MINIMAL DISTANCE MARTINGALE MEASURES AND OPTIMAL PORTFOLIOS CONSISTENT WITH OBSERVED MARKET PRICES

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Abstract In this paper we study derivative pricing under information on observed market prices of some derivatives. To this purpose we characterize minimal distance martingale measures under constraints on a finite number of random variables with respect to f-divergence distances in a general semimartingale setting. As a result a characterization of optimal portfolios of the underlyings and the given derivatives is obtained.

Key Words Derivative pricing, f-divergences, observed market prices.

1 INTRODUCTION

A common approach to derivative pricing in incomplete markets is to base the prices on a minimal distance martingale measure with respect to certain distances like L^2-distance (see [26] and [6]), Hellinger distance (see [19]), entropy distance (see [9]) and others.

This approach to derivative pricing takes into consideration the probabilistic model of the future behaviour of the underlyings, but not the information on derivative prices observed in the market. In order to include this information in the model we consider only those martingale measures which yield derivative prices consistent with the observed market prices. For derivative pricing we propose the minimal distance martingale measure consistent with observed market prices. A related idea of derivative pricing is studied by Kallsen in [17] and [16]. There derivative prices are considered such that it is optimal for an investor to hold a given non-zero position in the derivatives in order to maximize his expected utility. In the recent paper [18] the least favourable martingale measure consistent with observed market prices is proposed for derivative pricing. This pricing measure can actually be seen as minimal distance martingale measure consistent with observed market prices, see Section 4, and thus this proposal is consistent with our idea of integration of information on the market prices.

We consider the class of all f-divergence distances defined by strictly convex, differentiable functions f which includes the distances above and many further examples (see [21]). We obtain some necessary and some sufficient conditions for projections of the underlying measure on the set of martingale measures consistent with the observed prices of a finite number of derivatives in a general semimartingale market model.

A related problem is studied by Avellaneda (see [2]), who characterizes the probability measure which minimizes the relative entropy distance of the pricing measure to the class of all probability measures consistent with the observed market prices. However, the calibrated pricing measure obtained in this way is not necessarily a martingale measure.

The paper is organized as follows. In Section 2 we recall a characterization of f-projections on classes of distributions determined by inequality constraints. Based on this result we obtain in Section 3 characterizations of minimal distance martingale measures under constraints. We derive some necessary and some sufficient conditions for minimal distance martingale measures under constraints. In Section 4 we remind the notion of a minimax measure with respect to concave utility functions and convex sets of probability measures. Minimax martingale measures are equivalent to minimal distance martingale measures with respect to f-divergence distances induced by the convex conjugate of the utility function. As a consequence the characterizations of minimal distance martingale measures consistent with observed market prices are closely related to the determination of optimal portfolios, if one allows additionally constant positions in the derivatives with observed market prices.

2 f-DIVERGENCES AND MINIMAL DISTANCE MEASURES

In the following we recall a characterization of projections with respect to f-divergence distances on classes of distributions determined by inequality constraints. For a detailed discussion of f-divergence distances we refer to [21] or [28].

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 2.1 Let $Q \ll P$ and let $f : (0, \infty) \to \mathbb{R}$ be a convex function. Then the f-divergence distance between Q and P is defined as:

$$f(Q||P) := \begin{cases} \int f(\frac{dQ}{dP})dP & \text{, if the integral exists} \\ \infty & \text{, else} \end{cases}$$

where $f(0) = \lim_{x \downarrow 0} f(x)$.

Examples of f-divergence distances are the Kullback-Leibler or entropy distance for $f(x) = x \log x$, the total variation distance for $f(x) = |x - 1|$, the Hellinger distance for $f(x) = -\sqrt{x}$, the reverse relative entropy distance for $f(x) = -\log(x)$ and many others.

In the following we assume that f is a continuous, strictly convex and differentiable function. Let $\mathcal{K}$ be a convex set of probability measures on $(\Omega, \mathcal{F})$ dominated by P. A measure $Q^* \in \mathcal{K}$ is called a f-projection of P on $\mathcal{K}$ if

$$f(Q^*||P) = \inf_{Q \in \mathcal{K}} f(Q||P) =: f(\mathcal{K}||P).$$

Let F be a convex cone of real valued random variables on $(\Omega, \mathcal{F})$, i.e.

$$\{\sum_{i=1}^{k} \alpha_i f_i : \alpha_i \geq 0, f_i \in F\} = F,$$

and define the moment family determined by inequality constraints with respect to F as

$$\mathcal{K}_F := \{Q \ll P : F \subset L^1(Q) \text{ and } E_Q f \geq 0 \text{ for all } f \in F\}.$$

The following result was given in [23], Theorem 2:

Theorem 2.2 *Let $Q^* \ll P$ satisfy $f(Q^*||P) < \infty$ and assume that $d := E_{Q^*} f'(\frac{dQ^*}{dP})$ is finite.*

(i) $Q^* \in \mathcal{K}$ is the f-projection of P on $\mathcal{K}$ if and only if

$$E_{Q^*} f'\Big(\frac{dQ^*}{dP}\Big) \leq E_Q f'\Big(\frac{dQ^*}{dP}\Big) \textit{ for all } Q \in \mathcal{K} \textit{ with } f(Q||P) < \infty.$$

(ii) If $Q^* \in \mathcal{K}_F$ is the f-projection on $\mathcal{K}_F$, then

$$f'\Big(\frac{dQ^*}{dP}\Big) - d \in \bar{F}, \textit{ the } L^1(\Omega, \mathcal{F}, Q^*)\textit{-closure of } F.$$

(iii) If $Q^* \in \mathcal{K}_F$ such that $f'(\frac{dQ^*}{dP}) - d \in F$, then Q^* is the f-projection on $\mathcal{K}_F$.

