LIMITING DISTRIBUTION OF THE COLLISION RESOLUTION INTERVAL

P. Feldman* S. T. Rachev† L. Rüschendorf‡

Abstract

We use the theory of probability metrics to study the asymptotic normality of the collision resolution intervals in the CTM multi-access protocol under general conditions on the number of retransmitted messages and of new arrivals during the collision slots. Our main result establishes stability of the central limit theorem for the CTM algorithm. We provide extensive simulation results investigating the extent to which the mean of the collision resolution interval eventually becomes unstable for increasing values of n, the number of users who initially collide. The normal fit is numerically investigated and is shown to be quite satisfactory and stable with respect to moderate perturbations and $n \geq 50$.

1 Introduction

The Capetanakis-Tsybakov-Mikhailov (CTM) protocol is one of the more elegant solutions to the classical multiple-access problem in which a large (actually, infinite) population of users share a single communication channel. Throughput of this protocol is close to the throughput of the slotted Aloha protocol, and the CTM protocol, unlike slotted Aloha, is inherently stable. The "tree splitting protocols", of which the CTM protocol is an example, pose some interesting mathematical problems, and have been the subject of intensive study in recent years (see Aldous (1987), Aldous (1991), Bertsekas (1987), Borovkov (1988), Fayolle (1986), Fayolle (1985)).

We now briefly review the definition of the CTM protocol, see Bertsekas and Gallager (1987). Time is divided into slots of equal duration. During each slot, one of the following events occurs:

*Illgen Simulation Technology, 250 Storke Road, Suite 10, Santa Barbara, CA 93117, and Department of Computer Science, University of California, Santa Barbara, CA 93106, USA

†Department of Statistics and Applied Probability, University of California, Santa Barbara, CA 93106, USA

‡Institut für Mathematische Stochastik, Hebelstr. 27, 79104 Freiburg, Germany

Research supported by NATO, DFG and NSF Grants.

AMS 1991 subject classification: Primary 60 K 30, 90 B 12, 60 F 05, Secondary 60 B 10, 60 K 99

Key words and phrases: Multi-access problem, probability metrics, asymptotic normality

1. The slot is wasted because no one transmits.

2. Exactly one user transmits a message, in which case the message is successfully received.

3. The slot is wasted because two or more users transmit, interfering with each other. This is called a collision.

At the end of each slot, every user knows which of these three events occured (this is sometimes called "trinary feedback").

When a collision occurs, all users involved (those which transmitted during the slot) divide themselves into two groups on a random basis. Each user performs the equivalent of an independent coin toss in order to make its decision; p is the probability that a user selects the first group. Users in the first group re-transmit their messages during the slot following the one in which the collision occured; users in the second group defer their re-transmissions until all users in the first group have successfully transmitted their messages. If one of these groups contains more than one user, another collision will occur, in which case this group divides in the same way. Collisions are resolved on a LCFS basis, i.e. the most recent collision is resolved before any prior collisions.

Assume that new messages are generated according to a Poisson process with aggregate rate λ. Actually, users who have transmitted a message which collided do not generate any new messages until their message has been transmitted; however, since only a finite number of users are involved in any collision, the rate λ remains constant when the total user population is infinite.

Let L_n be the number of slots required for resolution of a collision between n users. L_n includes the slot in which the initial collision occured, plus the times for the two groups of users to transmit their messages. It is easily seen that

$$L_n \stackrel{d}{=} 1 + L_{I_n+X} + \tilde{L}_{n-I_n+Y}, \ n \geq 2 \tag{1}$$

with initial conditions $L_0 = L_1 = 1$, where $I_n \stackrel{d}{=} Bi(n,p)$ is the number of users who retransmit immediately, X is the number of new arrivals in the collision slot, and Y is the number of new arrivals during the slot in which the deferred retransmissions occur. $L_n \stackrel{d}{=} \tilde{L}_n$, and $X, Y, (L_n)_{n\geq 0}, \ (\tilde{L}_n)_{n\geq 0}$ are assumed to be mutually independent. For practical systems, the total number of users sharing a multiple-access channel might conceivably be as large as 10^3 or 10^4 but the number n of users involved in any collision would be at most a small fraction of this. Fayolle, et al. [6], showed that $\lim_{n\to\infty} EL_n/n$ exists only if

$$\log p / \log(1-p) \text{ is irrational;} \tag{2}$$

otherwise EL_n/n oscillates around a certain value. The non-linearity of L_n for $n \to \infty$ was pointed out in several of the above mentioned works. In a subsequent paper, Fayolle et al. (1986) proved the linearity of the variance of L_n under (2) and the finiteness of all moments of L_n. Confirming a conjecture of Massey (1981), Regnier and Jacquet (1989) proved that the variance of L_n is not linear for

