a}_n})}, \quad \overline{a}_n := a_n c_n \tag{1.5}$$

by a limiting Poisson process N. Here, $a_n = -F^{-1}(1-\frac{1}{n})$ are the normalizing constants ensuring convergence of the maxima M_n. Then we argue that the optimal choice problem of $\{X_1,\dots,X_n\}$ can be approximated by that of the limiting Poisson process and we also obtain asymptotically optimal stopping times and convergence of the optimal choice probabilities $v_n \to v_\infty$ as in the iid case in (1.2).

2 Asymptotics of the best choice problem

The aim of this paper is to prove the following approximation result.

Theorem 2.1 *If $P^{Y_1} \in D(\Psi_\alpha)$, $Y_i \le 0$, $w_F = 0$, $\alpha > 0$, and (c_i) satisfy (1.4) where $c > -\frac{1}{\alpha}$, then for the best choice problem for the discounted sequence $X_i = c_i Y_i$, $1 \le i \le n$ holds*

a) $v_n = \sup\{P(X_\tau = M_n);\ \tau\ \text{ a stopping time} \le n\}$
$\longrightarrow v_\infty = 0.58016\dots$

b) $T_n := \inf\left\{ i \le n;\ X_i \ge \overline{a}_n \left(\frac{b}{1-(\frac{i}{n})^{1+c\alpha}}\right)^{1/\alpha} (1+c\alpha)^{1/\alpha}, X_i = M_i \right\}$

is an asymptotically optimal stopping sequence, i.e.

$$P(X_{T_n} = M_n) \longrightarrow v_\infty. \tag{2.1}$$

Proof: The embedded point process $N_n = \sum_{i=1}^n \varepsilon_{(\frac{i}{n},\frac{X_i}{\overline{a}_n})}$ converges to a Poisson process $N = \sum_k \varepsilon_{(\tau_k,y_k)}$ with intensity measure μ defined by

$$\mu([0,t]\times[x,\infty)) = (-x)^\alpha \frac{t^{1+c\alpha}}{1+c\alpha}, \quad x \le 0. \tag{2.2}$$

For the proof note that $\frac{X_i}{\overline{a}_n} = \frac{c_i Y_i}{c_n a_n} =: \gamma_{n,\frac{i}{n}} \frac{Y_i}{a_n}$ and for $t \ne 0$ $(t_n,y_n) \to (t,y)$ implies that $R_n(t_n,y_n) := (t_n, y_n\gamma_{n,t_n}) \to (t, y\gamma_t) =: R_0(t,y)$ where $\gamma_{n,t_n} := \gamma_{n,\frac{[nt_n]}{n}}$ and $\gamma_t = t^{-c}$. Remind that $\widetilde{N}_n = \sum \varepsilon_{(\frac{i}{n},\frac{Y_i}{a_n})} \to \widetilde{N}$ a Poisson process with intensity $\widetilde{\mu} = \lambda_{[0,1]} \otimes \nu$, $\nu([x,0]) = (-x)^{-\alpha}$, $x < 0$ (convergence in distribution, see Resnick (1987, p. 210)). From uniform convergence of $\gamma_{n,\cdot} \to \gamma_\cdot$ on $[t,1]$ for $t > 0$ we conclude that $N_n = R_n\widetilde{N}_n \to N := R_0\widetilde{N}$ where R_n, R_0 operate on the points of the point process. N is a Poisson process with intensity $\mu = (\widetilde{\mu})^{R_0}$ which is easily calculated to be given by (2.2). (For details of this proof see Kühne and Rüschendorf (1998).)

The transformed point process $\widehat{N} = S(N)$, where $S(t,y) = (t^{1+c\alpha}, -\frac{(-y)^\alpha}{1+c\alpha}) = (S_1,S_2)$ is a Poisson process with intensity $\lambda_{[0,1]} \otimes \lambda_{(-\infty,0]}$. Since S_2 is strictly increasing the best choice problem for N can be reduced to that of $\widehat{N}$. From Gnedin (1996) we conclude that

$$\widehat{T} \;=\; \inf\left\{ S_1(\tau_k);\ S_2(y_k) > -\frac{b}{1-S_1(\tau_k)},\right.$$

$$S_2(y_k) = \max\{y_i;\ \tau_i \le \tau_k\} =: M^{\tau_k}\Big\} \;=\; S_1(T)$$

with (2.3)

$$T \;:=\; \inf\left\{\tau_k;\ y_k > \left(\frac{b}{1-\tau_k^{1+c\alpha}}\right)^{1/\alpha}(1+c\alpha)^{1/\alpha},\quad y_k = M^{\tau_k}\right\}$$

is an optimal stopping time for the best choice problem for $\widehat{N}$. Therefore, T is optimal for the best choice problem for the Poisson process N with probability of best choice

$$P\left(y_{K^T} = M = \max\{y_k\}\right) = v_\infty = 0.58016\ldots$$

(see (1.2)); here K^T denotes the stopping index of T.