Proposition 8.5 in [21] shows the existence of a f-projection of P on a convex class $\mathcal{K}$ under the assumptions that $\mathcal{K}$ is closed in the variational distance topology and $\lim_{x \to \infty} \frac{f(x)}{x} = \infty$. By Corollary 2.3 in [11] a f-projection is equivalent to P if $f'(0) = -\infty$ and if there exists a measure $Q \in \mathcal{K}$ with $Q \sim P$ and with finite distance $f(Q||P) < \infty$.

3 CHARACTERIZATION OF MINIMAL DISTANCE MARTINGALE MEASURES UNDER CONSTRAINTS

In the following we apply Theorem 2.2 to characterize f-projections on the set of martingale measures which fulfill some additional constraints. Our mathematical framework is as follows: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ is a filtered probability space, where the filtration is assumed to be right-continuous and $\mathcal{F} = \mathcal{F}_T$. Let S be a $\mathbb{R}^d$-valued semimartingale with deterministic S_0. Vector stochastic integrals are written as $\varphi \cdot S_t = \int_0^t \varphi_s \, dS_s$. (For the definition of a vector stochastic integral see [14].)

Let $\mathcal{M}$ ($\mathcal{M}_{loc}$) be the set of P-absolutely continuous (local) martingale measures and $\mathcal{M}^e$ ($\mathcal{M}^e_{loc}$) the subset of $\mathcal{M}$ ($\mathcal{M}_{loc}$) consisting of probability measures which are equivalent to P. Let $H_1, \dots, H_n$ be a finite set of $\mathcal{F}_T$-measurable random variables and $r \in \{0, \dots, n\}$.

We define the set of martingale measures with constraints as

$$\begin{aligned} \widehat{\mathcal{M}} \quad := \quad & \{Q \in \mathcal{M} : H_1, \dots, H_n \in L^1(Q),\ E_Q H_i \geq 0 \text{ for } 1 \leq i \leq r \\ & \text{and } E_Q H_i = 0 \text{ for } r+1 \leq i \leq n\}. \end{aligned}$$

The class $\widehat{\mathcal{M}}$ stands for the set of martingale measures consistent with some information on the prices of derivatives $H_1, \dots, H_n$. Notice that price information of the form $E_Q H_i \in [q_i, p_i]$ can also be described by inequality constraints as in the definition of $\widehat{\mathcal{M}}$. The sets $\widehat{\mathcal{M}}_{loc}, \widehat{\mathcal{M}}^e, \widehat{\mathcal{M}}^e_{loc}$ are defined analogously to $\widehat{\mathcal{M}}$. We assume throughout that

$$\widehat{\mathcal{M}}^e \neq \varnothing.$$

For a $\mathbb{R}^d$-valued local martingale N the class $L^1_{loc}(N)$ of predictable integrands is defined in [14]. For $Q \in \mathcal{M}_{loc}$ we denote by $L^1_{loc}(S, Q)$ the class of integrands $L^1_{loc}(S)$ which is defined with respect to the measure Q. The following theorem gives a necessary condition for the f-projection of P on $\widehat{\mathcal{M}}$.

Theorem 3.1 *Let $Q^* \in \widehat{\mathcal{M}}$ satisfy $f(Q^* || P) < \infty$. If Q^* is the f-projection of P on $\widehat{\mathcal{M}}$ and $d := E_{Q^*} f'(\frac{dQ^*}{dP})$ is finite, then*

$$f'\left(\frac{dQ^*}{dP}\right) = d + \varphi \cdot S_T + \sum_{i=1}^n \mu_i H_i \quad Q^*\text{-a.s.} \tag{3.1}$$

with $\mu_1, \dots, \mu_n \in \mathbb{R}$, such that $\mu_i \geq 0$ and $\mu_i(E_{Q^*} H_i) = 0$ for $1 \leq i \leq r$, and with some process $\varphi \in L^1_{loc}(S, Q^*)$, such that $\varphi \cdot S$ is a martingale under Q^*.

Proof. First we introduce a set G of random variables which determines $\widehat{\mathcal{M}}$ as a moment family. Define the set G' as

$$G' := \{\varphi \cdot S_T : \varphi^i = Y^i 1_{]s_i, t_i]},\ s_i < t_i,\ Y^i \text{ bounded } \mathcal{F}_{s_i}\text{-measurable}\}.$$

Let G be the convex cone generated by the set

$$G' \cup \{H_i : 1 \leq i \leq n\} \cup \{-H_i : r+1 \leq i \leq n\}.$$

Then we have the following characterization of $\widehat{\mathcal{M}}$.

$$\widehat{\mathcal{M}} \quad = \quad \{Q \ll P : G \subset L^1(Q) \text{ and } E_Q g \geq 0 \text{ for all } g \in G\}.$$

The necessary condition in Theorem 2.2 (ii) yields: $f'(\frac{dQ^*}{dP}) - d \in \bar{G}$. Corollary 2.5.2 in [29] (for a multidimensional version see Theorem 1.6 in [7]) implies that the $L^1(Q^*)$-closure of the vector space generated by G' is contained in

$$\{\varphi \cdot S_T : \varphi \in L^1_{loc}(S, Q^*), \text{ such that } \varphi \cdot S \text{ is a } Q^*\text{-martingale}\}.$$

According to Proposition 1.1 in [13] this result is valid without the assumption of a complete filtration. Extending Proposition I.3.3 in [25] from vector spaces to the class of closed convex cones one gets

$$f'\left(\frac{dQ^*}{dP}\right) \quad = \quad d + \varphi \cdot S_T + \sum_{i=1}^n \mu_i H_i \quad Q^*\text{-a.s.},$$

where $\mu_i \geq 0$ for $1 \leq i \leq r$. This implies by the definition of d that $\mu_i(E_{Q^*} H_i) = 0$ for $1 \leq i \leq r$. □

The following theorem is a variant of Theorem 3.1. It shows that the necessary condition in Theorem 3.1 is also valid for the set $\widehat{\mathcal{M}}_{loc}$ under the additional assumption that S is locally bounded.