$I_n \stackrel{d}{=} Bi(n,p),\ p = 1/2$, and $X = Y = 0$. In Jacquet/Regnier (1988) and Regnier/Jacquet (1989) the asymptotic normality of the standardized sequence $\{L_n\}$ (for $X = Y = 0$ or both Poisson) was established.

In this paper we examine the asymptotic normality of the law of L_n without the specific assumptions on the distribution type of I_n, X and Y provided that the variance of L_n is asymptotically linear. The proof is based on contraction properties of certain ideal probability metrics (see Rachev (1991), Rachev/Rüschendorf (1991), Rachev/Rüschendorf (1991)). In the second part of the paper we numerically investigate the influence of non-linearity in the case
$I_n \stackrel{d}{=} Bi(n,p),\ X = Y = 0$ and $p = \frac{1}{2}$. It turns out that $E\,L_n/n$, (Var $L_n)/n$ increase monotonically with n until n reaches a largest value ($n = 39,488$). After that the linearity breaks down in agreement with the theoretical results in Fayolle/Flajolet/Hofri/Jacquet (1985), Massey (1981), Regnier/Jacquet (1989). From the practical point of view it is of interest to consider the simple normalization

$$(3) \qquad Y_n = (L_n - \ell_n)/\sqrt{n}, \ \text{ with } \ \ell_n = E\,L_n$$

(corresponding to the normal domain of attraction in the CLT). A simulation study of the empirical d.f. of Y_n seems to confirm the normality for $10^2 \le n \le 10^4$. The "instability" of $E\,L_n/n$ and Var L_n/N and hence the non-normality of Y_n's law arise only in the regime of $n \gg 10^4$. But the order of magnitude of the instability is seen from our numerical results and simulation study to be extremely small (but existent in accordance with the theoretical results) and can be neglected from the practical point of view. This has the valuable consequence that in practical applications one can use just simple linear normalizations as in (3) and the normal approximation also for n extremely large.

Our main theoretical result indicates that normality holds if the variance behaves linearly and the number of retransmittances are not concentrated too much in the extremes. In this sense our result can be considered as a stability result for the asymptotic distribution. This idea of stability is confirmed by simulations for some cases of immigration in section 3. In our numerical study we detect the theoretically predicted instability but only for extremely large n and with a practically neglegable order of magnitude. Our simulation study confirms the stability in the standard model concerning dependence on p.

2 Asymptotic Normality of the Law of L_n

In this section we prove the asymptotic normality of Y_n (see (3)) under the following three conditions: for some $r \in (2,3]$,

(a) $E\,X^{r/2} + E\,Y^{r/2} < \infty$ and $\frac{I_n}{n} \xrightarrow{L^r} p \in (0,1)$;

(b) $\sigma_n^2 =$(Var $L_n)/n \to \sigma^2$; and

(c) $\sup_n E|Y_n|^r < \infty$ for some $r \in (2,3]$.

Note that the number of retransmitting users I_n is not necessarily binomial in our assumptions. This allows e.g. to consider departures from independence in the protocol.

Regnier and Jacquet (1989) showed that (a), (b), and (c) hold for $I_n \stackrel{d}{=} Bi(n,p)$, (2) and $X = Y = 0$. More generally (see Fayolle/Flajolet/Hofri/Jacquet (1985), Jacquet/Regnier (1988)) one can allow $X \stackrel{d}{=} Y \stackrel{d}{=} \mathrm{Pois}(\lambda)$.

