Comparison of option prices in semimartingale models

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Abstract. In this paper we generalize recent comparison results of El Karoui, Jeanblanc-Picqué, and Shreve (1998), Bellamy and Jeanblanc (2000) and Gushchin and Mordecki (2002) to d-dimensional exponential semimartingales S, S^*. Our main result gives sufficient conditions for the comparison of European options w.r.t. martingale pricing measures. The comparison is with respect to convex and also with respect to directionally convex functions. Sufficient conditions for these orderings are formulated in terms of the predictable characteristics of the stochastic logarithm of the stock price processes S, S^*. As examples we discuss the comparison of exponential semimartingales to multivariate diffusion processes, to stochastic volatility models, to Lévy processes, and to diffusions with jumps. As consequence we obtain extensions of several recent results on nontrivial price intervals. A crucial property in this approach is the propagation of convexity property. We develop a new approach to establish this property for several further examples of univariate and multivariate processes.

Key words: Contingent claim valuation, semimartingale model, price orderings, propagation of convexity

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1 Introduction

For the pricing of options in incomplete market models two general approaches have been suggested in the literature. One approach is to price the option by choosing an (equivalent) martingale pricing measure by some optimization criterion, like minimal risk, minimal distance to the underlying measure and/or fitting of some observed prices. In this way one gets several well established martingale measures like the minimal martingale, the q-optimal measure, the

minimal entropy measure, the Esscher–measure, or the variance minimal measure (see [37, 6, 29, 4, 11, 13]). A second general approach is the utility based indifference price and variants of it due to [21] and [5] (see also [28]). The utility indifference price is in one to one correspondence with the martingale pricing measure obtained by minimizing related f-divergence distances (see [13]).
If no additional assumptions (like an optimization criterion or utility approach) are made, the price of a European call option in an incomplete market model is not unique. It lies within the no-arbitrage price interval that is due to no-arbitrage considerations. For some classes of models the no-arbitrage interval coincides with the trivial pricing interval and the no-arbitrage principle is of no help for choosing a meaningful price. For the models with non-trivial no-arbitrage intervals a basic task is to identify a martingale measure under which the option price bounds the other option prices of the model that are evaluated w.r.t. all other martingale measures from above or from below. Thereto, one needs a comparison result for option prices of a model under different martingale measures.
Comparison results of this type also give an ordering between concrete distance minimizing martingale measures as pointed out above. One general idea is to parametrize the set of martingale measures and prove an ordering of prices in the parameter. After identifying the concrete martingale measures with the parameter, one obtains an ordering of prices calculated from these measures.
More generally, it is also of interest to compare different related models. If the parameters of one model bound the parameters of another in some sense, this should give an ordering of the corresponding option prices. In semimartingale models it is natural to consider the local characteristics of the semimartingale as parameters and to require an ordering for them. In fact, we will state that ordering the so-called differential characteristics of the stochastic logarithms of the considered semimartingales implies ordering of the option prices.
The following example illustrates the basic idea of the main comparison results in this paper. This example and also the further models considered in this paper belong to the class of stochastic exponential semimartingales which are in one to one relationship to exponential semimartingales (cp. Lemma A.1). For a given 1-dimensional semimartingale X the solution of the equation $S_t = s + \int_0^t S_{u-} dX_u$ is given by $S = s\mathcal{E}(X)$, where $\mathcal{E}$ denotes the stochastic exponential of X. In turn $X = \mathcal{L}og(S)$ is called the stochastic logarithm of S. For a semimartingale X the quadratic variation of the continuous martingale part X^c is also called Gaussian characteristic and denoted by $C = \langle X^c \rangle$. We assume that C has a representation of the form $C_t = \int_0^t c_u du$, where c is called the differential Gaussian characteristic of X. Now let $(S_t)_{t\in[0,T]}$ be a 1-dimensional stochastic volatility model defined by the SDE $dS_t = S_t \sigma_t dW_t$, where W is a P-Brownian motion and σ is an adapted process. Let $(S_t^*)_{t\in[0,T]}$ be the solution of a 1-dimensional diffusion with evolution $dS_t^* = S_t^* \sigma^*(t, S_t^*) dW_t^*$, where W^* is a P^*-Brownian motion. Observe that in this case $X = X^c$, and the Gaussian characteristics are given by $C_t = \langle X \rangle_t = \int_0^t \sigma_u^2 du$ and $C_t^* = \langle X^* \rangle_t = \int_0^t {\sigma^*}^2(u, S_u^*) du$. The differential Gaussian characteristics of the stochastic logarithms X, X^* of S, S^* are given by $c_t = \sigma_t^2$ and $c^* = {\sigma^*}^2$. El Karoui, Jeanblanc-Picqué, and Shreve (1998) show

that the comparison of the stochastic volatilities σ, σ^* of X, X^*, $\sigma_t \le \sigma^*(t, S_t)$ for λ-a.e. $t \in [0,T]$ P-a.s., λ the Lebesgue measure, implies an ordering of the prices of a European option with convex payoff function h

$$p := Eh(S_T) \le E^* h(S_T^*) := p^*,$$

where E (E^*) denotes the expectation w.r.t. P (P^*). The ordering of the volatilities is equivalent to the ordering of the differential Gaussian characteristics of the stochastic logarithms

$$c_t = \sigma_t^2 \le {\sigma^*}^2(t, S_t) = c^*(t, S_t),$$

for λ-a.e. $t \in [0,T]$ P-a.s.. So the tenor is the following: Essentially ordering of the differential characteristics of the stochastic logarithms X, X^* of two exponential semimartingale models S, S^* implies ordering of the prices of a European option $p \le p^*$ with convex payoff function. However, to establish this conclusion for general models, a technical condition needs to be satisfied, the “propagation of convexity property”.

Non-trivial bounds of the pricing interval are usually attained by Markovian models. Also in the previous example the upper bound process S^* is Markovian. The Markovian assumption for one of the two processes to be compared seems to be necessary. In Example 2.8 we show that if both models S, S^* are non-Markovian stochastic volatility models, then an ordering of the differential characteristics of the stochastic logarithms does not imply the expected ordering of the corresponding option prices.

The problem of deriving ordering results for option prices has been addressed in several recent papers (see [8, 20, 2, 16, 17, 18, 33]). The models leading to the trivial pricing interval have been characterized completely in [7, 24, 10, 26, 14]. The results for models with nontrivial pricing intervals and the corresponding comparison results are less complete. Comparison results for diffusion processes are discussed in El Karoui, Jeanblanc-Picqué, and Shreve (1998) and nontrivial bounds for stochastic volatility models are given in Frey and Sin (1999). Bellamy and Jeanblanc (2000) (see also [17]) prove that the price of an European call for a diffusion with jumps is bounded below by the corresponding Black–Scholes price and above by the trivial upper price (see also [3, 20] for alternative proofs). Finally, the comparison to Lévy processes is considered in Jakubenas (2002) and Gushchin and Mordecki (2002) and a nontrivial upper bound for discrete time models by the Cox–Ross–Rubinstein model is established in [14] and [35].

An important generalization of the technique introduced in El Karoui, Jeanblanc-Picqué, and Shreve (1998) and Bellamy and Jeanblanc (2000) has been established by Gushchin and Mordecki (2002) who derive a general comparison result for one-dimensional semimartingales to some Markov process w.r.t. convex ordering of terminal values. Essentially the comparison of local characteristics plus the important 'propagation of convexity' property of the Markov process imply convex ordering.

The role of convexity can be understood most easily in the following simple case. Let f be a convex function and let $S_t^* = S_0^* \mathcal{E}(\sigma \cdot W)_t$ be a univariate diffusion model with diffusion coefficient $\sigma, \mathcal{E}$ the stochastic exponential. Let E^* denote the corresponding expectation. Then the Black–Scholes price at time $t = 0$,

$$\mathcal{G}(s) = E^*(f(S_T^*) \mid S_0^* = s),$$

where E^* denotes the expectation, is a convex function in s. Let $S_t = s\mathcal{E}(\sigma \cdot W)_t \mathcal{E}(\phi \cdot M)_t$ be a diffusion with jumps model, with compensated Poisson martingale M and jump size ϕ. If the two stochastic exponentials are stochastically independent (as in the case of deterministic σ, ϕ and intensity) then one obtains by Jensen's inequality

$$\begin{aligned} \mathcal{G}(s) &= \mathcal{G}(E(s\mathcal{E}(\phi \cdot M)_T)) \leq E\mathcal{G}(s\mathcal{E}(\phi \cdot M)_T) \\ &= Ef(s\mathcal{E}(\phi \cdot M)_T \mathcal{E}(\sigma \cdot W)_T) = Ef(S_T) \end{aligned} \tag{1.1}$$

i.e. the price of the terminal option for the jump diffusion model dominates the price for the diffusion model. The argument for the comparison in (1.1) is valid also in the multidimensional case as soon as one has defined an analog of the stochastic exponential for this case. For this and some related models similar comparison results were given by Henderson and Hobson (2003) using related coupling arguments.

El Karoui, Jeanblanc-Picqué, and Shreve (1998) prove the essential "propagation of convexity" property, i.e. the convexity of the *backward functional*

$$\mathcal{G}(t, s) = E^*(f(S_T^*) \mid S_t^* = s) \tag{1.2}$$

in s for all t in a univariate diffusion model using the theory of stochastic flows. Bergman, Grundy, and Wiener (1996) prove this property for a one-dimensional diffusion and for a two-dimensional diffusion with level independent characteristics by PDE-arguments. For a partial extension to multivariate diffusions see Janson and Tysk (2004). A propagation of convexity result for univariate Markov processes is given in Martini (1999) and a typical Markov argument is also given in Gushchin and Mordecki (2002) for the one dimensional diffusion case.

In our paper we derive an extension of the comparison result of Gushchin and Mordecki (2002) to d-dimensional semimartingales. We consider the convex ordering and also a variant of the convex ordering – the directionally convex order – which has turned out to be of particular interest for risk measures. We also develop a new technique based on discrete approximation by Euler schemes to establish the propagation of convexity property for several uni- and multivariate processes, where the known techniques do not seem to be applicable. This new approach leads thus to more general comparison results also for one-dimensional processes. In the case of jump diffusions this approach allows to establish a general comparison result which extends the coupling based results essentially; for example it allows to consider the case of random jump sizes.

In section 2 of this paper we introduce the model assumptions and state the general comparison result. In section 3 we consider applications to the comparison of semimartingales to several classes of models like diffusions, stochastic volatility models, Lévy processes and diffusions with jumps in the multivariate case. A main part is to establish the propagation of convexity property for multivariate diffusions, diffusions with jumps and processes with independent increments. The result for diffusions with jumps is new also in the univariate case.

2 Comparison results

In this paper we derive comparison results for European options under various d-dimensional semimartingale models. Of particular interest in financial mathematics are (stochastic) exponential models. For the formulation of these models we introduce the following notation for d-dimensional (stochastic) exponentials and logarithms. Let $(S_t)_{t\in[0,T]}$ be a positive d-dimensional semimartingale on a stochastic basis $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\in[0,T]}, P)$. Then the stochastic logarithm of each component exists and we write $X = \mathcal{L}og(S)$ with $(X^1, \ldots, X^d) = (\mathcal{L}og(S^1), \ldots, \mathcal{L}og(S^d))$, where the superscript i denotes the ith coordinate, $i \leq d$. Similarly we define (stochastic) exponential and logarithm componentwise as $S = \mathcal{E}(X), S = e^{\bar{X}}, \bar{X} = \log(S)$, as well as products and quotients as $xy = (x^1y^1, \ldots, x^dy^d)$ and $x/y = \left(x^1/y^1, \ldots, x^d/y^d\right)$, respectively, $x, y \in \mathbb{R}^d$. We additionally assume that all semimartingales considered in this paper are special. A special semimartingale S has unique canonical decomposition $S = S_0 + M^S + B^S$, where M^S is a local martingale and B^S is a *predictable* càdlàg process with paths of finite variation. By $S \sim (B^S, C^S, \nu^S)$ we denote that the local predictable characteristics of S are given by B^S, C^S and ν^S. The characteristics of $X = \mathcal{L}og(S)$ we denote by $X \sim (B, C, \nu)$. In the case when $S, S_- > 0$, there is a one-to-one relationship between $X = \mathcal{L}og(S)$ and $\bar{X} = \log(S)$. The relationships between the local characteristics of S, X and $\bar{X}$ are given in Lemma A.1 and A.2 in the appendix. We consider semimartingales whose characteristics are *differentiable* with respect to the Lebesgue measure, i.e. for a semimartingale $Z \sim (B, C, \nu)$ there exists a predictable d-dimensional process $b = (b^i)_{i\leq d}$, a predictable process $c = (c^{ij})_{i,j\leq d}$ with values in the set of all symmetric, positive semidefinite $d \times d$-matrices and a transition kernel $K_{\omega,t}(dx)$ from $(\Omega \times [0,T], \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that

$$B^i = b^i \cdot t, \quad C^{ij} = c^{ij} \cdot t, \quad \nu(\omega; dt, dx) = K_{\omega,t}(dx)dt,$$

where $K_{\omega,t}(dx)$ satisfies $K_{\omega,t}(0) = 0$ and $\int K_{\omega,t}(dx)(|x|^2 \wedge 1) \leq 1$. We write $Z \sim (b, c, K)$ and call b, c, K the *differential* characteristics. Semimartingales with differentiable local characteristics are quasi-leftcontinuous and, therefore, discrete models are excluded from our considerations. For the notation and results on semimartingales and their characteristics we refer to Jacod and Shiryaev (2003).