Theorem 3.2 *Let S be locally bounded. Let $Q^* \in \widehat{\mathcal{M}}_{loc}$ satisfy $f(Q^*||P) < \infty$. If Q^* is the f-projection of P on $\widehat{\mathcal{M}}_{loc}$ and $d := E_{Q^*} f'(\frac{dQ^*}{dP})$ is finite, then*

$$f'\left(\frac{dQ^*}{dP}\right) = d + \varphi \cdot S_T + \sum_{i=1}^n \mu_i H_i \quad Q^*\text{-a.s.} \tag{3.2}$$

with $\mu_1, \ldots, \mu_n \in \mathbb{R}$, such that $\mu_i \geq 0$ and $\mu_i(E_{Q^*} H_i) = 0$ for $1 \leq i \leq r$, and with some process $\varphi \in L^1_{loc}(S, Q^*)$, such that $\varphi \cdot S$ is a martingale under Q^*.

Proof. Let G_{loc} be the convex cone generated by

$$\{\varphi \cdot S_T : \varphi^i = Y^i 1_{]s_i, t_i]} 1_{[0, \widehat{T}^i]}, s_i < t_i, Y^i \text{ bounded } \mathcal{F}_{s_i}\text{-measurable}, \widehat{T}^i \in \gamma^i\}$$
$$\cup \{H_i : \leq i \leq n\} \cup \{-H_i : r+1 \leq i \leq n\},$$

where $\gamma^i := \{\widehat{T}^i \text{ stopping time} : S^{i\,\widehat{T}^i} \text{ is bounded}\}$.
Then the convex cone G_{loc} determines $\widehat{\mathcal{M}}_{loc}$ as a moment family

$$\widehat{\mathcal{M}}_{loc} \quad = \quad \{Q \ll P : \widehat{G}_{loc} \subset L^1(Q) \text{ and } E_Q g \geq 0 \quad \forall g \in G_{loc}\}$$

The presentation (3.2) is then obtained as in the proof of Theorem 3.1. □

Remark. Theorems 3.1 and 3.2 are generalizations of the corresponding results in the case without constraints, see [11], [12], [20] and [24].

From Theorem 2.2 (i) we obtain the following sufficient condition for f-projections of P on $\widehat{\mathcal{M}}_{loc}$ ($\widehat{\mathcal{M}}$). We denote by $L(S)$ the set of predictable, S-integrable processes with respect to P (see [13]).

Theorem 3.3 *Let $Q^* \in \widehat{\mathcal{M}}_{loc}$ with $f(Q^*||P) < \infty$ such that for some process $\varphi \in L(S)$ and $\mu_1, \dots, \mu_n \in \mathbb{R}$ the following conditions hold:*

$$\begin{array}{ll} (i) & f'\Big(\dfrac{dQ^*}{dP}\Big) = d + \varphi \cdot S_T + \displaystyle\sum_{i=1}^n \mu_i H_i \quad P\text{-a.s.,} \\ (ii) & -\varphi \cdot S \text{ is bounded from below } P\text{-a.s.,} \\ (iii) & E_{Q^*}(\varphi \cdot S_T) = 0, \\ (iv) & \mu_i \geq 0,\ \mu_i(E_{Q^*} H_i) = 0 \text{ for } 1 \leq i \leq r. \end{array}$$

Then Q^* is the f-projection of P on $\widehat{\mathcal{M}}_{loc}$.

Proof. From Corollary 3.5 in [1] it follows from condition (ii) that $-\varphi \cdot S$ is a Q-local martingale and hence a Q-supermartingale for any $Q \in \widehat{\mathcal{M}}_{loc}$. Therefore,

$$\begin{aligned} E_Q f'\Big(\frac{dQ^*}{dP}\Big) &= d + E_Q(\varphi \cdot S_T) + \sum_{i=1}^n \mu_i E_Q H_i \\ &\geq d = E_{Q^*} f'\Big(\frac{dQ^*}{dP}\Big). \end{aligned}$$

Now the result follows from Theorem 2.2 (i). □

The following proposition shows that one can transform the minimization problem $\inf_{Q \in \widehat{\mathcal{M}}} f(Q||P)$ with respect to $\widehat{\mathcal{M}}$ into a minimization problem with respect to $\mathcal{M}$ including some penalty terms for violating the constraints. The coefficients γ_i in the penalty terms can be interpreted as Lagrange multipliers.