Theorem 1 *Under (a), (b), and (c) the distribution of Y_n is asymptotically $N(0,\sigma^2)$.*

Proof: From the definitions of L_n, Y_n, and (1), (3)

$$(4) \qquad Y_n \stackrel{d}{=} (\frac{I_n + X}{n})^{1/2}\, Y_{I_n+X} + (\frac{n - I_n + Y}{n})^{1/2}\, \tilde{Y}_{n-I_n+Y} + C_n(I_n, X, Y),$$

where $\tilde{Y}_n$ is an independent copy of Y_n and

$$C_n(k,m,\tilde{m}) := n^{-1/2}(1 + l_{k+m} + l_{n-k+\tilde{m}} - l_n).$$

Define a sequence of normal $N(0,\sigma_n^2)$-distributed independent r.v.'s Z_n, which are independent of (I_n), X and Y, and let

$$Z_n^* = (\frac{I_n + X}{n})^{1/2}\, Z_{I_n+X} + (\frac{n - I_n + Y}{n})^{1/2}\, \tilde{Z}_{n-I_n+Y} + C_n(I_n, X, Y),$$

where $\tilde{Z}_n$ is an independent version of Z_n.

Let μ_r be one of the following ideal metrics of order $r > 0$:

$$\begin{aligned}
\mu_r^{(1)}(X,Y) &= \sup\{|E(f(X) - f(Y))| : \|f^{(s)}\|_{\tilde{q}} \le 1\} \\
r &= s + 1/\tilde{p},\ s \in \mathbb{N},\ \tilde{p} \in [1,\infty],\ \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1, \\
\mu_r^{(2)}(X,Y) &= \sup_{t\in\mathbb{R}} |t|^{-r} |E\, e^{itX} - E\, e^{itY}|, \quad \text{and} \\
\mu_r^{(3)}(X,Y) &= \sup_{h\in\mathbb{R}} |h|^r \sup_{A\in\mathcal{B}(\mathbb{R})} |P(X + hN \in A) - P(Y + hN \in A)|,
\end{aligned}$$

where N is a standard normal r.v. independent of X and Y, see Aldous (1987), Rachev (1991) (Sect. 14.2) and Rachev/Rüschendorf (1991).

<u>Claim 1.</u> *(μ_r-closeness of Z_n^* and Y_n). Set $a_n = \mu_r(Z_n, Y_n)$ and suppose*

$$(5) \qquad a := \sup_n a_n < \infty.$$

Then $\sup_n \mu_r^{(i)}(Z_n^*, Y_n) \le a[p^{r/2} + (1-p)^{r/2}]$.

For $\mu_r = \mu_r^{(i)}$ $(i = 1,2,3)$,

$$\mu_r(Z_n^*, Y_n) \ \le \ \sum_{k,m,\tilde{m}} P(I_n = k,\ X = m,\ Y = \tilde{m})$$

$$\begin{aligned}
&\mu_r(\sqrt{\frac{k+m}{n}}\, Z_{k+m} + \sqrt{\frac{n-k+\tilde m}{n}}\, \tilde Z_{n-k+\tilde m} + c_n(k,m,\tilde m),\\
&\sqrt{\frac{k+m}{n}}\, Y_{k+m} + \sqrt{\frac{n-k+\tilde m}{n}}\, \tilde Y_{n-k+\tilde m} + c_n(k,m,\tilde m))\\
\le\ & \sum_{k,m,\tilde m} P(I_n = k,\, X = m,\, Y = \tilde m) a[(\frac{k+m}{n})^{r/2} + (\frac{n-k+\tilde m}{n})^{r/2}]\\
=\ & a\, E[(\frac{I_n + X}{n})^{r/2} + (\frac{n - I_n + Y}{n})^{r/2}].
\end{aligned}$$

Using assumption (a), the RHS of the above inequality converges to $a[p^{r/2} + (1-p)^{r/2}]$.

Claim 2. *(Condition (5) holds.)*

$$a \le C \sup_n (E|Y_n|^r + E|Z_n|^r) < \infty. \tag{6}$$

(Throughout the paper, C stands for an absolute constant.)
For $i = 1, 2$, or 3

$$\mu_r^{(i)}(X, Y) \le C(E|X|^r + E|Y|^r) < \infty$$

provided that $E(X^j - Y^j) = 0$ for $j = 1, 2$, (see, for example, Rachev (1991), chapters 14, 15) and thus (6) holds.