To compare European option prices under different models we assume that the *comparison process* S^* is a *Markovian* semimartingale. One motivation for this restriction is that in cases of nontrivial pricing intervals the upper resp. lower prices are typically given by Markov processes. The characteristics of $X^* = \mathcal{L}og(S^*)$ are of the form $X^* \sim (b^*(t, S^*_{t-}), c^*(t, S^*_{t-}), K^*_t(S^*_{t-}, \cdot))$. Throughout the paper we assume that the sets of equivalent martingale measures for S and S^* are not empty and we compare the evolution of the semimartingales under these equivalent martingale measures. Martingale measures that correspond to S are denoted by Q, those corresponding to S^* are denoted by Q^*. Expectations with respect to Q are denoted by E, those with respect to Q^* are denoted by E^*. We summarize the modelling framework in the following assumption.

Assumption MG *Assume that S, S^* (S^* Markovian) are positive d-dimensional martingales under measures Q and Q^*, respectively, and let the differential characteristics of their stochastic logarithms X, X^* be given as*

$$\begin{aligned} X_t(\omega) &\sim \big(0, c_t(\omega), K_{\omega,t}(\cdot)\big), && X_0 = 0, \\ X_t^*(\omega^*) &\sim \big(0, c^*(t, S_{t-}^*(\omega^*)), K_t^*(S_{t-}^*(\omega^*), \cdot)\big), && X_0^* = 0. \end{aligned}$$

In this paper we compare prices of European options on d underlying securities, when the payoff functions are convex or are directionally convex. We denote the set of convex functions by $\mathcal{F}_{\mathrm{cx}}$ and the set of directionally convex functions by $\mathcal{F}_{\mathrm{dcx}}$. Directionally convex functions are characterized by second differences: For $g : \mathbb{R}^d \to \mathbb{R}$ define the difference operator $\Delta_i^\varepsilon f(x) = f(x+\varepsilon e_i) - f(x)$, where e_i is the i-th unit vector and $\varepsilon > 0$. Then $g : \mathbb{R}^d \to \mathbb{R}$ is directionally convex iff $\Delta_i^\varepsilon \Delta_j^\delta f(x) \geq 0$, for all $x \in \mathbb{R}^d$, all $1 \leq i, j \leq n$ and all $\varepsilon, \delta > 0$. If $f \in \mathcal{F}, \mathcal{F} = \mathcal{F}_{\mathrm{cx}}$ (or $\mathcal{F} = \mathcal{F}_{\mathrm{dcx}}$) is twice continuously differentiable, the (directional) convexity is characterized as follows. $f \in C^2$ is convex if and only if the Hesse form $\mathrm{D}^2 f(x) = (\mathrm{D}_{ij}^2 f(x))_{i,j \leq d}$ is positive semidefinite for all $x \in \mathbb{R}^d$; $f \in C^2$ is directionally convex if and only if $\mathrm{D}_{ij}^2 f(x) \geq 0$, for all $i, j \leq d$ and all $x \in \mathbb{R}^d$. For random vectors U, V the (directionally) convex order is defined by

$$\begin{aligned} U \leq_{\mathrm{cx}} V &\Longleftrightarrow Ef(U) \leq Ef(V), \forall f \in \mathcal{F}_{\mathrm{cx}}, \\ U \leq_{\mathrm{dcx}} V &\Longleftrightarrow Ef(U) \leq Ef(V), \forall f \in \mathcal{F}_{\mathrm{dcx}}, \end{aligned} \tag{2.3}$$

such that the integrals exist. We compare the terminal values S_T, S_T^* with respect to these orders, when $(S_t)_{t\in[0,T]}, (S_t^*)_{t\in[0,T]}$ satisfy Assumption MG. As basic reference for results on convex type stochastic orders we refer to Müller and Stoyan (2002).

Let $\mathcal{H} \in C^{1,2}([0,T] \times \mathbb{R}^d) \to \mathbb{R}$. In the following lemma we establish that if the process $\mathcal{H}(t, S_t^*)$ is a local $(\mathcal{A}_t^*)$-martingale under Q^* and if the jump measure μ^* satisfies an integrability condition, then $\mathcal{H}(t,s)$ satisfies a general version of the Kolmogorov backward equation

$$\mathrm{D}_t \mathcal{H}(t,s) + \frac{1}{2} \sum_{i,j \leq d} \mathrm{D}_{ij}^2 \mathcal{H}(t,s) s^i s^j c^{*ij}(t,s) + \int_{(-1,\infty)^d} (\Lambda \mathcal{H})(t,s,x) K_t^*(s, dx) = 0, \tag{2.4}$$

where $(\Lambda\mathcal{H})(t,s,x) := \mathcal{H}(t, s(1+x)) - \mathcal{H}(t,s) - \sum_{i\leq d} \mathrm{D}_i \mathcal{H}(t,s) s^i x^i$. For $W^*(\omega^*, t, s) := \mathcal{H}(t, S_{t-}^*(\omega^*) + s) - \mathcal{H}(t, S_{t-}^*(\omega^*)) - \sum_{i\leq d} \mathrm{D}_i \mathcal{H}(t, S_{t-}^*(\omega^*)) s^i$ we denote by $W^* * \mu_t^{S^*}$ the integral

$$W^* * \mu_t^{S^*} := \int_{[0,t]\times\mathbb{R}^d} W^*(\cdot, u, s) \mu^{S^*}(du, ds), \tag{2.5}$$

and by $\mathcal{A}_{\mathrm{loc}}^+$ we denote the set of locally integrable increasing processes.

Lemma 2.1. *Assume that $\mathcal{H} \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and let S^* satisfy Assumption MG. Assume that $\mathcal{H}(t, S_t^*)$ is a local $(\mathcal{A}_t^*)$-martingale under Q^*. If $|W^*| * \mu^{S^*} \in \mathcal{A}_{\mathrm{loc}}^+$, or if $\mathcal{H}(t,\cdot)$ is convex, then $\mathcal{H}(t,s)$ satisfies the Kolmogorov backward equation (2.4).*

The proof is given in the appendix.
More specifically, for $g \in \mathcal{F}, \mathcal{F} \in \{\mathcal{F}_{\mathrm{cx}}, \mathcal{F}_{\mathrm{dcx}}\}$, we consider functions $\mathcal{H}$ given by the backward functional $\mathcal{G}(t,s) = E^*(g(S_T^*)|S_t^* = s)$ in (1.2) for a Markovian comparison process S^*. Then $\mathcal{G}(t,s) = \int g(y)P_{t,T}^*(s,dy)$, where $P_{t,T}^*(s,dy)$ is the transition probability of S^*. Throughout the paper we postulate enough smoothness of $P_{t,T}^*(s,dy)$ to imply that $\mathcal{G} \in C^{1,2}$.

Assumption SC(g) *Let $g : \mathbb{R}^d \to \mathbb{R}$. S^* satisfies the smoothness condition SC(g), if the pricing functional $\mathcal{G}(t,s) = \int g(y)P_{t,T}^*(s,dy)$ is in $C^{1,2}([0,T]\times\mathbb{R}^d)$. Similarly, S^* satisfies the smoothness condition SC($\mathcal{F}_0$) for some $\mathcal{F}_0 \subset \mathcal{F}$, if SC(g) holds for all $g \in \mathcal{F}_0$.*

As second crucial assumption we need the *propagation of (directional) convexity*, i.e. the condition that $g \in \mathcal{F}$ implies that $\mathcal{G}(t,\cdot) \in \mathcal{F}, \forall t \in [0,T]$. In other words the convexity of $\mathcal{G}(T,\cdot) = g(\cdot)$ is propagated to earlier time points t.

Assumption P(g) *Let $g \in \mathcal{F}$, $\mathcal{F} \in \{\mathcal{F}_{\mathrm{cx}}, \mathcal{F}_{\mathrm{dcx}}\}$. S^* satisfies the propagation of (directional) convexity property P(g), if $\mathcal{G}(t,\cdot) \in \mathcal{F}$. Similarly, S^* satisfies the propagation of (directional) convexity property P($\mathcal{F}_0$) for some $\mathcal{F}_0 \subset \mathcal{F}$, if P(g) holds for all $g \in \mathcal{F}_0$.*

In the following we extend one-dimensional convex comparison results of El Karoui, Jeanblanc-Picqué, and Shreve (1998), Bellamy and Jeanblanc (2000) and Gushchin and Mordecki (2002) to multivariate semimartingales. We additionally consider also the directionally convex order. Comparison conditions on the differential characteristics of the stochastic logarithms of S, S^* imply a comparison of the terminal values. The basic idea of the approach in the papers mentioned above to derive comparison results is the introduction and study of the *backward linking process* $\mathcal{G}(t,S_t)$ which relates both processes S, S^* in a suitable way. This basic idea is also essential in the following development of various comparison results. We need conditions concerning integrability and boundedness of the backward linking process $\mathcal{G}(t,S_t)$.

Assumption BIC(g) *Let $g : \mathbb{R}^d \to \mathbb{R}$. S, S^* satisfy the boundedness and integrability condition BIC(g), if the backward linking process $\mathcal{G}(t,S_t)$ is bounded from below and $E\mathcal{G}(t,S_t) < \infty$. Similarly, S, S^* satisfy the boundedness and integrability condition BIC($\mathcal{F}_0$) for some $\mathcal{F}_0 \subset \mathcal{F}$, if BIC(g) holds for all $g \in \mathcal{F}_0$.*

Assumption CD(g) *Let $g : \mathbb{R}^d \to \mathbb{R}$. S, S^* satisfy the integrability condition CD(g), if the backward linking process $\mathcal{G}(t,S_t)$ is a process of class (D). Similarly, S, S^* satisfy the integrability condition CD($\mathcal{F}_0$) for some $\mathcal{F}_0 \subset \mathcal{F}$, if CD(g) holds for all $g \in \mathcal{F}_0$.*

The comparison result for directionally convex payoff functions is as follows.

Theorem 2.2 (Directionally convex order, $S \leq_{\mathbf{dcx}} S^*$). *For $g \in \mathcal{F}_{\mathrm{dcx}} \cap C^2$ assume that S, S^* satisfy Assumptions MG and BIC(g) (or CD(g)). Additionally, assume that S^* satisfies Assumptions SC(g) and P(g) and let further $|W| * \mu_t^S, |W^*| * \mu_t^{S^*} \in \mathcal{A}_{\mathrm{loc}}^+$.*

Then the comparison of the differential characteristics of the stochastic logarithms

$$c_t^{ij}(\omega) \le c^{*ij}(t, S_{t-}(\omega)), \tag{2.6}$$

$$\int\limits_{(-1,\infty)^d} f(t, S_{t-}(\omega), x) K_{\omega,t}(dx) \le \int\limits_{(-1,\infty)^d} f(t, S_{t-}(\omega), x) K_t^*(S_{t-}(\omega), dx), \tag{2.7}$$