With $M_{n,\ell,m} = \max\{X_\ell, X_{\ell+1},\ldots,X_m\}$, and $M_{s,t} = \max\{y_k;\ s \le \tau_k \le t\}$ and using that T_n are threshold stopping times, we obtain as in Kühne and Rüschendorf (1998) that

$$\left(\frac{T_n}{n}, \frac{X_{T_n}}{\overline{a}_n}, \frac{M_{n,1,T_n-1}}{\overline{a}_n}, \frac{M_{n,T_n+1,n}}{\overline{a}_n}\right) \xrightarrow{\mathcal{D}} \left(T, y_{K^T}, M_{0,T-}, M_{T+,1}\right). \tag{2.4}$$

Since $P(y_{K^T} = \max\{M_{0,T-}, M_{T+,1}\}) = 0$ this implies for the stopping time T_n defined in the theorem

$$\begin{aligned} & P\left(X_{T_n} = M_n\right) = P\left(X_{T_n} \ge \max\{M_{n,1,T_n-1}, M_{T_n+1,n}\}\right) \\ \longrightarrow\ & P\left(y_{K^T} \ge \max\{M_{0,T-}, M_{T+,1}\}\right) = P\left(y_{K^T} = M\right) = v_\infty \end{aligned} \tag{2.5}$$

So we finally have to prove that v_∞ is an upper bound for the asymptotic probability of the best choice problem for $X_1,\ldots,X_n$. The optimal stopping times S_n for the best choice problem for $X_1,\ldots,X_n$ are given by

$$\begin{aligned} S_n \;=\; \inf\{i;\ 1 \le i \le n, & P\left(X_i = M_n | X_1,\ldots,X_i\right) \\ & \ge \sup\{P(X_\tau = M_n | X_1,\ldots,X_i);\ \tau \in \Gamma_i^n\}\} \end{aligned} \tag{2.6}$$

where Γ_i^n is the set of stopping times τ with $i \le \tau \le n$ (see Chow, Robbins and Siegmund (1971)).

For $i \le n-1$ holds on $\{X_i = M_i\}$

$$\begin{aligned} P\{X_i = M_n | X_1,\ldots,X_i) &= P\left(X_i = M_n | X_i\right) \\ &= P\left(\max\{X_{i+1},\ldots,X_n\} \le X_i | X_i\right) \\ &= F_{X_{i+1}\vee\ldots\vee X_n}(X_i). \end{aligned}$$

On $\{X_i < M_i\}$ we have $P(X_i = M_n | X_1,\ldots X_i) = 0$ while a.s. $\sup\{P(X_\tau = M_n | X_1,\ldots,X_i);\ \tau \in \Gamma_{i+1}^n\} > 0$.

For a stopping time $\tau \geq i+1$ holds on $\{X_i = M_i\}$ by the independence assumption

$$\begin{aligned} P(X_\tau = M_n | X_1, \dots, X_i) &= P(X_\tau = X_{i+1} \vee \dots \vee X_n \geq X_i | X_i) \\ &= (P(X_\tau = X_{i+1} \vee \dots \vee X_n \geq \cdot))(X_i) \\ &=: h_i^\tau(X_i). \end{aligned}$$

Let $k_i = \max\{h_i^\tau; \tau \in \Gamma_{i+1}^n\}$ then $k_i(X_i) = \sup\{P(X_\tau = M_n | X_1, \dots, X_i), \ \tau \in \Gamma_{i+1}^n\}$. Since k_i is antitone and $F_{X_{i+1} \vee \dots \vee X_n}$ is isotone, there exists a monotonically nonincreasing sequence $(u_{n,i})_{1 \leq i \leq n}$ such that

$$\begin{aligned} S_n &= \inf\left\{i \leq n; \ F_{X_{i+1} \vee \dots \vee X_n}(X_i) \geq k_i(X_i)\right\} \\ &= \inf\left\{i \leq n; \ X_i \geq u_{n,i}\right\}. \end{aligned} \tag{2.7}$$

In conclusion S_n is a threshold stopping time.

It remains to prove that

$$\lim_{n \to \infty} \sup P(X_{S_n} = M_n) \leq v_\infty = P(y_{_{K^T}} = M), \quad M = \max\{y_k, k \in \mathbb{N}\}. \tag{2.8}$$

(2.8) then implies asymptotic optimality of T_n. Assume that for some subsequence $(n') \subset \mathbb{N}$, $v' = \lim_{n'} P(X_{S_{n'}} = M_{n'}) > v_\infty$. Then by the monotonicity of $(u_{n,i})$ there exists a further subsequence $(n'') \subset (n')$ such that the thresholds converge, $u_{n'',i} \to u_i$. Convergence of the thresholds implies convergence of the stopping times and values (see Kühne and Rüschendorf (1998) or Kennedy and Kertz (1992));

$$\left(\frac{S_{n''}}{n''}, \frac{X_{n'', S_{n''}}}{a_{n''}}\right) \xrightarrow{\mathcal{D}} (S', y_{_{K^{S'}}}), \tag{2.9}$$

S' a stopping time for the limiting Poisson process N. It also implies convergence of the probabilities of best choice (as in (2.5))

$$P(X_{S_{n''}} = M_{n''}) \to P(y_{_{K^{S'}}} = M) = v' > v_\infty, \tag{2.10}$$

a contradiction to the optimality of T for the best choice problem for N.

Together this implies that T_n is asymptotically optimal for the best choice problem and $v_n = P(X_{T_n} = M_n) \to v_\infty = P(y_{_{K^T}} = M)$. □

Remarks:

a) The proof carries over to the cases where $P^{Y_1} \in D(\Lambda)$ resp. $D(\Phi_\alpha), \alpha > 1$. The corresponding point process convergence result is stated in Kühne and Rüschendorf (1998) or in Kühne (1997).

b) Extreme value theory for sequences in the presence of a 'trend' as in (1.4) has been discussed in de Haan and Verkade (1987). The assumption on the sequence (c_i) that $\gamma_t = t^{-c}, c \in \mathbb{R}$ is implied by the existence of the limit $\gamma_t = \lim \frac{c_{[nt]}}{c_n}$. This condition ensures convergence of the point processes N_n to a Poisson process with intensity of the product form. The optimal choice problem for this Poisson process can be reduced to the case considered in Gnedin (1996) and by our approximation argument we obtain that asymptotically the optimal probability of best choice in the discounted sequence is identical to the optimal probability of best choice in the limiting Poisson case. □

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