Proposition 3.4 *Let S and $H_1, \dots, H_n$ be bounded and $\inf_{Q \in \widehat{\mathcal{M}}} Ef(\frac{dQ}{dP}) < \infty$. Assume the existence of a measure $Q_0 \in \widehat{\mathcal{M}}$ such that $E_{Q_0} H_i > 0$ for $1 \leq i \leq r$ and $f(Q_0||P) < \infty$. Furthermore assume that there exists a neighbourhood V of $(0, \dots, 0) \in \mathbb{R}^{n-r}$ such that for all $v \in V$ there exists an element $Q \in \mathcal{M}$ with $f(Q||P) < \infty$ and $(E_Q H_{r+1}, \dots, E_Q H_n) = v$.*

Then $Q^* \in \widehat{\mathcal{M}}$ is a f-projection of P on $\widehat{\mathcal{M}}$, i.e.

$$f(Q^*||P) = \inf_{Q \in \widehat{\mathcal{M}}} f(Q||P),$$

if and only if there are $\gamma_1, \dots, \gamma_n \in \mathbb{R}$ such that

$$f(Q^*||P) + \sum_{i=1}^n \gamma_i E_{Q^*} H_i \quad = \quad \inf_{Q \in \mathcal{M}} \{f(Q||P) + \sum_{i=1}^n \gamma_i E_Q H_i\}$$

and $\gamma_i \leq 0, \gamma_i(E_{Q^*} H_i) = 0$ for $1 \leq i \leq r$.

Proof. 1. Since S is bounded it follows that the set $\mathcal{M}$ of absolutely continuous martingale measures is closed in variation. As $H_1, \ldots, H_n$ are bounded this is also true for $\mathcal{M}' := \{Q \in \mathcal{M} : E_Q H_i = 0$ for $r+1 \leq i \leq n\}$. Since $\mathcal{M}$, $\mathcal{M}'$ are convex, they are also closed in $\sigma(L^1, L^\infty)$, if one identifies $Q \in \mathcal{M}$ with its Radon-Nikodym density with respect to P (see for example Proposition IV.3.1 in [25]). We define a function $B : \mathcal{M}' \to \mathbb{R}^r$ by

$$B(Q) \quad = \quad (-E_Q H_1, \ldots, -E_Q H_r).$$

Obviously the component mappings of B are convex and continuous with respect to $\sigma(L^1, L^\infty)$. The optimization problem is to minimize the lower semicontinuous functional $f(\cdot||P)$ (see Theorem 1.47 in [21]) over $\widehat{\mathcal{M}}$,

$$\inf_{Q \in \widehat{\mathcal{M}}} f(Q||P).$$

This problem can be written in the form

$$\inf_{\substack{Q \in \mathcal{M}' \\ BQ \leq 0}} f(Q||P),$$

where $BQ \leq 0$ is understood componentwise.

The assumption on the existence of an inner point Q_0 in $\mathcal{M}'$ allows to apply a Lagrange multiplier theorem (see [8], Theorem III.5.1), which results in the following equivalence:

$Q^* \in \widehat{\mathcal{M}}$ is a f-projection of P on $\widehat{\mathcal{M}}$ if and only if there are $\gamma_i \leq 0$ such that

$$f(Q||P) + \sum_{i=1}^{r} \gamma_i E_{Q^*} H_i \quad = \quad \inf_{Q \in \mathcal{M}'} \{f(Q||P) + \sum_{i=1}^{r} \gamma_i E_Q H_i\}, \qquad (3.3)$$

and $\gamma_i(E_{Q^*} H_i) = 0$ for $1 \leq i \leq r$.

2. Next we follow a similar line of argument to handle the equality constraints in the right-hand side of (3.3). Since the component mappings of B are continuous and linear, the mapping $J : \mathcal{M} \to \mathbb{R}$, defined by $J(Q) := f(Q||P) + \sum_{i=1}^{r} \gamma_i E_Q H_i$, is lower semicontinuous and convex. We define a function $B' : \mathcal{M} \to \mathbb{R}^{n-r}$ by

$$B'(Q) := (E_Q H_{r+1}, \ldots, E_Q H_n).$$

The optimization problem

$$\inf_{Q \in \mathcal{M}'} J(Q)$$

can be written as

$$\inf_{\substack{Q \in \mathcal{M} \\ B'(Q)=0}} J(Q).$$

The perturbation function $\Phi : \mathcal{M} \times \mathbb{R}^{n-i_0} \to \bar{\mathbb{R}}$ is chosen as

$$\Phi(Q,v) :=:= \begin{cases} J(Q), & \text{if } Q \in \mathcal{M} \text{ and } B'(Q) = v \\ \infty, & \text{otherwise.} \end{cases}$$

For $v \in \mathbb{R}^{n-r}$ define $h(v) := \inf_{Q \in \mathcal{M}} \Phi(Q,v)$. By assumption $h(0)$ is finite. Observe that $\mathcal{M}$ is closed, J is lower semicontinuous and convex, and B' is linear and continuous. Therefore, the function h is convex (see [8], Lemma III.5.2, Lemma III.2.1). From Theorem 23.4 in [22] it follows that h is subdifferentiable in 0. Hence by Proposition III.3.2 in [8] we obtain the following equivalence:

$Q^* \in \mathcal{M}'$ solves $\inf_{Q \in \mathcal{M}'} J(Q)$ if and only if there are $\gamma_{r+1}, \dots, \gamma_n \in \mathbb{R}$ such that

$$f(Q^*||P) + \sum_{i=1}^{n} \gamma_i E_{Q^*} H_i \quad = \quad \inf_{Q \in \mathcal{M}} \{ f(Q||P) + \sum_{i=1}^{n} \gamma_i E_Q H_i \}.$$

□

Remark. Following the line of arguments of the proof of Theorem 7.1 in [11] one verifies that under the additional assumption $f'(\frac{dQ^*}{dP}) \in L^1(Q^*)$ the coefficients $\gamma_1, \dots, \gamma_n$ in Proposition 3.4 correspond to $-\mu_1, \dots, -\mu_n$, where μ_i are the coefficients of the characterization of Q^* in Theorem 3.1.