$\lambda\!\!\!\lambda \times Q$-a.e., for all $f : \mathbb{R}_+ \times \mathbb{R}_+^d \times (-1, \infty)^d \to \mathbb{R}$ with $f(t, s, \cdot) \in \mathcal{F}_{\mathrm{dcx}}$ such that the integrals exist, implies

$$Eg(S_T) \le Eg(S_T^*).$$

Proof. The main point of the proof is to show that the backward linking process $\mathcal{G}(t, S_t)$ is an $(\mathcal{A}_t)$-supermartingale under Q. Then it follows that

$$Eg(S_T) = E\mathcal{G}(T, S_T) \le \mathcal{G}(0, 1) = E^* g(S_T^*).$$

As $|W| * \mu^S \in \mathcal{A}_{\mathrm{loc}}^+$, Itô's formula implies that $\mathcal{G}(t, S_t)$ is a semimartingale with evolution

$$\begin{aligned}\mathcal{G}(t, S_t) = \mathcal{G}(0,1) + M_t + \int\limits_{[0,t]} \mathrm{D}_t \mathcal{G}(u, S_{u-}) du \\ + \frac{1}{2} \sum_{i,j \le d} \int\limits_{[0,t]} \mathrm{D}_{ij}^2 \mathcal{G}(u, S_{u-}) dC_u^{S_{ij}} + W * \mu_t^S,\end{aligned}$$

where $M_t := \sum_{i \le d} \int\limits_{[0,t]} \mathrm{D}_i \mathcal{G}(u, S_{u-}) dS_u^i$ is a one-dimensional local $(\mathcal{A}_t)$-martingale under Q. As $|W| * \mu^S \in \mathcal{A}_{\mathrm{loc}}^+$, there is a local $(\mathcal{A}_t)$-martingale $\hat{M}$ such that

$$\begin{aligned}\mathcal{G}(t, S_t) = \mathcal{G}(0,1) + M_t + \hat{M}_t + \int\limits_{[0,t]} \mathrm{D}_t \mathcal{G}(u, S_{u-}) du \\ + \frac{1}{2} \sum_{i,j \le d} \int\limits_{[0,t]} \mathrm{D}_{ij}^2 \mathcal{G}(u, S_{u-}) dC_u^{S_{ij}} + W * \nu_t^S,\end{aligned}$$

where $W * \nu_t^S$ is defined as in (2.5) with μ^{S^*} replaced by ν^S. Using Lemma A.2 we obtain in terms of differential characteristics of X that $\mathcal{G}(t, S_t) = \mathcal{G}(0,1) + M_t + \hat{M}_t + A_t$, where

$$\begin{aligned}A_t := \int\limits_{[0,t]} \Big\{ \mathrm{D}_t \mathcal{G}(u, S_{u-}) + \frac{1}{2} \sum_{i,j \le d} \mathrm{D}_{ij}^2 \mathcal{G}(u, S_{u-}) S_{u-}^i S_{u-}^j c^{ij}(u, S_{u-}) \\ + \int\limits_{(-1,\infty)^d} (\Lambda \mathcal{G})(u, S_{u-}, x) K_u(S_{u-}, dx) \Big\} du\end{aligned}$$

is predictable and of finite variation. As the backward functional $\mathcal{G}(t, s)$ satisfies the Kolmogorov backward equation of Lemma 2.1 we obtain

$$\begin{aligned}A_t = \int\limits_{[0,t]} \Big\{ \frac{1}{2} \sum_{i,j \le d} \mathrm{D}_{ij}^2 \mathcal{G}(u, S_{u-}) S_{u-}^i S_{u-}^j (c_u^{ij} - c^{*ij}(u, S_{u-})) \\ + \int\limits_{(-1,\infty)^d} (\Lambda \mathcal{G})(u, S_{u-}, x)(K_u(dx) - K_u^*(S_{u-}, dx)) \Big\} du\end{aligned} \tag{2.8}$$

First we show that the backward linking process $\mathcal{G}(t,S_t)$ is a *local* $(\mathcal{A}_t)$-supermartingale under Q. As A_t is predictable and of finite variation, A_t is of integrable variation. Due to the comparison assumption in (2.6) and the directional convexity of $\mathcal{G}$ in the space variable, the first term of the integrand of A_t is non-positive. Let $(\omega,u)\in(\Omega,[0,T])$ be fixed and define the function

$$\Upsilon(x) := \Lambda\mathcal{G}(u,S_{u-}(\omega),x). \tag{2.9}$$

As $\mathrm{D}^2_{ij}\Upsilon(x) = S^i_{u-}S^j_{u-}\mathrm{D}^2_{ij}\mathcal{G}(u,S_{u-}(1+x))\geq 0$ for all x, $\Upsilon(\cdot)$ is directionally convex. Therefore, (2.7) implies that also the second term of the integrand of A_t is non-positive. This yields $-A_t\in\mathcal{A}^+_{\mathrm{loc}}$ and it follows that $\mathcal{G}(t,S_t)$ is a local $(\mathcal{A}_t)$-supermartingale under Q.
It remains to prove that $\mathcal{G}(t,S_t)$ is a supermartingale. In the case that $\mathcal{G}(t,S_t)$ is bounded below with $E\mathcal{G}(t,S_t)<\infty,\forall t$, M_t is bounded below and, therefore, is an $(\mathcal{A}_t)$-supermartingale under Q. It follows that $\mathcal{G}(t,S_t)$ is a supermartingale, as it is integrable. In the case that $\mathcal{G}(t,S_t)$ is a process of class (D) we consider a localizing sequence τ_n for $\mathcal{G}(t,S_t)$. As for all $t\in[0,T]$ we have Q-a.s. that $(\mathcal{G}(t,S_t))^{\tau_n}\longrightarrow\mathcal{G}(t,S_t), n\to\infty$, and $\mathcal{G}(t,S_t)$ is of class (D), the convergence takes place in L^1 and therefore $\mathcal{G}(t,S_t)$ is an $(\mathcal{A}_t)$-supermartingale under Q. $\square$

Remark 2.3. 1. Theorem 2.2 is formulated for directionally convex functions that are smooth. If the conditions of the theorem hold true for a generating class $\mathcal{F}_0\subset\mathcal{F}_{\mathrm{dcx}}\cap C^2$ of the directionally convex order it follows that $S_T\leq_{\mathrm{dcx}} S^*_T$. In particular, one obtains as consequence an ordering result for European call options.

2. As seen in the proof of Theorem 2.2 it is sufficient to assume

$$\int_{(-1,\infty)^d}(\Lambda\mathcal{G})(t,S_{t-}(\omega),x)K_{\omega,t}(dx)\leq\int_{(-1,\infty)^d}(\Lambda\mathcal{G})(t,S_{t-}(\omega),x)K^*_t(S_{t-}(\omega),dx),$$

for $\lambda\!\!\!\lambda\times Q$-a.e. (t,ω), instead of the directionally convex ordering condition on the jumps in (2.7).

If the inequalities on the characteristics of X and X^* in Theorem 2.2 are reversed, we obtain a similar comparison result. In this case we have to prove that the backward linking process $\mathcal{G}(t,S_t)$ is an $(\mathcal{A}_t)$-submartingale under Q. The boundedness assumption on $\mathcal{G}$ no longer is useful for this conclusion as boundedness of $\mathcal{G}(t,s)$ from above does not make sense for $\mathcal{G}(t,\cdot)\in\mathcal{F}_{\mathrm{dcx}}$.

Theorem 2.4 (Directionally convex order, $S^*\leq_{\mathrm{dcx}}S$). *For $g\in\mathcal{F}_{\mathrm{dcx}}\cap C^2$ assume that S,S^* satisfy Assumptions MG and CD(g). Additionally, assume that S^* satisfies Assumptions SC(g) and P(g) and let further $|W|*\mu^S_t, |W^*|*\mu^{S^*}_t\in\mathcal{A}^+_{\mathrm{loc}}$.*
Then the comparison of the differential characteristics of the stochastic logarithms

$$c^{*ij}(t,S_{t-}(\omega))\leq c^{ij}_t(\omega),$$

$$\int_{(-1,\infty)^d}f(t,S_{t-}(\omega),x)K^*_t(S_{t-}(\omega),dx)\leq\int_{(-1,\infty)^d}f(t,S_{t-}(\omega),x)K_{\omega,t}(dx)$$

$\lambda \times Q$*-a.e., for all* $f : \mathbb{R}_+ \times \mathbb{R}_+^d \times (-1,\infty)^d \to \mathbb{R}$ *with* $f(t,s,\cdot) \in \mathcal{F}_{\mathrm{dcx}}$ *such that the integrals exist, implies*

$$E^* g(S_T^*) \le E g(S_T).$$

Proof. Similar to the proof of Theorem 2.2 we show that the backward linking process $\mathcal{G}(t,S_t)$ is an $(\mathcal{A}_t)$-submartingale under Q. Then it follows that

$$E^* g(S_T^*) = \mathcal{G}(0,1) \le E\mathcal{G}(T,S_T) = E g(S_T).$$

Proceeding as in the proof of Theorem 2.2, the backward linking process has representation $\mathcal{G}(t,S_t) = \mathcal{G}(0,1) + M_t + \hat{M}_t + A_t$, with A_t given by (2.8). It follows from the ordering of the characteristics of X and X^* that $A_t \in \mathcal{A}_{\mathrm{loc}}^+$ and, therefore, $\mathcal{G}(t,S_t)$ is a local $(\mathcal{A}_t)$-submartingale under Q.
As $\mathcal{G}(t,S_t)$ is assumed to be a process of class (D), the arguments of the second case of the proof of Theorem 2.2 apply. Therefore, $\mathcal{G}(t,S_t)$ is an $(\mathcal{A}_t)$-submartingale under Q. □

Similar comparison results hold true also for the convex order. By $\le_{\mathrm{psd}}$ we denote the positive semidefinite partial ordering on the set $M_+(d,\mathbb{R})$ of real positive semidefinite $d \times d$-matrices.

Theorem 2.5 (Convex order, $S \le_{\mathrm{cx}} S^*$). *For* $g \in \mathcal{F}_{\mathrm{cx}} \cap C^2$ *assume that* S, S^* *satisfy Assumptions MG and BIC(g) (or CD(g)). Additionally, assume that* S^* *satisfies Assumptions SC(g) and P(g).*
Then the comparison of the differential characteristics of the stochastic logarithms

$$\begin{aligned} (c_t^{ij}(\omega))_{i,j\le d} &\le_{\mathrm{psd}} (c^{*ij}(t,S_{t-}(\omega)))_{i,j\le d}, \\ \int\limits_{(-1,\infty)^d} f(t,S_{t-},x) K_{\omega,t}(dx) &\le \int\limits_{(-1,\infty)^d} f(t,S_{t-},x) K^*(t,S_{t-}(\omega),dx), \end{aligned} \tag{2.10}$$

$\lambda \times Q$*-a.e., for all non-negative* $f : \mathbb{R}_+ \times \mathbb{R}_+^d \times (-1,\infty)^d \to \mathbb{R}$ *with* $f(t,s,\cdot) \in \mathcal{F}_{\mathrm{cx}}$ *such that the integrals exist, implies*

$$E g(S_T) \le E^* g(S_T^*).$$

Proof. We proceed as in the proof of Theorem 2.2. Again we first show that $\mathcal{G}(t,S_t)$ is a local $(\mathcal{A}_t)$-supermartingale under Q. The evolution of $\mathcal{G}(t,S_t)$ is given by (2.8) and we have to show that $-A_t \in \mathcal{A}_{\mathrm{loc}}^+$. Due to the positive semidefiniteness of the symmetric matrix $\big(c^{*ij} - c^{ij}\big)_{i,j\le d} := \big(c^{*ij}(t,S_{t-}(\omega)) - c_t^{ij}(\omega)\big)_{i,j\le d}$ for fixed (ω,t), its spectral decomposition is given by $\big(c^{*ij} - c^{ij}\big)_{i,j\le d} = \big(\sum\limits_{k\le d} \lambda_k a_k^i a_k^j\big)_{i,j\le d}$, where $\lambda_k \ge 0$ are the eigenvalues and a_k are the eigenvectors of the matrix $(c^* - c)$. Therefore the first term of the integrand of A_t takes the form $-\frac{1}{2} \sum\limits_{k\le d} \lambda_k \sum\limits_{i,j\le d} \mathrm{D}_{ij}^2 \mathcal{G}(u,S_{u-}) S_{u-}^i a_k^i S_{u-}^j a_k^j$, and is non-positive due to the propagation of convexity property in Assumption $\mathrm{P}(g)$ and the characterization of the convex order. Also by the convexity assumption on $\mathcal{G}(t,\cdot)$, $(\Lambda\mathcal{G})(u,S_{u-}(\omega),x)$ is non-negative and $\Upsilon(x) := \Lambda\mathcal{G}(u,S_{u-},x) \in \mathcal{F}_{\mathrm{cx}}$.

Furthermore, by the ordering of the jump-compensators, the second term in the integrand of A_t is non-positive. Therefore, $-A_t \in \mathcal{A}^+_{\mathrm{loc}}$ and $\mathcal{G}(t,S_t)$ is a local $(\mathcal{A}_t)$-supermartingale under Q. The $(\mathcal{A}_t)$-supermartingale property under Q follows similar to the proof of Theorem 2.2. $\square$

Remark 2.6. 1. Theorem 2.2 is formulated for convex functions that are smooth. If the conditions of the theorem hold true for for a generating class $\mathcal{F}_0 \subset \mathcal{F}_{\mathrm{cx}} \cap C^2$ of the convex order it follows that $S_T \leq_{\mathrm{cx}} S^*_T$. In particular, one obtains as consequence an ordering result for European call options.