Claim 3. *(Asymptotic normality of Z_n^*.) For $n \to \infty$, $b_n = \mu_r(Z_n, Z_n^*) \to 0$.*

We consider the case $\mu_r = \mu_r^{(1)}$ only. Let κ_r be the r-th pseudomoment,

$$\kappa_r(X, Y) = r \int_{\mathbb{R}} |x|^{r-1} |F_X(x) - F_Y(x)| dx.$$

Then since the mean and variance of Z_n coincide with those of Z_n^* ($\mu_r(Z_n^*, Y_n) < \infty$ implies $E((Z_n^*)^j - Y_n^j) = 0,\ j = 1, 2$) it follows that $b_n \le C\, \kappa_r(Z_n, Z_n^*)$.

Recall that $(Z_n)_{n\ge1}$ is independent of $(I_n)_{n\ge1}$ and X, Y. Let N_o denote a standard normal r.v. independent of (I_n) and X, Y. Consequently,

$$\begin{aligned}
Z_n^* &= \sqrt{\frac{I_n + X}{n}}\, Z_{I_n+X} + \sqrt{\frac{n - I_n + Y}{n}}\, \tilde Z_{n-I_n+Y} + C_n(I_n, X, Y)\\
&\stackrel{d}{=} (\frac{I_n + X}{n} \sigma^2_{I_n+X} + \frac{n - I_n + Y}{n} \sigma^2_{n-I_n+Y})^{1/2}\, N_o + C_n(I_n, X, Y)\\
&=: \eta_n\, N_o + C_n(I_n, X, Y).
\end{aligned}$$

From assumptions (a), (b) we get the convergence of η_n in probability:

$$\eta_n \xrightarrow{P} (p\,\sigma^2 + (1-p)\sigma^2)^{1/2} = \sigma.$$

Since $Z_n^* = \eta_n\, N_o + C_n(I_n, X, Y)$ has the same mean and variance as $Z_n \stackrel{d}{=} \sigma_n\, N_o$,

$$\begin{aligned}
\sigma_n^2 &= E(\eta_n\, N_o + C_n(I_n, X, Y))^2\\
&= E\,\eta_n^2 + E(C_n(I_n, X, Y))^2.
\end{aligned}$$

As $\eta_n \xrightarrow{L^2} \sigma$ we conclude that $C_n(I_n, X, Y) \xrightarrow{P} 0$. This implies that $b_n = \mu_r(Z_n, Z_n^*) \to 0$.

With $a_n = \mu_r(Z_n, Y_n) \le \mu_r(Z_n^*, Y_n) + b_n$ and $\bar{a} = \overline{\lim}\, a_n$ we finally obtain from claims 1 - 3

Claim 4. $\bar{a} = 0$.

To prove the claim choose $n_o = n_o(\varepsilon)$ $(\varepsilon > 0)$, so that $a_k \le \bar{a} + \varepsilon$ for $k > n_o$. Then for $n \ge n_o$ as in the proof of claim 1,

$$\begin{aligned} a_n \;\le\; & \mu_r(Z_n^*, Y_n) + b_n \le \Big(\sum_{k=0}^{n_o-1} + \sum_{k=n-n_o}^{n}\Big) P(I_n = k) \sup_{\substack{0 \le k \le n_o - 1,\\ n-n_o \le k \le n}} (a_{k+X} + a_{n-k+Y}) \\ & \times E\Big[\Big(\frac{k+X}{n}\Big)^{r/2} + \Big(\frac{n-k+Y}{n}\Big)^{r/2}\Big] \\ & + \sum_{k=n_o}^{n-n_o-1} P(I_n = k) E\Big[\Big(\frac{k+X}{n}\Big)^{r/2}(\bar{a}+\varepsilon) + \Big(\frac{n-k+Y}{n}\Big)^{r/2}(\bar{a}+\varepsilon)\Big] + b_n. \end{aligned}$$

Recall Claim 2, $a = \sup_n a_n < \infty$, and thus, as $n \to \infty$,

$$\begin{aligned} \bar{a} \;\le\; & \limsup_n \Big(\sum_{k=0}^{n_o-1} + \sum_{k=n-n_o}^{n}\Big) P(I_n = k) 2a\, E(X^{r/2} + Y^{r/2}) \\ + \; & (\bar{a}+\varepsilon)(p^{r/2} + (1-p)^{r/2}) + \limsup b_n \\ = \; & 0 + (\bar{a}+\varepsilon)(p^{r/2} + (1-p)^{r/2}) + 0. \end{aligned}$$