4 RELATIONSHIP TO PORTFOLIO OPTIMIZATION

Minimal distance martingale measures are closely related to minimax martingale measures and hence to utility maximization problems. We briefly restate the notion of a minimax measure and some results about minimax measures for general convex models as introduced in [11], Section 4, in order to point out the relationship to portfolio optimization. In the following, as in Section 2, we denote by $\mathcal{K}$ a general convex set of probability measures on $(\Omega, \mathcal{F})$ dominated by P.

A utility function $u: \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is assumed to be strictly increasing, strictly concave, continuously differentiable in dom$(u) := \{x \in \mathbb{R} \mid u(x) > -\infty\}$ and to satisfy

$$u'(\infty) \quad = \quad \lim_{x \to \infty} u'(x) = 0, \tag{4.1}$$

$$u'(\bar{x}) \quad = \quad \lim_{x \downarrow \bar{x}} u'(x) = \infty \tag{4.2}$$

for $\bar{x} := \inf\{x \in \mathbb{R} \mid u(x) > -\infty\}$. (This implies that either dom$(u) = (\bar{x}, \infty)$ or dom$(u) = [\bar{x}, \infty)$.)

We denote by I the inverse of the derivative of u. Assumption (4.1) implies that $I(0) = \infty$. The convex conjugate function $u^* : \mathbb{R}_+ \to \mathbb{R}$ of u is defined by

$$u^*(y) \quad := \quad \sup_{x \in \mathbb{R}} \{u(x) - xy\} = u(I(y)) - yI(y).$$

For $Q \in \mathcal{K}$ and $x > \bar{x}$ we define

$$U_Q(x) := \sup\{Eu(Y) : Y \in L^1(Q), E_Q Y \leq x, Eu(Y)^- < \infty\}. \tag{4.3}$$

The value $U_Q(x)$ can be interpreted as the maximal expected utility which can be achieved with endowment x, if the market prices are computed by Q. If $E_Q(I(\lambda \frac{dQ}{dP}))$ is finite for all $\lambda > 0$, then $U_Q(x)$ has the following well-known representation (see for example [11], Lemma 4.1):

$$U_Q(x) \quad = \quad E\Big[u\Big(I\Big(\lambda_Q(x)\frac{dQ}{dP}\Big)\Big)\Big], \tag{4.4}$$

where $\lambda_Q(x)$ is chosen such that $EI(\lambda_Q(x)\frac{dQ}{dP}) = x$.
The random variable $I(\lambda_Q(x)\frac{dQ}{dP})$ can be interpreted as optimal contingent claim which is financeable under the pricing measure Q.

Definition 4.1 A measure $Q^* = Q^*(x) \in \mathcal{K}$ is called minimax measure with respect to endowment x and model $\mathcal{K}$ if it minimizes $Q \mapsto U_Q(x)$ over all $Q \in \mathcal{K}$, i.e.,

$$U_{Q^*}(x) \quad = \quad U(x) := \inf_{Q \in \mathcal{K}} U_Q(x).$$

We refer to [3], [10] and [11] for further information about minimax measures. We denote by $u^*_{\lambda_0}(\cdot||\cdot)$ the f-divergence distance corresponding to $f(x) = u^*(\lambda_0 x)$.

Under the assumptions:

$$\exists x > \bar{x} \text{ with } U(x) < \infty, \tag{4.5}$$

$$E_Q I\Big(\lambda\frac{dQ}{dP}\Big) < \infty \, \forall \lambda > 0 \, \forall Q \in \mathcal{K} \tag{4.6}$$

one gets the following result (see Proposition 4.3 in [11]):

Proposition 4.2 *Let $x > \bar{x}$.*

(i) Let $\lambda_0 > 0$ such that $\lambda_0 \in \partial U(x)$. If $Q^* \in \mathcal{K}$ is a $u^*_{\lambda_0}$-projection of P on $\mathcal{K}$, then Q^* is a minimax measure for $\mathcal{K}$ and $\lambda_0 = \lambda_{Q^*}(x)$.

(ii) If $Q^* \in \mathcal{K}$ is a minimax measure, then Q^* is a $u^*_{\lambda_{Q^*}(x)}$-projection of P on $\mathcal{K}$ and $\lambda_{Q^*}(x) \in \partial U(x)$.

This result shows that minimax measures can be determined by distance minimization and conversely a $u^*_{\lambda_0}$-projection has an alternative interpretation in the sense of utility-maximization. Notice that for the standard utility functions like $u(x) = \frac{x^p}{p}$ $(p \in (-\infty, 1) \setminus \{0\})$, $u(x) = \log x$ and $u(x) = 1 - e^{-px}$ $(p \in (0, \infty))$ the minimax measure does not depend on x respectively the u^*_λ-projection does not depend on λ.

Since $U(x)$ typically is not known explicitly it is of interest to be able to determine some $\lambda_0 \in \partial U(x)$. In the next proposition we give a sufficient condition to imply $\lambda_0 \in \partial U(x)$.

According to the relationship between minimax measures and u_λ^*-projections in Proposition 4.2 the preceding results induce also necessary and sufficient conditions for a minimax measure in $\widehat{\mathcal{M}}$ respectively in $\widehat{\mathcal{M}}_{loc}$.

Assume that the conditions 4.5, 4.6 hold true for $\widehat{\mathcal{M}}$ ($\widehat{\mathcal{M}}_{loc}$).