2. Again it is sufficient to assume

$$\int\limits_{(-1,\infty)^d} (\Lambda\mathcal{G})(t,S_{t-}(\omega),x)K_{\omega,t}(dx) \leq \int\limits_{(-1,\infty)^d} (\Lambda\mathcal{G})(t,S_{t-}(\omega),x)K^*_t(S_{t-}(\omega),dx),$$

for $\lambda\!\!\!\lambda \times Q$-a.e. (t,ω), instead of the convex ordering condition on the jumps in (2.10).

As in the case of the directionally convex ordering, under an integrability assumption on $\mathcal{G}(t,S_t)$ the inequalities on the differential characteristics of X and X^* may be reversed to obtain $E^*g(S^*_T) \leq Eg(S_T)$. We omit the proof, which is a combination of the proofs of Theorems 2.4 and 2.5.

Theorem 2.7 ($S^* \leq_{\mathrm{cx}} S$). *For $g \in \mathcal{F}_{\mathrm{cx}} \cap C^2$ assume that S, S^* satisfy Assumptions MG and CD(g). Additionally, assume that S^* satisfies Assumptions SC(g) and P(g).*
Then the comparison of the differential characteristics of the stochastic logarithms

$$(c^{*ij}(t,S_{t-}(\omega)))_{i,j\leq d} \leq_{\mathrm{psd}} (c^{ij}_t(\omega))_{i,j\leq d},$$

$$\int\limits_{(-1,\infty)^d} f(t,S_{t-},x)K^*_t(S_{t-}(\omega),dx) \leq \int\limits_{(-1,\infty)^d} f(t,S_{t-},x)K_{\omega,t}(dx),$$

$\lambda\!\!\!\lambda \times Q$-a.e., for all non-negative $f : \mathbb{R}_+ \times \mathbb{R}^d_+ \times (-1,\infty)^d \to \mathbb{R}$ with $f(t,s,\cdot) \in \mathcal{F}_{\mathrm{cx}}$ such that the integrals exist, implies

$$E^*g(S^*_T) \leq Eg(S_T).$$

The Markovian assumption on the comparison process S^* seems to be necessary for comparison results as in Theorems 2.2–2.7. This is already known from the comparison results for random vectors. A nice example demonstrating this in the context of diffusion processes was communicated to us by J. Kallsen.

Example 2.8 (Larger volatility does not imply a larger price in SV models). *Let $\bar{\sigma} \in \mathbb{R}_+$ and W be a one-dimensional Brownian motion on a stochastic basis $(\Omega, \mathcal{A}, (\mathcal{A}_t), Q)$. Let $\bar{S}$ be the solution of*

$$d\bar{S}_t = \bar{\sigma}\bar{S}_t dW_t, \quad \bar{S}_0 = 1.$$

For $K > 1$ we define an $(\mathcal{A}_t)$-adapted process σ^*_t as

$$\sigma^*_t := \begin{cases} \bar{\sigma}, \max\limits_{u\leq t} \bar{S}_u \leq K, \\ 0, \textit{ else.} \end{cases}$$

Let S^* be a solution of $dS_t^* = \sigma_t^* S_t^* dW_t, \quad S_0^* = 1$. Let $0 < \tilde{\sigma} < \bar{\sigma}$ and $T_0 \in [0, T]$ be fixed. Define an $(\mathcal{A}_t)$-adapted process σ_t as

$$\sigma_t := \begin{cases} \tilde{\sigma}, \max\limits_{u \leq t} \bar{S}_u \leq K, \ 0 \leq t \leq T_0, \\ \bar{\sigma}, \max\limits_{u \leq t} \bar{S}_u \leq K, \ T_0 < t \leq T, \\ 0, \ \textit{otherwise}, \end{cases}$$

and let S be a solution of $dS_t = \sigma_t S_t dW_t, \quad S_0 = 1$. Then $\sigma_t \leq \sigma_t^*$ and S, S^* are not Markovian as their volatilities are path-dependent.
Let C be a European call with strike K. The price for that call option is zero under the S^* model. But as there is a positive probability for paths with $S_{T_0} > S_{T_0}^*$, we have $Q(S_T > K) > 0$. Therefore, the call price is positive with respect to the S model although the volatility σ of S is smaller than the volatility σ^* of S^*.

3 Applications

In this section we give applications of the ordering results in section 2. The main point to check is the propagation of (directional) convexity property for the Markovian comparison process S^*. Subsections 3.1 and 3.2 are concerned with the propagation of convexity property for multivariate diffusions and diffusions with jumps, respectively. We first prove a convexity result for the corresponding Euler approximation scheme and then obtain the propagation of convexity property also for the limit. In the case of diffusions with jumps this result is also new in the univariate case. As examples we consider convex comparison of diffusions with stochastic volatility models and of diffusions with jumps with non-Markovian martingales. This implies also convex comparison between Lévy driven diffusions for pointwise ordered local volatilities.
In subsection 3.3 we show that the stochastic exponential of a process with independent increments (PII) may serve as Markovian comparison process S^* that satisfies the propagation of (directional) convexity property. As examples we give convex and directionally convex ordering results of the stochastic exponentials of compound Poisson processes. Some convex comparison results of exponential normal inverse Gaussian and variance gamma models are also stated.

3.1 Convex comparison with a multivariate diffusion

In this subsection we apply the convex comparison results in Theorems 2.5 and 2.7 to the case where the comparison process S^* is a multivariate diffusion. Similar results are given in different generality for one-dimensional diffusions in the literature. El Karoui, Jeanblanc-Picqué, and Shreve (1998) compare option prices under diffusion models to option prices where the underlying process is a stochastic volatility model. The ordering is implied by an ordering on the corresponding volatilities. The proof uses PDE arguments and the propagation of convexity property. Frey and Sin (1999) prove a special case where the volatility of the diffusion is given by a constant $\sigma_{\max}$ that dominates the volatility of the stochastic volatility model. Then the option price under the stochastic volatility model is bounded above by the Black–Scholes price with volatility $\sigma_{\max}$. Bellamy and Jeanblanc (2000) compare a diffusion to a diffusion with jumps, using similar arguments as in El Karoui, Jeanblanc-Picqué, and Shreve (1998). Gushchin and Mordecki (2002) extend this result to the case where the jump-diffusion is not Markovian. They show that a diffusion may serve as Markovian comparison process and then apply their general comparison theorem. Hobson (1998) proves that option prices under two different diffusion models are ordered, if the diffusion coefficients are ordered pointwise, using coupling arguments. Henderson (2002) uses the coupling method to give an ordering result on options in stochastic volatility models, when the volatility is a diffusion that is driven by a Brownian motion W, independent of the Brownian motion B driving S, S^*. Henderson, Hobson, Howison, and Kluge (2003) extend this result to the case where W and B are correlated, using PDE techniques. See also Bergman, Grundy, and Wiener (1996) as an early reference on option price monotonicity. The comparison results proved via PDE techniques make use of the propagation of convexity property. For one-dimensional diffusions this property is proved in Bergman, Grundy, and Wiener (1996) using PDE techniques, in El Karoui, Jeanblanc-Picqué, and Shreve (1998) via the theory of stochastic flows and in Hobson (1998) via coupling. See also Martini (1999) for the proof of this property for Markovian and martingalian semigroups. The proofs of these papers do not seem to be applicable to jump processes or in the multivariate case. A partial extension to multivariate diffusions is given in Janson and Tysk (2004). We introduce a new method of proof that relies on ordering of Markov chains that approximate the comparison process.

Let $g \in \mathcal{F}_{\text{cx}}$ and assume that W^* is a d-dimensional Brownian motion on a stochastic basis $(\Omega^*, \mathcal{A}^*, (\mathcal{A}_t^*)_{t\in[0,T]}, Q^*)$. For $\sigma^* : [0,T] \times \mathbb{R}^d \to M_+(d, \mathbb{R})$ let S^* be the unique strong solution of the SDE

$$dS_t^* = \sigma^*(t, S_t^*)dW_t^*, \quad S_0^* = 1. \tag{3.11}$$

Additionally, we assume that Assumptions MG and SC(g) are satisfied, i.e. that S, S^* are positive martingales and that the backward functional $\mathcal{G} \in C^{1,2}$. We assume throughout this section that BIC(g) or CD(g) is satisfied, depending on which condition is needed for the comparison theorems of the previous section.

For the proof of the propagation of convexity, we use an Euler scheme for S^* and prove the propagation of convexity for the approximating Markov chain. Let $t_0 \in [0,T]$ and discretize $[t_0, T]$ into $K+1$ equidistant points $t_i := i\frac{T-t_0}{K} + t_0, i \in \{0, \dots, K\}$. We denote the Euler scheme of $(S_t^*)_{t\in[t_0,T]}, S_{t_0}^* = s$, by

$$\begin{aligned} \tilde{S}_{K,t_{i+1}}^* &= \tilde{S}_{K,t_i}^* + \sigma^*(t_i, \tilde{S}_{K,t_i}^*)(W_{t_{i+1}}^* - W_{t_i}^*), \quad i \in \{0, \dots, K-1\}, \\ \tilde{S}_{K,t_0}^* &= s. \end{aligned} \tag{3.12}$$

Under some additional assumptions on the diffusion coefficient and the payoff function $g \in \mathcal{F}_{\text{cx}}$, the terminal value of the Euler scheme $\tilde{S}_{K,T}^*$ converges in distribution to S_T^* as $K \to \infty$, independent of the starting point $S_{t_0}^* = \tilde{S}_{K,t_0}^* = s$. We call this the approximation property and refer to Kloeden and Platen (1992) and to Liu and Li (2000) for weak approximation results.

Assumption AP(g) *Let $g : \mathbb{R}^d \to \mathbb{R}$. The Euler scheme $\tilde{S}^*_K$ satisfies the approximation property AP(g), if*

$$\tilde{\mathcal{G}}_K(t,s) := E^*(g(\tilde{S}^*_{K,T})|\tilde{S}^*_{K,t} = s) \to \mathcal{G}(t,s), \quad \forall t \in [0,T], s \in \mathbb{R}^d,$$

as $K \to \infty$.

A sufficient condition for the propagation of convexity of the Euler scheme is the $\leq_{\mathrm{cx}}$-monotonicity of the corresponding transition operator. To prove this we assume that $\sigma^* : [t_0, T] \times \mathbb{R}^d \to M_+(d,\mathbb{R})$ is convex in the second component with the positive semidefinite partial ordering $\leq_{\mathrm{psd}}$ on $M_+(d,\mathbb{R})$. By $W \sim N(\mu, \Sigma)$ we denote that W is normally distributed with expectation vector μ and covariance matrix Σ.

Lemma 3.1 (A convex ordering result for Markov operators). *Let S be a d-dimensional random vector that is independent of $W \sim N(0, I)$, where I is the identity, and let $\sigma : \mathbb{R}^d \to M_+(d,\mathbb{R})$. If σ is convex, then the Markov operator $\mathcal{T}$ on the set of probability measures with state space $(\mathbb{R}^d, \mathcal{B}^d)$ defined by*

$$\mathcal{T}S \stackrel{d}{=} S + \sigma(S)W$$

is $\leq_{\mathrm{cx}}$-monotone, i.e. $S_1 \leq_{\mathrm{cx}} S_2$ implies $\mathcal{T}S_1 \leq_{\mathrm{cx}} \mathcal{T}S_2$.

Proof. Let S_1, S_2 be d-dimensional random vectors that are independent of W and satisfy $S_1 \leq_{\mathrm{cx}} S_2$. Due to Strassen's Theorem there are random vectors $\hat{S}_i \stackrel{d}{=} S_i$, $i = 1, 2$, on a probability space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$ such that $\hat{E}(\hat{S}_2|\hat{S}_1) = \hat{S}_1$, where $\hat{E}$ denotes the expectation with respect to $\hat{P}$. We assume without loss of generality that S_1, S_2 are these versions. Then, for $f \in \mathcal{F}_{\mathrm{cx}}$, Jensen's inequality implies

$$\begin{aligned} Ef(\mathcal{T}S_2) &= Ef(S_2 + \sigma(S_2)W) = EE(f(S_2 + \sigma(S_2)W)|S_1, W) \\ &\geq Ef(S_1 + E(\sigma(S_2)|S_1)W) = E^{S_1} Ef(s_1 + E(\sigma(S_2)|s_1)W), \end{aligned}$$

where the last equality follows from conditioning on $S_1 = s_1$ and E^{S_1} denotes the expectation with respect to the distribution of S_1. As the convex order is stable under mixtures, it remains to prove that

$$E(\sigma(S_2)|s_1)W \geq_{\mathrm{cx}} \sigma(s_1)W. \tag{3.13}$$

As $\sigma \in \mathcal{F}_{\mathrm{cx}}$, Jensen's inequality implies

$$g(s_1) := E(\sigma(S_2)|s_1) \geq_{\mathrm{psd}} \sigma(s_1),$$

and therefore it follows that $g(s_1)^T g(s_1) \geq_{\mathrm{psd}} \sigma(s_1)^T \sigma(s_1)$, where the superscript T denotes the transpose. As $\sigma(s_1)W \sim N(0, \sigma(s_1)^T\sigma(s_1))$, $g(s_1)W \sim N(0, g(s_1)^T g(s_1))$, this implies $E(\sigma(S_2)|s_1)W = g(s_1)W \geq_{\mathrm{cx}} \sigma(s_1)W$ (cp. Müller and Stoyan (2002, Theorem 3.4.7)). □

Propagation of convexity of $\mathcal{G}$ follows from the convexity of the transition operator of $\tilde{S}^*_K$ and the approximation property.