Since $r > 2$, $p^{r/2} + (1-p)^{r/2} < 1$ which implies that $\bar{a} = 0$ and thus the proof of the theorem is complete since μ_r convergence implies weak convergence. $\square$

Remarks:

a) Theorem 1 shows a remarkable stability of the central limit theorem for L_n. It says that the central limit theorem can be expected if the variance behaves approximatively linear and even true under protocols which are not based on a binomial number of retransmitting users. It is clear from Fayolle/Flajolet/Hofri/Jacquet (1985), Jacquet/Regnier (1988), Regnier/Jacquet (1989) that in examples it is not easy to obtain the asymptotic behaviour of the first moments. Our method of proof separates this problem and establishes a general structural stability property concerning the asymptotic distribution. This should be of some interest for the application of the algorithm, too. This stability is not clear or expected from the methods which established the central limit theorem up to now in some very special cases. The progress in this paper is achieved by the use of ideal probability metrics which reflect the structure of the algorithm. For some related examples of this method cf. Rachev and Rüschendorf (1991).

b) In a subsequent paper (to be published elsewhere) we find that the linearity of Var (L_n) can be weakened to the condition that Var $(L_n) = nG(n)$, where $G(n)$

is regularly varying of order 1, i.e. $G(tn)/G(n) \longrightarrow 1$ for all $t > 0$ as $n \to \infty$ and bounded away from zero and infinity (which is of interest for the case that $\log p/\log(1-p)$ is rational). If the random retransmittance distribution leads to extremely small or large groups, then different normalizations of Var (L_n) are necessary and nonnormal limiting distributions arise. More precisely we prove that $Y_n = \dfrac{L_n - l_n}{\lambda_n} \xrightarrow{w} Z_{\lambda,\alpha}$ a symmetric α-stable r.v. with characteristic function $E\exp\{itZ_{\lambda,\alpha}\} = \exp\{-(\lambda t)^\alpha\}$ with scaling factor $\lambda > 0$ and index of stability $\alpha \in (0, 2]$ if

a') <u>(group size condition)</u> $\dfrac{I_n}{n} \xrightarrow{L^q} p, 0 < p < 1$ for $q = \dfrac{r}{\alpha}$ and $E(X^q + Y^q) < \infty$.

b') <u>(variability condition)</u> $\lambda_n^\alpha = nG(n)$ for some slowly varying function G bounded away from 0 and ∞.

c') <u>(tail condition)</u> $E(Y_n^j - Z_{\lambda,\alpha}^j) = 0, j = 1, ..., s$ and $\sup_n \kappa_r(X_n, Z_{\lambda,\alpha}) < \infty$ where $\kappa_r(X, Y) = \int |x|^{r-1}|F_X(x) - F_Y(x)|dx$ is the r-th difference pseudomoment, $r > \max\{1, \alpha\}$.

In the normal case $\alpha = 2$, a') is identical to a), b') is equivalent to $\sigma_n^2 = nG(n)$ and c') is equivalent to c). So the tail and variability conditions in a'), c') essentially determine the limiting index of stability and we find that the behaviour of the approximative linearity of the variance in this paper is essentially necessary for asymptotic normality.

□

3 Numerical Results

In the first part of this section we study numerically the extent of non-linearity of EL_n, Var L_n in the special case of (1.1) where $X = Y = 0, I_n \stackrel{d}{=} Bi(n, p), \log p/\log(1-p)$ rational. Up to now there is theoretical but no numerical evidence for the non-linearity and its magnitude in the literature. Consequently we check the influence of this non-linearity on the asymptotic "normality" of Y_n – the normalized version of L_n (cf. (3)) – which is motivated from the practical point of view, since it is much simpler to use the linear normalization than the exact ones. Finally, we investigate stability of the limiting distribution w.r.t. p and present some simulations in the case where X and Y are not zero.