Proposition 4.3 *Let $Q^* \in \widehat{\mathcal{M}}$ ($\widehat{\mathcal{M}}_{loc}$), $\lambda > 0$ satisfy $u_\lambda^*(Q^*||P) < \infty$. Assume that for some $\varphi \in L(S)$ and constants $\mu_1, \dots, \mu_n \in \mathbb{R}$ the following conditions hold:*

$$
\begin{array}{ll}
(i) & I\Big(\lambda \dfrac{dQ^*}{dP}\Big) = x + \varphi \cdot S_T + \displaystyle\sum_{i=1}^{n} \mu_i (H_i - E_{Q^*} H_i) \quad P\text{-a.s.}, \\
(ii) & \varphi \cdot S \text{ is bounded from below } P\text{-a.s.}, \\
(iii) & E_{Q^*}(\varphi \cdot S_T) = 0, \\
(iv) & \mu_i \leq 0 \text{ and } \mu_i(E_{Q^*} H_i) = 0 \text{ for } 1 \leq i \leq r.
\end{array}
$$

Then Q^* is a minimax measure for $\widehat{\mathcal{M}}$ ($\widehat{\mathcal{M}}_{loc}$) and x and $\lambda \in \partial U(x)$.

Proof. Since $E_{Q^*} I(\lambda \frac{dQ^*}{dP}) = x$ one gets that $\lambda = \lambda_{Q^*}(x)$. Moreover, since $(u_\lambda^*)'(x) = -\lambda I(\lambda x)$, we conclude from Theorem 3.3 that Q^* is the u_λ^*-projection of P on $\widehat{\mathcal{M}}$ ($\widehat{\mathcal{M}}_{loc}$) and that the condition of Theorem 2.2 (i) is fulfilled. Hence for all measures $Q \in \widehat{\mathcal{M}}$ ($\widehat{\mathcal{M}}_{loc}$) satisfying $u_\lambda(Q||P) < \infty$ one gets $E_Q I(\lambda \frac{dQ^*}{dP}) \leq x$. This implies that

$$
U_{Q^*}(x) \quad = \quad E\Big[u\Big(I\Big(\lambda \frac{dQ^*}{dP}\Big)\Big)\Big] \leq U_Q(x).
$$

Assumption (4.6) implies that (see Lemma 4.1 in [11])

$$
\{Q \in \widehat{\mathcal{K}} : u_\lambda^*(Q||P) < \infty\} \quad = \quad \{Q \in \mathcal{K} : U_Q(x) < \infty\}.
$$

Therefore, Q^* is a minimax measure for x and $\widehat{\mathcal{M}}$ ($\widehat{\mathcal{M}}_{loc}$). From Proposition 4.2 we conclude that $\lambda = \lambda_{Q^*}(x) \in \partial U(x)$. □

In the following we point out the relationship between minimal distance martingale measures under constraints and portfolio optimization. We consider a market model which consists of $d+1$ assets. We assume that the assets $0, \dots, d$ are modeled by the $\mathbb{R}^{d+1}$-valued semimartingale $S = (S^0, \dots, S^d)$ and suppose without loss of generality that the price of asset 0 is constant, i.e., $S_t^0 \equiv 1$. Assume that additionally to the price process S also the prices $\{p_1, \dots, p_n\}$ of a finite set of contingent claims $\{H_1, \dots, H_n\}$ are known at time $t = 0$. We suppose that one can buy or sell these contingent claims in $t = 0$.

We call a pair (φ,μ) of a predictable, S-integrable, $\mathbb{R}^{d+1}$-valued process φ and a vector $\mu\in\mathbb{R}^n$ an admissible strategy if

$$\sum_{i=0}^d \varphi_t^i S_t^i \quad = \quad x-\sum_{i=1}^n \mu_i p_i + \varphi\cdot S_t$$

for any $t\in\mathbb{R}_+$ and the process $\varphi\cdot S$ is bounded from below. The set of admissible strategies is denoted by $\mathcal{A}$.
We define an *optimal portfolio strategy* as a strategy $(\widehat{\varphi},\widehat{\mu})\in\mathcal{A}$ which maximizes

$$(\varphi,\mu)\mapsto E\Big(u\Big(x+\varphi\cdot S_T+\sum_{i=1}^n \mu_i(H_i-p_i)\Big)\Big) \tag{4.7}$$

over all $(\varphi,\mu)\in\mathcal{A}$. We denote by $\widehat{\mathcal{M}}$ the class of martingale measures consistent with the observed market prices of the derivatives $H_1,\dots,H_n$,

$$\widehat{\mathcal{M}}:=\{Q\in\mathcal{M}: H_i\in L^1(Q) \text{ and } E_Q H_i=p_i,\, 1\le i\le n\}.$$

Optimal portfolio strategies can be obtained from a representation of the $u^*_{\lambda_0}$-projection Q^* of P on $\widehat{\mathcal{M}}$ as in Theorem 3.1. Assume that the conditions 4.5, 4.6 hold true for $\widehat{\mathcal{M}}$.

Theorem 4.4 *Let $Q^*\in\widehat{\mathcal{M}}^e$ be the $u^*_{\lambda_0}$-projection of P on $\widehat{\mathcal{M}}$ for some $\lambda_0\in\partial U(x)$, $\lambda_0>0$.*

(i) There exist constants $\widehat{\mu}_1,\dots,\widehat{\mu}_n\in\mathbb{R}$ and a process $\widehat{\varphi}\in L(S)$ such that:

$$I\Big(\lambda_0\frac{dQ^*}{dP}\Big)=x+\widehat{\varphi}\cdot S_T+\sum_{i=1}^n\widehat{\mu}_i(H_i-p_i). \tag{4.8}$$

(ii) If representation (4.8) holds and if the stochastic integral $\widehat{\varphi}\cdot S$ is bounded from below, then $(\widehat{\varphi},\widehat{\mu})$ is an optimal portfolio strategy (where $\widehat{\varphi}_t^0:=x+\widehat{\varphi}\cdot S_t-\sum_{i=1}^n\widehat{\mu}_i p_i-\sum_{i=1}^d\widehat{\varphi}_t^i S_t$).