Theorem 3.2 (Propagation of convexity, diffusion case). *Let $g \in \mathcal{F}_{\mathrm{cx}}$, S^* be a d-dimensional diffusion and assume that the Euler scheme $\tilde{S}^*_K$ of S^* satisfies Assumption AP(g). If $\sigma^*(t,\cdot)$ is convex for all $t \in [0,T]$, then propagation of convexity property P(g) holds, i.e.*

$$\mathcal{G}(t,\cdot) \in \mathcal{F}_{\mathrm{cx}}, \quad \forall t \in [0,T].$$

Proof. We consider the Euler approximation scheme $\tilde{S}^*_K$ defined in (3.12) with interpolation points t_i and define the corresponding transition operator by $\mathcal{T}_{t_i} S \overset{d}{=} S + \sigma(t_i, S) W$, where $W \overset{d}{=} W^*_{t_i+1} - W^*_{t_i}$. Then for $t_0 \in [0,T]$ we have by the Markov property of this scheme

$$\tilde{\mathcal{G}}_K(t_0, y) = E^*(g(\tilde{S}^*_{K,T}) | \tilde{S}^*_{K,t_0} = y) = E^* g(\mathcal{T}_{t_{K-1}} \dots \mathcal{T}_{t_0} y).$$

For $y_1, y_2 \in \mathbb{R}^d$ and $\alpha \in (0,1)$ let Y be a Bernoulli random vector with distribution $P^Y = \alpha \varepsilon_{\{y_1\}} + (1-\alpha)\varepsilon_{\{y_2\}}$. Then

$$\alpha y_1 + (1-\alpha) y_2 = EY \leq_{\mathrm{cx}} Y.$$

Using the $\leq_{\mathrm{cx}}$-monotonicity of the operator $\mathcal{T}_t$ for all $t \in [0,T]$ from Lemma 3.1 we obtain

$$\begin{aligned} \tilde{\mathcal{G}}_K(t_0, \alpha y_1 + (1-\alpha) y_2) &= \tilde{\mathcal{G}}_K(t_0, EY) = E^* g(\mathcal{T}_{t_{K-1}} \dots \mathcal{T}_{t_0} EY) \\ &\leq E^* g(\mathcal{T}_{t_{K-1}} \dots \mathcal{T}_{t_0} Y) = \tilde{\mathcal{G}}_K(t_0, Y). \end{aligned} \tag{3.14}$$

Taking expectations on both sides of (3.14) implies convexity of $\tilde{\mathcal{G}}_K(t_0, \cdot)$. The approximation property AP then implies that $\mathcal{G}(t_0, \cdot) \in \mathcal{F}_{\mathrm{cx}}$. □

The diffusion process S^* in (3.11) has differential characteristics $S^*_t \sim \big(0, \sigma^{S^*}(t, S^*_t)(\sigma^{S^*}(t, S^*_t))^T, 0\big)$, where superscript T is the transposition. Therefore, by Lemma A.2, the stochastic logarithm $X^* = \mathcal{L}og(S^*)$ has differential characteristics $(0, c^*(t, S^*_t), 0)$, with

$$c^{*ij}(t, S^*_t(\omega)) = \sum_{k \leq d} \frac{\sigma^{S^* ik}(t, S^*_t(\omega))}{S^{*i}_t(\omega)} \frac{\sigma^{S^* jk}(t, S^*_t(\omega))}{S^{*j}_t(\omega)}. \tag{3.15}$$

Using this representation and the propagation of convexity property for multivariate diffusions, we obtain a convex comparison of a diffusion model to a stochastic volatility model. Following Hofmann, Platen, and Schweizer (1992), a general class of one-dimensional stochastic volatility models is given by

$$\begin{aligned} \frac{dS_t}{S_t} &= \sigma(t, S_t, v_t) dW_t, \\ dv_t &= b(t, S_t, v_t) dt + \eta_1(t, S_t, v_t) dW_t + \eta_2(t, S_t, v_t) dB_t, \end{aligned}$$

where W and B are independent Brownian motions. This includes the models of Hull and White (1987), Stein and Stein (1991), Wiggins (1987), Scott (1987) and Heston (1993), see also Frey (1997) for an overview over stochastic volatility models. In terms of the characteristics of $X = \mathcal{L}og(S)$, we have $X \sim \big(0, (\sigma(t, S_t, v_t))^2, 0\big)$. More generally, we say that S is a d-dimensional stochastic volatility model if

$$X \sim (0, c, 0) \tag{3.16}$$

for an adapted predictable process c with values in $M_+(d, \mathbb{R})$. Under an ordering condition on the diffusion coefficients, a diffusion S^* is an upper bound for a stochastic volatility model S.

Theorem 3.3 (Convex comparison of SV model to diffusion, upper bound). *Let $g \in \mathcal{F}_{\mathrm{cx}} \cap C^2$. Assume that $S = \mathcal{E}(X)$ is a d-dimensional stochastic volatility model, where X has differential characteristics given by (3.16) and that S^* is a d-dimensional diffusion with diffusion coefficient function c^* given by (3.15). Let S, S^* satisfy Assumptions MG and BIC(g) (or CD(g)) and assume that S^* satisfies Assumption SC(g) and that the Euler scheme $\tilde{S}^*_K$ of S^* satisfies Assumption AP(g).*
If $\sigma^*(t, \cdot)$ is convex, for all $t \in [0, T]$, then the comparison of the differential characteristics of the stochastic logarithms

$$(c_t^{ij}(\omega))_{i,j \le d} \le_{\mathrm{psd}} (c^{*ij}(t, S_{t-}(\omega)))_{i,j \le d}, \quad \lambda\!\!\!\lambda \times Q\text{–}a.e.,$$

implies

$$Eg(S_T) \le E^* g(S_T^*).$$

Proof. This follows from Theorem 2.5 and Proposition 3.2. □

As proved in Bellamy and Jeanblanc (2000) in the univariate case, a diffusion with jumps is riskier than a diffusion, if the volatility of the diffusion with jumps is not smaller than the volatility of the pure diffusion. We generalize this to multivariate models and non-Markovian jump processes. We say that S is a d-dimensional diffusion with jumps, if

$$X = \mathcal{L}og(S) \sim (0, c, K). \tag{3.17}$$

Theorem 3.4 (Convex comparison of SV model with jumps to diffusion, lower bound). *Let $g \in \mathcal{F}_{\mathrm{cx}} \cap C^2$. Assume that $S = \mathcal{E}(X)$ is a d-dimensional stochastic volatility model with jumps, where X has differential characteristics given by (3.17), and that S^* is a d-dimensional diffusion with diffusion coefficient function c^* given by (3.15). Let S, S^* satisfy Assumptions MG and CD(g) and assume that S^* satisfies Assumption SC(g) and that the Euler scheme $\tilde{S}^*_K$ of S^* satisfies AP(g).*
If $\sigma^*(t, \cdot)$ is convex, then the comparison of the differential characteristics of the stochastic logarithms

$$(c^{*ij}(t, S_{t-}(\omega)))_{i,j \le d} \le_{\mathrm{psd}} (c_t^{ij}(\omega))_{i,j \le d}, \quad \lambda\!\!\!\lambda \times Q\text{–}a.e.,$$

implies

$$E^* g(S_T^*) \le Eg(S_T).$$

Proof. This follows from Theorem 2.7 and Proposition 3.2. □

Remark 3.5. In the one-dimensional case, the comparison of c and c^* reduces to a pointwise comparison. If S^* is a solution of the SDE

$$\frac{dS^*}{S^*} = \sigma^*(t, S_t^*) dW_t^*, \quad S_0^* = 1,$$

then $c_t^* = (\sigma^*(t, S_t^*))^2$. Under the assumptions of Theorem 3.3, a sufficient condition for the upper bound is

$$c_t(\omega) \le (\sigma^*(t, S_t(\omega)))^2, \quad \text{for } \lambda\!\!\!\lambda \times Q - \text{a.e. } (t, \omega). \tag{3.18}$$

This is already stated in El Karoui, Jeanblanc-Picqué, and Shreve (1998). Under the assumptions of Theorem 3.4 a sufficient condition for the lower bound is

$$(\sigma^*(t, S_t(\omega)))^2 \leq c_t(\omega), \quad \text{for } \lambda\!\!\lambda \times Q - \text{a.e. } (t, \omega). \tag{3.19}$$

For $K = 0$ the lower bound result is also stated in El Karoui, Jeanblanc-Picqué, and Shreve (1998). Bellamy and Jeanblanc (2000) establish the lower bound for deterministic K.

3.2 Convex comparison with a multivariate diffusion with jumps

We generalize the results of the previous subsection to the case where the Markovian comparison process S^* is a multivariate diffusion with jumps. For the one-dimensional case some comparison results are given in the literature. Bellamy and Jeanblanc (2000) compare a diffusion with a diffusion with jumps, where the jumps are driven by a Poisson process with deterministic intensity. Henderson and Hobson (2003) consider two cases. In the first case, the jumps are driven by a Poisson random measure and all involved parameters are deterministic functions of time. Therefore, the jump and the diffusion part are independent and coupling arguments apply, see also our introduction. In the second case they obtain a convex comparison result for a diffusion with jumps where the intensity of the Poisson random measure is Markovian and there is only one constant jump size. Møller (2003) applies the cut criterion to prove a convex ordering result for univariate diffusions with positive jumps that are assumed to have independent increments. In the following we extend these results. We give sufficient conditions under which d-dimensional diffusions with jumps generated by a marked point process are comparable to non-Markovian martingales.

Let $g \in \mathcal{F}_{\text{cx}}$ and assume that W^* is a d-dimensional Brownian motion on a stochastic basis $(\Omega^*, \mathcal{A}^*, (\mathcal{A}_t^*)_{t\in[0,T]}, Q^*)$. Assume that N^* is a Poisson random measure on $[0,T] \times \mathbb{E}$, where the mark space $\mathbb{E}$ is $\mathbb{R}$ or $\mathbb{R}^d$. Let N^* have deterministic intensity $\lambda^*(dy)dt$ and $\lambda^*(\mathbb{E}) < \infty$. For $\sigma^* : [0,T] \times \mathbb{R}^d \to M_+(d, \mathbb{R})$ and $\phi^* : [0,T] \times \mathbb{R}^d \times \mathbb{E} \to \mathbb{R}^d$ let S^* be a unique strong solution of

$$\begin{aligned} dS_t^* &= \sigma^*(t, S_t^*)dW_t^* + \phi^*(t, S_{t-}^*, y)(N^*(dt, dy) - \lambda^*(dy)dt), \\ S_0^* &= 1 \end{aligned} \tag{3.20}$$

Additionally, we assume as in the diffusion case that Assumptions MG and SC(g) are satisfied, i.e. that S, S^* are positive martingales and the backward functional $\mathcal{G} \in C^{1,2}$. Again we assume throughout this subsection that BIC(g) or CD(g) is satisfied, depending on which condition is needed for the comparison theorems of section 2.