3.1 Computation of the Moments

From (1) we readily obtain the following deterministic recursions:

(i) *Recursion for the mean* $l_n = E\,L_n$:

$$(7)\quad l_n = \frac{1}{1-p^n-q^n}[1+(p^n+q^n)l_0 + \sum_{k=1}^{n-1}\binom{n}{k}p^k\,q^{n-k}(l_k+l_{n-k})],\ n\geq 2$$

with $q = 1-p,\ l_0 = l_1 = 1$.

(ii) *Recursion for* $s_n = E\,L_n^2$:

$$(8)\qquad s_n = \frac{1}{1-p^n-q^n}\{(p^n+q^n)[1+s_0 2l_0+2l_n+2l_0l_n] + \sum_{k=0}^{n-1}\binom{n}{k}p^k\,q^{n-k}[1+s_k+s_{n-k}+2l_k+2l_{n-k}+2l_kl_{n-k}]\},\ n\geq 2,$$

$$s_0 = s_1 = 1.$$

(iii) *Recursion for* $m_n = E\,L_n^3$:

$$(9)\qquad m_n = \frac{1}{1-p^n-q^n}[(p^n+q^n)(3l_0s_n+3s_n+3s_0l_n+6l_0l_n+3l_n + m_0+3s_0+3l_0+1)+M_n],\ n\geq 2$$

with

$$M_n := \sum_{k=1}^{n-1}\binom{n}{k}p^k\,q^{n-k}(m_{n-k}+3l_ks_{n-k}+3s_{n-k}+3s_kl_{n-k}+6l_kl_{n-k} + 3l_{n-k}+m_k+3s_k+3l_k+1),\quad \text{and}\quad m_0 = m_1 = 1.$$

(iv) *Recursion for* $f_n = E\,L_n^4$:

$$(10)\qquad f_n = \frac{1}{1-p^n-q^n}[(p^n+q^n)(1+f_o+4m_n+4m_o] + 6\,s_n+6\,s_o+4\ell_n+4\ell_o+8\ell_n\,m_o+8\ell_om_n + 4\ell_ns_o+4\ell_os_n+8\ell_o\ell_n)+F_n],\ n\geq 2,$$

with

$$F_n := \sum_{k=1}^{n-1}\binom{n}{k}p^kq^{n-k}(1+f_k+f_{n-k}+ + 4m_k+4m_{n-k}+6s_k+6s_{n-k}+4\ell_k+4\ell_{n-k}+8\ell_km_{n-k} + 8\ell_{n-k}m_k+4\ell_ks_{n-k}+4\ell_{n-k}s_k+8\ell_k\ell_{n-k})$$

and $f_o = f_1 = 1$.

(v) *Formula for* $E\,Y_n^4,\ Y_n = \frac{L_n-l_n}{\sqrt{n}}$:

$$(11)\qquad E\,Y_n^4 = \frac{1}{n^2}(f_n-3l_n^4+6s_nl_n^2-4m_nl_n).$$

A numerical study for the mean and variance of L_n for $n \leq 20$ is provided in Regnier and Jacquet (1985). We evaluated recursions for the first through fourth moments using a variety of different precisions on different machines. For large values of n, computation of the moments is susceptible to numerical roundoff error, and high precision is required. Initially, we computed moments for several different values of p, carrying the recursions out to about n =3000[1]. For values of n up to 1000, these results were compared with the sample moments from the simulation runs, and showed close agreement.

Initial investigation of the behaviour of the mean l_n of L_n at $p = 0.5$ failed to show the predicted instability of ℓ_n/n. The normalized value l_n/n seemed to converge rapidly, reaching a value of about 2.885 for $n = 2400$, and showing no variation out to 7 decimal places with further increase in n. The increments $l_n/n - l_{n-1}/(n-1)$ were observed to always be positive, another indication of convergence. This apparent disagreement between theory and experiment was sufficiently disturbing that an additional, larger computer run was deemed necessary. In order to push into a regime of yet higher n and also obtain greater accuracy, a run was made on a Cray Y-MP (NSF, San Diego) using double-precision arithmetic (about 28 significant digits). The first moment was computed for values of n up to 250,000, and the new results demonstrated the predicted instability. At $n = 38,488$, a negative increment appears, and subsequently, values of the increment oscillate in a sinusoidal fashion, with a peak magnitude of about 1×10^{-10}. The behaviour of the increments is shown graphically in Figure 1 on a logarithmic scale.