Proof. 1. Theorem 3.1 and the identity $\lambda_0=\lambda_{Q^*}(x)$ (see Proposition 4.2) imply the existence of the representation in (4.8).
2. By the definition of $\widehat{\varphi}_t^0$ and observing that S^0 is assumed to be identical 1 it holds that $\sum_{i=0}^d\widehat{\varphi}_t^i S_t^i=x-\sum_{i=1}^n\widehat{\mu}_i p_i+\widehat{\varphi}\cdot S_t$ for any $t\in\mathbb{R}_+$ and hence $(\widehat{\varphi},\widehat{\mu})\in\mathcal{A}$. From Corollary 3.5 in [1] it follows that $\varphi\cdot S$ is a Q^*-local martingale and hence a Q^*-supermartingale for any $(\varphi,\mu)\in\mathcal{A}$. Therefore,

$$E_{Q^*}\Big(x+\varphi\cdot S_T+\sum_{i=1}^n\mu_i(H_i-p_i)\Big)\quad\le\quad x.$$

Since $I(\lambda_0 \frac{dQ^*}{dP})$ is the optimal contingent claim which is financeable under the pricing measure Q^* and the endowment x (see (4.4)) we conclude that $(\widehat{\varphi}, \widehat{\mu})$ is an optimal portfolio strategy. $\square$

Remarks. 1. If the utility-function is finite on $(-\infty, \infty)$, i.e. $\bar{x} = -\infty$, then the condition that $\widehat{\varphi} \cdot S$ is bounded from below is not fulfilled in general. In this case one has to choose a suitable extended concept of admissible strategies in order to solve utility maximization problems. This issue is discussed in [24], [5] and [15].

2. Theorem 4.4 shows that if derivative prices are computed by the minimax (respectively, minimal distance) measure Q^* for $\widehat{\mathcal{M}}$ and x, then the optimal contingent claim can be duplicated by a strategy $(\widehat{\varphi}, \widehat{\mu})$. Hence although it may be profitable to invest in the derivatives $H_1, \ldots, H_n$ it turns out that one cannot increase the maximal expected utility in comparison to the strategy $(\widehat{\varphi}, \widehat{\mu})$ by trading in further derivatives. This shows that Q^* yields consistent derivative prices in the sense of [18].

3. There is a close relationship of the portfolio optimization problem (4.7) to utility-based hedging of one contingent claim H_1:

$$\sup_{(\varphi,-1)\in\mathcal{A}} E(u(x_0 + \varphi \cdot S_T - H_1)). \tag{4.9}$$

If for $k = 1$ and $\widehat{\mu} = -1$ the strategy $\widehat{\varphi}$ in representation (4.8) is such that $\widehat{\varphi} \cdot S$ is bounded from below, then $\widehat{\varphi}$ turns out to be the optimal portfolio strategy for the utility-based hedging problem (4.9) with initial endowment $x_0 = x - p_1$. Hence problem (4.7) is closely related to problem (4.9). In the portfolio optimization problem (4.7) we have a fixed initial price of the derivative, in the utility-based hedging problem (4.9) we have a fixed number of derivatives in the set of allowed portfolios. Due to this relationship, results of this paper are closely related to results on utility-based hedging. The problem of utility-based hedging has been studied recently in [4], [5], [11] and [15].

Example. We consider a discrete-time market model. Let $S = (S_0, S_1, \ldots, S_T)$ be the price of a risky asset where $S_0 \in \mathbb{R}, S_0 > 0$ and $S_t = S_0 \prod_{s=1}^t X_s$ where $X_1, \ldots, X_T$ are $(0, \infty)$-valued random variables. Moreover the price p of one derivative H is given. Theorem 3.3 gives as sufficient conditions for a measure $Q^* \in \widehat{\mathcal{M}}$, where

$$\widehat{\mathcal{M}} := \{Q \in \mathcal{M} : E_Q H = p\},$$

to minimize the relative entropy (corresponding to $f(x) = x \log x$) over all measures in $\widehat{\mathcal{M}}$:

(i) $\quad \dfrac{dQ^*}{dP} = \dfrac{e^{\varphi \cdot S + \mu H}}{E\left(e^{\varphi \cdot S + \mu H}\right)},$

(ii) $\quad \varphi \cdot S$ is bounded from below.

In the discrete-time setting condition (ii) implies condition (iii) in Theorem 3.3. Condition (i) can be written in the following equivalent way:

$$\frac{dQ^*}{dP} = \frac{e^{\sum_{i=1}^T \gamma_i (X_i - 1) + \mu H}}{E\left(e^{\sum_{i=1}^T \gamma_i (X_i - 1) + \mu H}\right)},$$

where γ_i is $\mathcal{F}_{i-1}$-measurable and $\mu \in \mathbb{R}$. The random variable γ_i describes the amount of money invested at time i in the risky asset.