To establish the propagation of convexity property P(g) we proceed as in the diffusion case. We prove the propagation of convexity for the Euler scheme corresponding to S^* and then make use of the approximation property. For $t_0 \in [0,T]$ we discretize $[t_0, T]$ into $K+1$ equidistant points $t_i := i\frac{T-t_0}{K} + t_0, i \in$

$\{0, \dots, K\}$ and denote the Euler scheme of S^* by

$$\begin{aligned}
\tilde{S}^*_{K,t_{i+1}} &= \tilde{S}^*_{K,t_i} + \sigma^*(t_i, \tilde{S}^*_{K,t_i})(W^*_{t_{i+1}} - W^*_{t_i}) \\
&\quad + \phi^*(t_i, S^*_{t_i}, Y)\tilde{N}^* - E^Y \phi^*(t_i, S_{t_i}, Y)\lambda^*(\mathbb{E})\Delta t_i, \qquad (3.21) \\
\tilde{S}^*_{K,t_0} &= s,
\end{aligned}$$

where $\tilde{N}^*$ is binomial with $P^{\tilde{N}^*} = (1 - \lambda^*(\mathbb{E})\Delta t_i)\varepsilon_{\{0\}} + \lambda^*(\mathbb{E})\Delta t_i \varepsilon_{\{1\}}$, Y has distribution $\frac{\lambda^*(dy)}{\lambda^*(\mathbb{E})}$ on $(\mathbb{E}, \mathcal{E})$, $E^Y f(\cdot, \cdot, Y) := \frac{1}{\lambda^*(\mathbb{E})} \int f(\cdot, \cdot, y)\lambda^*(dy)$ and $\Delta t_i = t_{i+1} - t_i$. We refer to Liu and Li (2000) for conditions that imply the approximation property for this case.
Again, we assume that $\sigma^*(t, \cdot)$ is convex and additionally we assume that the jump coefficient satisfies one of the following conditions.

(J1) $\mathbb{E} = \mathbb{R}^d$ and $\phi^* : \mathbb{R}_+ \times \mathbb{R}^d_+ \times \mathbb{R}^d \to \mathbb{R}^d$ factorizes into $\phi^*(t, s, y) := \varphi^*(t, s)y$, where $\varphi^* : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}_+$, $\varphi^*(t, \cdot)$ is convex, $\forall t \in [t_0, T]$.

(J2) $\mathbb{E} = \mathbb{R}$ and $\phi^* : \mathbb{R}_+ \times \mathbb{R}^d_+ \times \mathbb{R} \to \mathbb{R}^d$ factorizes into $\phi^*(t, s, y) := \varphi^*(t, s)y$, where $\varphi^* : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d_+$, $\varphi^*(t, \cdot)$ is affine-linear, $\forall t \in [t_0, T]$.

(J3) $d = 1$, $\mathbb{E} = \mathbb{R}$, $N \geq 0$ and $\phi^* : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ factorizes into $\phi^*(t, s, y) := \sum_{i \leq m} \varphi^*_i(t, s)\psi^*_i(y)$, where $\varphi^*_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is convex, $\forall t \in [t_0, T], i \leq m$ and $\psi^*_i : \mathbb{R} \to \mathbb{R}_+$ is non-decreasing, $\forall i \leq m$.

Theorem 3.6 (Propagation of convexity, diffusion with jumps case). *Let $g \in \mathcal{F}_{\mathrm{cx}}$, S^* be a d-dimensional diffusion with jumps and assume that the Euler scheme $\tilde{S}^*_K$ of S^* satisfies Assumption AP(g). If $\sigma^*(t, \cdot)$ is convex for all $t \in [0, T]$ and the jump part satisfies one of the conditions (J1)–(J3) then the propagation of convexity property P(g) holds, i.e.*

$$\mathcal{G}(t, \cdot) \in \mathcal{F}_{\mathrm{cx}}, \quad \forall t \in [0, T].$$

Proof. The proof uses similar to that of Theorem 3.2 the Euler approximation scheme $\tilde{S}^*_K$ in 3.21. The main part that needs to be established is the $\leq_{\mathrm{cx}}$-monotonicity of the corresponding Markov operator $\mathcal{T}_{t_i} S \stackrel{d}{=} S + \sigma(t_i, S)W + \phi(t_i, S, Y)N - E^Y \phi(t_i, S, Y)EN$. This is the content of the following lemma. $\square$

Lemma 3.7 (A convex ordering result for Markov operators). *Let S, W, N, Y be independent integrable random variables, where S, W are $\mathbb{R}^d$-valued, $W \sim N(0, I)$ and N has values in $\mathbb{R}$. Assume that $\sigma : \mathbb{R}^d \to M_+(d, \mathbb{R})$ is convex and consider the Markov operator $\mathcal{T} S \stackrel{d}{=} S + \sigma(S)W + \phi(S, Y)N - E^Y \phi(S, Y)EN$.*
Then $\mathcal{T}$ is $\leq_{\mathrm{cx}}$-monotone, if one of the following conditions holds true

1. *Y has values in $\mathbb{R}^d$ and $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ factorizes into $\phi(s, y) = \varphi(s)y$, where $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ is convex.*
2. *Y has values in $\mathbb{R}$ and $\phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ factorizes into $\phi(s, y) = \varphi(s)y$, where $\varphi : \mathbb{R}^d \to \mathbb{R}^d_+$, $\varphi(t, \cdot)$ is affine-linear.*

3. *$d = 1$, Y has values in $\mathbb{R}$, $N \geq 0$ and ϕ factorizes into $\phi(s,y) = \sum_{i\leq m} \varphi_i(s)\psi_i(y)$, where $\varphi_i : \mathbb{R} \to \mathbb{R}_+$ are convex, and $\psi_i : \mathbb{R} \to \mathbb{R}_+$ are non-decreasing, $i \leq m$.*

Proof. 1./2. Assume that S_1, S_2, are d-dimensional random vectors that are independent of W, N and Y and satisfy $S_1 \leq_{\mathrm{cx}} S_2$. Due to Strassen's Theorem we choose without loss of generality S_1, S_2 such that $E(S_2|S_1) = S_1$. Let the situation in 1 or 2 be given. For $f \in \mathcal{F}_{\mathrm{cx}}$ Jensen's inequality implies

$$\begin{aligned} Ef(\mathcal{T}S_2) &= E\left(Ef(S_2 + \sigma(S_2)W + \varphi(S_2)(YN - EYN))|S_1, W, Y, N\right) \\ &\geq Ef(E(S_2|S_1) + E(\sigma(S_2)|S_1)W + E(\varphi(S_2)|S_1)(YN - EYN)) \\ &\geq E^{S_1} Ef(s_1 + E(\sigma(S_2)|s_1)W + E(\varphi(S_2)|s_1)(YN - EYN)). \end{aligned}$$

Due to Lemma 3.1 and stability of the convex order under convolutions and mixtures it suffices to prove that

$$C := E(\varphi(S_2)|s_1)(YN - EYN) \geq_{\mathrm{cx}} \varphi(s_1)(YN - EYN) =: B.$$

1. For convex $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ Jensen's inequality implies that $\vartheta(s_1) := E(\varphi(S_2)|s_1) - \varphi(s_1) \in \mathbb{R}_+$. For $j \leq d$ we define $R_j = \vartheta(s_1)(Y_jN - EY_jN)$. Then it follows from convexity of f that

$$Ef(C) \geq Ef(B) + E\langle \nabla f(B), R\rangle,$$

where $R = (R_1, \ldots, R_d)$, ∇ is the gradient and $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^d$. From $R = \frac{\vartheta(s_1)}{\varphi(s_1)} B$ it follows that $E\langle \nabla f(B), R\rangle = \frac{\vartheta(s_1)}{\varphi(s_1)} E\langle \nabla f(B), B\rangle$. Due to a characterization result of optimal couplings in Rüschendorf and Rachev (1990, Theorem 1.) it follows that $(B, \nabla f(B))$ is an optimal ℓ_2-coupling. This implies $E\langle \nabla f(B), B\rangle \geq \langle E\nabla f(B), EB\rangle = 0$, as $EB = 0$.
2. From affine-linearity of φ it follows that $E(\varphi(S_2)|s_1) = \varphi(s_1)$ and, therefore, $C \overset{d}{=} B$ and thus the ordering conclusion.
3. Let $d = 1$ and $\phi(s,y) = \sum\limits_{i\leq m} \varphi_i(s)\psi_i(y)$. As for $f \in \mathcal{F}_{\mathrm{cx}}$ Jensen's inequality implies

$$Ef(\mathcal{T}S_2) \geq E^{S_1} Ef(s_1 + E(\sigma(S_2)|s_1)W + \sum_{i\leq m} E(\varphi_i(S_2)|s_1)(\psi_i(Y)N - E\psi_i(Y)N)),$$

it suffices to prove that

$$\begin{aligned} C &:= \sum_{i\leq m} E(\varphi_i(S_2)|s_1)(\psi_i(Y)N - E\psi_i(Y)N) \\ &\geq_{\mathrm{cx}} \sum_{i\leq m} \varphi_i(s_1)(\psi_i(Y)N - E\psi_i(Y)N) =: B. \end{aligned}$$

Due to Jensen's inequality $\vartheta_i := E(\varphi_i(S_2)|s_1) - \varphi_i(s_1)$ is non-negative. For $f \in \mathcal{F}_{\mathrm{cx}} \cap C^2$ and $R := \sum_{i\leq m} \vartheta_i(\psi_i(Y)N - E\psi_i(Y)N)$, convexity of f implies

$$Ef(C) \geq Ef(B) + Ef'(B)R.$$

To prove that $Ef'(B)R$ is non-negative, we make use of some results on association of random vectors (cp. Müller and Stoyan (2002, Theorems 3.10.5, 3.10.7.)).

As Y, N are independent random variables, (Y, N) is associated. From monotonicity of $\psi_i \geq 0$ it follows that $\Psi_i(y,n) := \psi_i(y)n$, $n \geq 0$, is non-decreasing in (y,n), $\forall i \leq m$, and, therefore $N \geq 0$ implies that

$$Z := (Z_1, \dots, Z_m) = (\Psi_1(Y,N), \dots, \Psi_m(Y,N))$$

is associated, thus $\bar{Z} = Z - EZ$ is associated (see Müller and Stoyan (2002, Theorem 3.10.7.)). From non-negativity of $g_i(s_1)$ and $\vartheta_i(s_1)$ it follows that $(B,R) = (\sum_{i \leq m} g_i(s_1)\bar{Z}_i, \sum_{i \leq m} \vartheta_i(s_1)\bar{Z}_i)$ is non-decreasing in $\bar{Z}$ and, therefore, is associated. Thus, $EF_1(B,R)F_2(B,R) \geq EF_1(B,R)EF_2(B,R)$ for all non-decreasing $F_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $k = 1,2$. As $f \in \mathcal{F}_{\mathrm{cx}}$, f' is non-decreasing and with $F_1(B,R) := f'(B)$, $F_2(B,R) := R$ it follows that $Ef'(B)R \geq Ef'(B)ER = 0$. $\square$

By the propagation of convexity property, the convex comparison result of Theorems 2.5 and 2.7 can be applied to the case, where the Markovian comparison process S^* is a d-dimensional diffusion with jumps. The characteristics of $X^* = \mathcal{L}og(S^*)$ are of the form

$$c^{*ij}(t, S_t^*(\omega)) = \sum_{k \leq d} \frac{\sigma^{S^*ik}(t, S_t^*(\omega))}{S_t^{*i}(\omega)} \frac{\sigma^{S^*jk}(t, S_t^*(\omega))}{S_t^{*j}(\omega)}.$$

$$K_t^*(s,G) = \int \mathbb{1}_{G \setminus \{0\}} \left(\frac{\phi^*(t,s,y)}{s} \right) \lambda^*(dy), \quad G \in \mathcal{B}^d,$$

as $K_t^{S^*}(s,G) = \int \mathbb{1}_{G\setminus\{0\}}\big(\phi^*(t,s,y)\big)\lambda^*(dy) = \int \mathbb{1}_{G\setminus\{0\}}(z)\lambda^{*\phi^*(t,s,\cdot)}(dz)$ and using Lemma A.2.

We consider the one-dimensional SDE

$$\frac{dS_t^*}{S_t^*} = \sigma^*(t, S_t^*)dW_t^* + \phi^*(t, S_{t-}^*, y)(N^*(dt,dy) - \lambda^*(dy)dt).$$

In this case we have $K_t^*(s,G) = \int \mathbb{1}_{G\setminus\{0\}}(z)\lambda^{*\phi^*(t,s,\cdot)}(dz), G \in \mathcal{B}$, and therefore

$$X^* \sim \left(0, (\sigma^*(t,S_t^*))^2, \lambda^{*\phi^*(t,S_{t-}^*,\cdot)}\right). \tag{3.22}$$

Assume similarly that S is a stochastic volatility model with jumps with evolution

$$\frac{dS_t}{S_t} = \sigma_t dW_t + \phi(t, S_t, y)(N(dt,dy) - \lambda(dy)dt),$$

where σ_t is an adapted process and $\lambda(\mathbb{E}) < \infty$. Then

$$X \sim \left(0, \sigma_t^2, \lambda^{\phi(t,S_{t-},\cdot)}\right), \tag{3.23}$$

and convex comparison of a one-dimensional diffusion with jumps to a stochastic volatility model with jumps reads as follows.