The condition that $Q^* \in \widehat{\mathcal{M}}$ leads to $T+1$ recursive nonlinear equations for the parameters γ_i, μ. First γ_T is determined dependent on the parameter μ by the equation:

$$E\left((X_T - 1)\, e^{\gamma_T (X_T - 1) + \mu H} | \mathcal{F}_{T-1}\right) = 0.$$

Then γ_{T-1} is determined by

$$E\left((X_{T-1} - 1)\, e^{\gamma_{T-1}(X_{T-1}-1) + \gamma_T (X_T - 1) + \mu H} | \mathcal{F}_{T-2}\right) = 0.$$

Finally γ_1 is determined by

$$E\left((X_1 - 1)\, e^{\sum_{i=2}^T \gamma_i (X_i - 1) + \gamma_1 (X_1 - 1) + \mu H} | \mathcal{F}_0\right) = 0.$$

According to a generalized Bayes formula (see for example [27], page 438-439) this procedure ensures that Q^* is a martingale measure.

The parameter μ is determined by the moment constraint $E_{Q^*} H = p$. One has finally to check condition (ii) for $\varphi_i := \frac{\gamma_i}{S_0 \prod_{s \leq i-1} X_s}$. Then as consequence Q^* as constructed above is the minimal distance measure for $\widehat{\mathcal{M}}$ and $f(x) = x \log x$. Moreover (φ, μ) is an optimal portfolio strategy for the exponential utility function $u(x) = -e^{-x}$. Notice that for the exponential utility function the optimal portfolio strategy is independent of the initial endowment, as $e^{y+z} = e^y e^z$.

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REFERENCES

[1] J. Ansel and C. Stricker. Couverture des actifs contingents et prix maximum. *Annales de l'Institut Henri Poincaré* **30** (1994), 303–315.

[2] M. Avellaneda. Minimum-entropy calibration of asset-pricing models. *International Journal of Theoretical and Applied Finance* **1** (1998), 447–472.

[3] F. Bellini and M. Frittelli. On the existence of minimax martingale measures. Technical Report 2, Universita di Milano, 1998.

[4] J. Cvitanić, W. Schachermayer, and H. Wang. Utility maximization in incomplete markets with random endowment. Preprint, to appear in Finance and Stochastics, 1999.

[5] F. Delbaen, P. Grandits, Th. Rheinländer, D. Samperi, M. Schweizer, and Ch. Stricker. Exponential hedging and entropic penalties. Preprint, 2000.

[6] F. Delbaen and W. Schachermayer. The variance optimal martingale measure for continuous processes. *Bernoulli* **9** (1996), 81–105.

[7] F. Delbaen and W. Schachermayer. A compactness principle for bounded sequences of martingales. In R. C. Dalang, M. Dozzi, and F. Russo, editors, *Seminar on Stochastic Analysis, Random Fields and Applications*, *Progress in Probability* **45** (1999), 137–173. Birkhäuser, Basel.

[8] I. Ekeland and R. Temam. *Convex analysis and variational problems*. North-Holland publishing company, Amsterdam, 1976.

[9] M. Frittelli. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance* **10** (2000), 39–52.

[10] M. Frittelli. Optimal solutions to utility maximization and to the dual problem. Preprint, 2000.

[11] T. Goll and L. Rüschendorf. Minimax and minimal distance martingale measures and their relationship to portfolio optimization. Preprint, to appear in Finance and Stochastics, 2000.

[12] P. Grandits and T. Rheinländer. On the minimal entropy martingale measure. Preprint, 1999.

[13] J. Jacod. *Calcul Stochastique et Problemes de Martingales*. Lecture Notes in Mathematics **714** (1979), Springer-Verlag.

[14] J. Jacod. Intégrales stochastiques par rapport a une semi-martingale vectorielle et changements de filtration. In *Séminaire de Probabilités XIV, 1978/79*, Lecture Notes in Mathematics **784** (1980), 161–172, Springer-Verlag.

[15] Y. Kabanov and C. Stricker. On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper. Preprint, 2001.

[16] J. Kallsen. Duality links between portfolio optimization and derivative pricing. Technical Report 40/1998, Mathematische Fakultät Universität Freiburg i. Br., 1998.

[17] J. Kallsen. *Semimartingale Modelling in Finance*. Dissertation Universität Freiburg i. Br., 1998.

[18] J. Kallsen. Utility-based derivative pricing in incomplete markets. Preprint, 2000.

[19] U. Keller. *Realistic modelling of financial derivatives*. Dissertation Universität Freiburg i. Br., 1997.

[20] D. Kramkov and W. Schachermayer. A condition on the asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability* **9** (1999), 904–950.

[21] F. Liese and I. Vajda. *Convex Statistical Distances*. Teubner, Leipzig, 1987.

[22] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970.

[23] L. Rüschendorf. Projections of probability measures. *Statistics* **18** (1987), 123–129.

[24] W. Schachermayer. Optimal investment in incomplete markets when wealth may become negative. Preprint, to appear in Annals of Applied Probability, 1999.

[25] H.H. Schaefer. *Topological Vector Spaces*. Springer, Berlin, 1971.

[26] M. Schweizer. Approximation pricing and the variance-optimal martingale measure. *Annals of Probability* **24** (1996), 206–236.

[27] A. Shiryaev. *Essentials of Stochastic Finance*. World Scientific, Singapore, 1999.

[28] I. Vajda. *Theory of statistical inference and information*. Kluwer Academic Publishers, Dordrecht, 1989.

[29] M. Yor. Sous-espaces denses dans L^1 ou H^1 et représentation des martingales. In *Séminaire de Probabilités XII, 1978*, Lecture Notes in Mathematics **649** (1978), 265–309, Springer-Verlag.