Theorem 3.8 (Convex comparison of diffusions with jumps to SV model with jumps). *Let $g \in \mathcal{F}_{\mathrm{cx}} \cap C^2$. Assume that $S = \mathcal{E}(X), S^* = \mathcal{E}(X^*)$ are one-dimensional processes described by (3.23) and (3.22), respectively. Let S, S^* satisfy Assumption MG and assume that S^* satisfies Assumption SC(g) and that the Euler scheme $\tilde{S}_K^*$ of S^* satisfies Assumption AP(g). Additionally assume that $\sigma^*(t,\cdot) \in \mathcal{F}_{\mathrm{cx}}$, for all $t \in [0,T]$, and the jump part satisfies one of the conditions (J1)–(J3).*

1. *(Upper bound) Let Assumption BIC(g) or CD(g) be satisfied. If for $\lambda\!\!\!\lambda \times Q$-a.e. (t,ω)*

$$\begin{aligned} \sigma_t^2(\omega) \leq &\quad (\sigma^*(t, S_t(\omega)))^2\,, \\ \lambda^{\phi(t,S_{t-}(\omega),\cdot)} \leq_{\mathrm{cx}} &\ \lambda^{*\phi^*(t,S_{t-}(\omega),\cdot)}, \end{aligned} \tag{3.24}$$

then $Eg(S_T) \leq E^* g(S_T^*)$.

2. *(Lower bound) Let Assumption CD(g) be satisfied. If for $\lambda\!\!\!\lambda \times Q$-a.e. (t,ω)*

$$\begin{aligned} (\sigma^*(t, S_t(\omega)))^2 \leq &\quad \sigma_t^2(\omega), \\ \lambda^{*\phi^*(t,S_{t-}(\omega),\cdot)} \leq_{\mathrm{cx}} &\ \lambda^{\phi(t,S_{t-}(\omega),\cdot)}, \end{aligned}$$

then $E^* g(S_T^*) \leq Eg(S_T)$.

Proof. For $f(t,s,\cdot) \in \mathcal{F}_{\mathrm{cx}}$ the ordering in (3.24) implies ordering of the jump measures

$$\begin{aligned} &\int f(t, S_{t-}, y) K_t(dy) = \int f(t, S_{t-}, \phi(t, S_{t-}, y)) \lambda(dy) \\ &\leq \int f(t, S_{t-}, \phi^*(t, S_{t-}, y)) \lambda^*(dy) = \int f(t, S_{t-}, y) K_t^*(S_{t-}, dy). \end{aligned}$$

Due to Theorem 3.6 the propagation of convexity property $\mathrm{P}(g)$ is satisfied, and therefore the result follows from Theorem 2.5 . □

Theorem 3.8 also allows to compare one-dimensional Lévy driven SDEs with pointwise ordered volatilities in the convex sense. We have to pose some additional assumptions on the Lévy measure.

Let $(L_t)_{t\in[0,T]}$ be a Lévy process such that $E|L_t| < \infty, \forall t \in [0,T]$ and assume that L_t has no drift component. Then the Lévy–Itô decomposition of L is of the form

$$L_t = cB_t + \int x N(t, dx) - t \int x \lambda(dx),$$

with $c \in \mathbb{R}^+$, with a Brownian motion B and where $N(dt, dx)$ is a Poisson random measure with deterministic compensator $\lambda(dx)dt$. Assume that S, S^* are solutions of the SDEs

$$\begin{aligned} dS_t &= \sigma(t, S_t) dL_t, \quad S_0 = 1, \\ dS_t^* &= \sigma^*(t, S_t^*) dL_t,\ S_0^* = 1. \end{aligned} \tag{3.25}$$

Corollary 3.9 (Comparison of Lévy driven diffusions). *Let $g \in \mathcal{F}_{\mathrm{cx}}$ and assume that S, S^* are solutions of the Lévy driven SDEs (3.25). Let S, S^* satisfy Assumptions MG and BIC(g) (or CD(g)) and assume that $\lambda(E) < \infty$ and $\int y \lambda(dy) = 0$. Let further S^* satisfy Assumption SC(g) and assume that the Euler scheme $\tilde{S}_K^*$ of S^* satisfies Assumption AP(g).*

If $\sigma^* : [0,T] \times \mathbb{R} \to \mathbb{R}$ is convex in the second component and if for $\lambda\!\!\!\lambda \times Q$-a.e. (ω, t) it holds true that

$$\sigma(t, S_t(\omega)) \leq \sigma^*(t, S_t(\omega)),$$

then $Eg(S_T) \leq E^* g(S_T^*)$.

Proof. The characteristics of the stochastic logarithms of S, S^* are of the form $X \sim \left(0, \sigma(t, S_t), \lambda^{\sigma(t,S_t)\cdot}\right)$ and $X^* \sim \left(0, \sigma^*(t, S_t^*), \lambda^{\sigma^*(t,S_t^*)\cdot}\right)$, respectively. The result follows from an application of Theorem 3.8 with $\phi(t,s,y) := \sigma(t,s)y$ and $\phi^*(t,s,y) := \sigma^*(t,s)y$. □

3.3 Comparison of processes with independent increments

In financial modelling one often uses exponential Lévy models $e^{\bar{X}_t}$, $(\bar{X}_t)$ a Lévy process. In this subsection we consider d-dimensional processes $S = \mathcal{E}(X)$, $S^* = \mathcal{E}(X^*)$ where the stochastic logarithms X, X^* have independent increments (PII) and compare European options with payoffs $g \in \mathcal{F}$, $\mathcal{F} \in \{\mathcal{F}_{\text{cx}}, \mathcal{F}_{\text{dcx}}\}$, for the underlyings S and S^*. We assume that Assumptions MG and SC(g) are satisfied and that Assumptions BIC(g) or CD(g) hold true to insure suitable integrability of the backward linking process $\mathcal{G}(t, S_t)$, depending on which condition is needed in the comparison theorems of section 2. From Lemma A.1 it follows that $\bar{X} = \log \mathcal{E}(X)$, $\bar{X}^* = \log \mathcal{E}(X^*)$ are PII iff X, X^* are PII. In Lemma 3.11 we establish that propagation of (directional) convexity property P(g) holds true for the exponential PII case. Thus, our general comparison results of section 2 apply also to exponential Lévy models.

Propagation of (directional) convexity for the exponential PII case follows from the following representation of the backward functional $\mathcal{G}(t, s)$.

Lemma 3.10 (Representation of the backward functional). *Let $(X_t^*)_{t\in[0,T]}, X_0^* = 0$, be a d-dimensional martingale with independent increments (PII) and $\Delta X^* > -1$. Let $S^* = \mathcal{E}(X^*)$, $t_0 \in [0,T]$ and assume that $g : \mathbb{R}^d \to \mathbb{R}$ is s.th. $\mathcal{G}(t_0, s) < \infty$, $\forall s$. Then there is a PII $(X_t^{*t_0})_{t\in[0,T]}$ with*

$$\mathcal{G}(t_0, s) = E^* g(s\mathcal{E}(X^{*t_0})_T).$$

Proof. Let $\bar{X}^* := \log \mathcal{E}(X^*)$ and define for $t_0 \in [0,T]$ the PII

$$\bar{X}_t^{*t_0} := \begin{cases} 0, & 0 \le t \le t_0 \\ \bar{X}_t^* - \bar{X}_{t_0}^*, & t_0 < t \le T. \end{cases}$$

Due to the one to one relationship between the stochastic and the ordinary exponential there is a PII $(X_t^{*t_0})_{t\in[0,T]}$ such that for $t \in (t_0, T]$

$$\mathcal{E}(X^{*t_0})_t = \exp(\bar{X}_t^{*t_0}) = \exp(\bar{X}_t^* - \bar{X}_{t_0}^*) = S_t^* \frac{1}{S_{t_0}^*},$$

and $\mathcal{E}(X^{*t_0})_t = 1, \forall t \in [0, t_0]$. By the independence of the increments of X^{*t_0} we obtain $E^* g(s\mathcal{E}(X^{*t_0})_T) = E^*(g(S_{t_0}^* \mathcal{E}(X^{*t_0})_T) | S_{t_0}^* = s) = E^*(g(S_T^*) | S_{t_0}^* = s)$. □

In the considered case of exponential PII models, propagation of (directional) convexity is a corollary to the previous representation result.

Lemma 3.11 (Propagation of (directional) convexity). *Let $g \in \mathcal{F}$, $\mathcal{F} \in \{\mathcal{F}_{\text{cx}}, \mathcal{F}_{\text{dcx}}\}$ and assume that X^* is PII with $\Delta X^* > -1$.
Then $S^* = \mathcal{E}(X^*)$ satisfies P(g).*

As a first example we consider the case where S, S^* are stochastic exponentials of compound Poisson processes. For $\lambda, \lambda^* < \infty$ and probability measures R, R^* with mass on $(-1, \infty)^d$ let $X = \mathcal{L}og(S)$ and $X^* = \mathcal{L}og(S^*)$ with

$$X \sim (0, 0, \lambda R), \quad X^* \sim (0, 0, \lambda^* R^*). \tag{3.26}$$

A (directionally) convex comparison of R and R^* implies the corresponding comparison of S_T and S_T^*. For the directionally convex comparison we additionally have to assume that for all $t \in [0,T]$

$$\int\limits_{[0,t]} \int\limits_{(-1,\infty)^d} |\Lambda \mathcal{G}(u, S_{u-}, x)| \lambda^* R^*(dx) du \in \mathcal{A}_{\text{loc}}^+, \tag{3.27}$$

for the Kolmogorov backward equation to hold (see Lemma 2.1).

Theorem 3.12 ((Directionally) convex comparison of stochastic exponentials of compound Poisson processes). *Let $\mathcal{F}_0 \subset \mathcal{F} \cap C^2$, $\mathcal{F} \in \{\mathcal{F}_{\text{cx}}, \mathcal{F}_{\text{dcx}}\}$, be a generating class of the (directionally) convex order. Assume that $S = \mathcal{E}(X)$, $S^* = \mathcal{E}(X^*)$ are stochastic exponentials of compound Poisson processes X, X^* with characteristics (3.26) that satisfy Assumption MG. Let S^* satisfy Assumption SC($\mathcal{F}_0$) and for $\mathcal{F} = \mathcal{F}_{\text{dcx}}$ additionally assume that (3.27) holds true for all $g \in \mathcal{F}_0$.*

1. *(Upper bound) Let S, S^* satisfy Assumption BIC($\mathcal{F}_0$).*
   (a) *If $\lambda = \lambda^*$ and $R \leq_{\mathcal{F}} R^*$, then $S_T \leq_{\mathcal{F}} S_T^*$.*
   (b) *If $\lambda \leq \lambda^*$ and $R = R^*$ with $ER = 0$, then $S_T \leq_{\text{cx}} S_T^*$.*
2. *(Lower bound) Let S, S^* satisfy Assumption CD($\mathcal{F}_0$).*
   (a) *If $\lambda = \lambda^*$ and $R^* \leq_{\mathcal{F}} R$, then $S_T^* \leq_{\mathcal{F}} S_T$.*
   (b) *If $\lambda^* \leq \lambda$ and $R = R^*$ with $ER = 0$, then $S_T^* \leq_{\text{cx}} S_T$.*

Proof. The proof is an easy application of the comparison Theorems 2.2–2.7. Parts (1b) and (2b) follow from the fact that $\lambda \leq \lambda^*$ and $ER = 0$ imply $\lambda R \leq_{\text{cx}} \lambda^* R$. □

Remark 3.13. If $\lambda \leq \lambda^*, R \leq_{\text{cx}} R^*$ and $ER = 0$, then Theorem 3.12 implies $S_T \leq_{\text{cx}} S_T^*$. An analogue result is true for the lower bound.

Theorem 3.12 implies an ordering result for Lévy processes with infinite Lévy measures by approximation. For an infinite Lévy measure F we denote by F_n the truncated measure $F_n(dx) := \mathbb{1}_{\{|x|>\varepsilon_n\}} F(dx)$, $\varepsilon_n > 0$, which is a finite Lévy measure.

Corollary 3.14 (Comparison of stochastic exponentials of pure jump Lévy processes with infinite Lévy measures). *Let $\mathcal{F}_0 \subset \mathcal{F} \cap C^2$, $\mathcal{F} \in \{\mathcal{F}_{\text{cx}}, \mathcal{F}_{\text{dcx}}\}$, be a generating class of the (directionally) convex order. For Lévy measures F, F^* with infinite total mass let $X \sim (0,0,F)$, $X^* \sim (0,0,F^*)$ and denote the truncated versions by F_n, F_n^*. Let $S_n = \mathcal{E}(X_n)$, $S_n^* = \mathcal{E}(X_n^*)$ satisfy Assumption MG, assume that S^* satisfies Assumption SC($\mathcal{F}_0$) and for $\mathcal{F} = \mathcal{F}_{\text{dcx}}$ additionally assume that (3.27) holds true for all $g \in \mathcal{F}_0$. If $\varepsilon_n, \varepsilon_n^* \downarrow 0$ are such that $\|F_n\| = \|F_n^*\| < \infty$ and $F_n \leq_{\mathcal{F}} F_n^*$, then $S_T \leq_{\mathcal{F}} S_T^*$.*

Ordering results for (stochastic) exponentials of Lévy processes with infinite Lévy measures are also obtained by combining one of the general comparison results of Theorems 2.2–2.7 with the propagation of (directional) convexity property for stochastic exponentials of PII from Lemma 3.11 (and Lemma A.1 in the ordinary exponential case). As example we consider multivariate exponential generalized hyperbolic models (see Barndorff-Nielsen (1977)). v. Hammerstein (2003) derived the Lévy densities of generalized hyperbolic distributions. In the special cases of normal inverse Gaussian (NIG $=\mathrm{NIG}_d(\alpha,\beta,\delta,\mu,\Delta)$) and variance gamma (VG $=\mathrm{VG}_d(\lambda,\alpha,\beta,\mu,\Delta)$) distributions they are given by

$$\ell_{\mathrm{NIG}}(x) = \frac{2\alpha^{d/2}e^{x\beta^T}}{\left(2\pi\sqrt{x(\Delta^{-1}x)^T}\right)^{d/2}}\frac{\delta\sqrt{\alpha}}{\sqrt{2\pi}\,(x(\Delta^{-1}x)^T)^{1/4}}K_{\frac{d+1}{2}}\left(\alpha\sqrt{x(\Delta^{-1}x)}\right),$$
$$\ell_{\mathrm{VG}}(x) = \frac{2\lambda\alpha^{d/2}e^{x\beta^T}}{\left(2\pi\sqrt{x(\Delta^{-1}x)^T}\right)^{d/2}}K_{\frac{d}{2}}(\alpha\sqrt{x(\Delta^{-1}x)}).$$

As in the case of NIG the scaling parameter δ appears multiplicatively in the Lévy density, comparison of two Lévy densities with scaling parameters $\bar{\delta}\leq\bar{\delta}^*$ implies convex ordering of the corresponding exponential Lévy models $S = e^{\bar{X}} \leq_{\mathrm{cx}} S^* = e^{\bar{X}^*}$. An analogue ordering result for VG models in the parameter λ holds true.

A Appendix

In Lemma A.1 the relationship between stochastic and ordinary exponential in terms of characteristics is established. This is a multivariate extension of Goll and Kallsen (2000, Lemma A.8) and Jacod and Shiryaev (2003, Theorem II.8.10). For a truncation function $h:\mathbb{R}^d\to\mathbb{R}^d$ we denote by $X\sim(B,C,\nu)_h$ that X has drift characteristic $B=B(h)$, that depends on the truncation function h, Gaussian characteristic C and jump characteristic ν (cp. Jacod and Shiryaev (2003, Definition II.2.6). Lemma A.1 in particular implies that $X=\mathcal{L}og(S)$ is PII iff $\bar{X}=\log(S)$ is PII, as the characteristics of X are deterministic iff the characteristics of $\bar{X}$ are deterministic.

Lemma A.1. *1. Let $X\sim(B,C,\nu)_h$ be a d-dimensional semimartingale and $\bar{X}:=\log\mathcal{E}(X)$. Then the characteristics $(\bar{B},\bar{C},\bar{\nu})_{\bar{h}}$ of $\bar{X}$ are given by*

$$\begin{aligned}\bar{B} &= B-\frac{\mathrm{diag}(C)}{2}+(\bar{h}(\log(1+x))-h(x))*\nu,\\ \bar{C} &= C,\\ \bar{\nu}([0,t]\times G) &= \int\limits_{[0,t]\times(-1,\infty)^d}\mathbb{1}_G(\log(x+1))\nu(du,dx),\quad G\in\mathcal{B}^d,\end{aligned}$$

where $\mathrm{diag}(C):=(C^{11},\dots,C^{dd})$ is the diagonal of the matrix C.

2. *Let $\bar{X} \sim (\bar{B}, \bar{C}, \bar{\nu})_{\bar{h}}$ be a d-dimensional semimartingale and $X := \mathcal{L}og(e^{\bar{X}})$. Then the characteristics $(B, C, \nu)_h$ of X are given by*

$$\begin{aligned}
B &= \bar{B} + \frac{\operatorname{diag}(\bar{C})}{2} + (h(e^x - 1) - \bar{h}(x)) * \bar{\nu}, \\
C &= \bar{C}, \\
\nu([0,t] \times G) &= \int\limits_{[0,t]\times\mathbb{R}^d} \mathbb{1}_G(e^x - 1)\bar{\nu}(du, dx), \quad G \in \mathcal{B}((-1,\infty)^d).
\end{aligned}$$

Proof. As the multivariate (stochastic) exponential (resp. logarithm) is defined componentwise, the arguments of the proof of the univariate case apply to every component of the multivariate characteristics and the result follows from

$$\bar{X}^i = X^i - X_0^i - \frac{1}{2}\langle X^{c,i}, X^{c,i}\rangle + (\log(1 + x^i) - x^i) * \mu^{X^i}.$$

The appearance of the diagonal of C in the drift part $\bar{B}$ is due to that representation. The proof of part 2 is similar to that of the first part. $\square$

Lemma A.2. *1. Let $X \sim (B, C, \nu)_h$ be a d-dimensional semimartingale with $X_0 = 0$ and $\Delta X > -1$. Then $S := \mathcal{E}(X) \sim (B^S, C^S, \nu^S)_{h^S}$ is a positive d-dimensional semimartingale with characteristics*

$$\begin{aligned}
B_t^{S,i} &= \int\limits_{[0,t]} S_{u-}^i dB_u^i + \int\limits_{[0,t]\times(-1,\infty)^d} (h^{S,i}(S_{u-}x) - S_{u-}^i h(x^i))\nu(du, dx), \\
C_t^{S,ij} &= \int\limits_{[0,t]} S_{u-}^i S_{u-}^j dC_u^{ij}, \\
\nu^S([0,t] \times G) &= \int\limits_{[0,t]\times(-1,\infty)^d} \mathbb{1}_G(S_{u-}x)\nu(du, dx), \quad G \in \mathcal{B}^d.
\end{aligned}$$

2. *Let $S \sim (B^S, C^S, \nu^S)_{h^S}$ be a d-dimensional semimartingale with $S_0 = 1$ and $S, S_- > 0$. Then $X := \mathcal{L}og S \sim (B, C, \nu)_h$ is a d-dimensional semimartingale with $\Delta X > -1$ and*

$$\begin{aligned}
B_t^i &= \int\limits_{[0,t]} \frac{1}{S_{u-}^i} dB_u^{S,i} + \int\limits_{[0,t]\times\mathbb{R}^d} \left\{ h^i\left(\frac{s}{S_{u-}}\right) - \frac{1}{S_{u-}^i} h^{S,i}(s) \right\} \nu^S(du, ds), \\
C_t^{ij} &= \int\limits_{[0,t]} \frac{1}{S_{u-}^i} \frac{1}{S_{u-}^j} dC_u^{S,ij}, \\
\nu([0,t] \times G) &= \int\limits_{[0,t]\times\mathbb{R}^d} \mathbb{1}_G\left(\frac{s}{S_{u-}}\right) \nu^S(du, ds), \quad G \in \mathcal{B}^d.
\end{aligned}$$

Proof. 1. This is proved in Goll and Kallsen (2000, Example 4.3) in the case where X is a d-dimensional Lévy process. The proof for semimartingales is similar.

2. As $S = \mathcal{E}(X)$ is defined componentwise, X is given as $X^i = \frac{1}{S^i_-} \cdot S^i, i \leq d$. For the Gaussian characteristic it follows that $C^{ij} = \langle X^{i,c}, X^{j,c} \rangle = \frac{1}{S^i_-}\frac{1}{S^j_-} \cdot C^{S,ij}$. To compute the jump compensator let $f : (0,\infty)^d \to \mathbb{R}^d$ be defined by $f(s) = (\log(s^1), \dots, \log(s^d))^T$. Itô's Lemma for characteristics (see e.g. Goll and Kallsen (2000, corollary A.6)) implies $\bar{\nu}([0,t] \times G) = \int\limits_{[0,t]\times\mathbb{R}^d} \mathbb{1}_G \left(\log\left(\frac{S_{u-}+s}{S_{u-}}\right)\right) \nu^S(du, ds)$. From part 2 of Lemma A.1 we obtain $\nu([0,t] \times G) = \int\limits_{[0,t]\times\mathbb{R}^d} \mathbb{1}_G \left(\frac{s}{S_{u-}}\right) \nu^S(du, ds)$.

It remains to compute B_t. In the same manner as for the jump compensator we use Itô's Lemma to obtain for $G \in \mathcal{B}^d$ and $i \leq d$

$$\begin{aligned}\bar{B}^i_t = &\int\limits_{[0,t]} \frac{1}{S^i_{u-}} dB^{S,i}_u - \frac{1}{2} \int\limits_{[0,t]} \left(\frac{1}{S^i_{u-}}\right)^2 dC^{S,ii}_u \\ &+ \int\limits_{[0,t]\times\mathbb{R}^d} \left\{\bar{h}^i(\log(S_{u-}+s) - \log(S_{u-})) - \frac{1}{S^i_{u-}} h^{S,i}(s)\right\} \nu^S(du, ds).\end{aligned}$$

Applying part 2 of Lemma A.1 this yields after a bit of calculus

$$B^i_t = \int\limits_{[0,t]} \frac{1}{S^i_{u-}} dB^{S,i}_u + \int\limits_{[0,t]\times\mathbb{R}^d} \left\{h^i\left(\frac{s}{S_{u-}}\right) - \frac{1}{S^i_{u-}} h^{S,i}(s)\right\} \nu^S(du, ds). \qquad \square$$

Proof of Lemma 2.1 1. Assume that $|W^*| * \mu^{S^*} \in \mathcal{A}^+_{\text{loc}}$ and let $t \in (0, T]$. Similar to the proof of Theorem 2.2 Itô's lemma and Lemma A.2 imply that $\mathcal{G}(t, S^*_t)$ is a semimartingale with evolution $\mathcal{G}(t, S^*_t) = \mathcal{G}(0, 1) + M^*_t + M^{**}_t + A^*_t$, where M^*, M^{**} are local $(\mathcal{A}_t)$-martingales under Q^* and

$$\begin{aligned}A^*_t := &\int\limits_{[0,t]} \Big\{\mathrm{D}_t\mathcal{G}(u, S^*_{u-}) + \frac{1}{2}\sum_{i,j\leq d} \mathrm{D}^2_{ij}\mathcal{G}(u, S^*_{u-}) S^{*i}_{u-} S^{*j}_{u-} c^{*ij}(u, S^*_{u-}) \\ &+ \int\limits_{(-1,\infty)^d} (\Lambda\mathcal{G})(u, S_{u-}, x) K^*_u(S^*_{u-}, dx)\Big\} du.\end{aligned}$$

is predictable and of finite variation. As $\mathcal{G}(t, S^*_t)$ a local $(\mathcal{A}_t)$-martingale it follows by the uniqueness of the representation of a special semimartingale, that A^*_t is a predictable local martingale with finite variation starting at zero and therefore is zero.

2. We show that $\mathcal{G}(t, \cdot) \in \mathcal{F}_{\text{cx}}$ implies the integrability condition $|W^*| * \mu^{S^*} \in \mathcal{A}^+_{\text{loc}}$. Then the result follows from the first part. As $\mathcal{G}(t, S^*_t)$ is a local martingale under Q^*, it is a special semimartingale. The process

$$\int\limits_{[0,t]} \mathrm{D}_t\mathcal{G}(u, S^*_{u-}) du + \frac{1}{2}\sum_{i,j\leq d} \int\limits_{[0,t]} \mathrm{D}^2_{ij}\mathcal{G}(u, S^*_{u-}) dC^{S^*_{ij}}_u$$

is of finite variation and predictable and therefore is in $\mathcal{A}_{\text{loc}}$. Due to a representation result for special semimartingales (see Jacod and Shiryaev (2003, Proposition I.4.23)) it follows that $W^* * \mu^{S^*} \in \mathcal{A}_{\mathrm{loc}}$. Convexity of $\mathcal{G}(t,\cdot)$ implies $W^*(\omega^*, t, s) \geq 0$. Therefore, $|W^*| * \mu^{S^*} = W^* * \mu^{S^*} \in \mathcal{A}^+_{\mathrm{loc}}$. □

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