[1]Examining (7) - (9), it is clear that the required computation grows as n^2. However, one can achieve a substantial improvement in run time by discarding terms in which the binomial coefficient is very small (say, more than four standard deviations from the peak of the distribution); in this case, computation grows as $n^{3/2}$.

Figure 1. Increments of $l_n/n, p = 0.5$

$n =$ number of users who initially collide.

Based on recursions (7) - (10) the numerical results for evaluation of l_n/n, $\Delta(l_n/n) := \frac{l_n}{n} - \frac{l_{n-1}}{n-1}$, $\mathrm{Var}_n := \mathrm{Var}(L_n/\sqrt{n})$ and $\Delta(\mathrm{Var}_n) := \mathrm{Var}_n -$ Var_{n-1} are shown below:

Table 1. numerical results $p = 0.5,\ L_0 = L_1 = 1$

n	l_n/n	$\Delta(l_n/n)$	Var_n	$\Delta(\mathrm{Var}_n)$
2	2.5000D+00	1.5000D+00	4.0000D+00	4.000D+00
3	2.5556D+00	5.5556D-02	3.2593D+00	-7.4074D-01
4	2.6310D+00	7.5397D-02	3.3832D+00	1.2396D-01
5	2.6838D+00	5.2857D-02	3.3875D+00	4.2812D-03
10	2.7853D+00	1.0985D-02	3.3832D+00	1.1672D-04
100	2.8754D+00	1.0113D-04	3.3834D+00	9.1046D-07
500	2.8834D+00	4.0528D-06	3.3834D+00	-8.5624D-08
1000	2.8844D+00	1.0224D-06	3.3834D+00	-4.1963D-08
5000	2.8852D+00	3.4639D-08	3.3834D+00	2.1844D-08
10000	2.8853D+00	7.3428D-09	3.3835D+00	-4.1539D-07

On the other hand the change of the initial conditions disturbes the value of l_n/n and Var_n.

Table 2. $p = 0.5,\ L_0 = L_1 = 0$

n	l_n/n	$\Delta(l_n/n)$	Var_n	$\Delta(\mathrm{Var}_n)$
2	1.0000E+00	1.000E+00	1.000E+00	1.000E+00
3	1.1111E+00	1.1111E-01	8.1481E-01	-1.8519E-01
4	1.1905E+00	7.9365E-02	8.4580E-01	3.0990E-02
5	1.2419E+00	5.1429E-02	8.4688E-01	1.0703E-03
10	1.3427E+00	1.1048E-02	8.4579E-01	2.9179E-05
100	1.4327E+00	1.0107E-04	8.4586E-01	2.2762E-07
500	1.4407E+00	4.0304E-06	8.4586E-01	-2.1503E-08
1000	1.4417E+00	1.0117E-06	8.4586E-01	-1.2471E-08

Table 1 confirms the stability of $\frac{l_n}{n} \approx 2.88$, $\mathrm{Var}_n \approx 3.38$ for moderate $n \in (10^2, 10^4)$, and $p = 0.5$. Slight perturbation of p around 0.5 does not change the overall stability of $\frac{l_n}{n}$ for practically relevant n, see Figure 2.

Figure 2. $|\Delta(l_n/n)|,\ p = 0.499$

Summarizing our numerical findings it appears that for reasonably large $n \geq 100$ and $p = 0.5$ the non-linearity of ℓ_n/n, and Var_n is not observed in a practical relevant magnitude, also in this range of values of n the behavior of ℓ_n and Var_n is stable with respect to p. We check the adequacy of the normal approximation of the distribution of Y_n by producing histograms and sample moments of L_n by Monte-Carlo simulation.

3.2 Simulation Results

A full simulation of the protocol would require storage and manipulation of state information for all users involved in collisions. However, there is a simpler approach which avoids actual implementation of the CTM protocol, and requires less memory and much less computation. Our simulation simply carries out the stochastic recursion $L_n \stackrel{d}{=} 1 + L_{I_n} + \tilde{L}_{n-I_n}$, $I_n \stackrel{d}{=} Bi(n, \frac{1}{2})$ directly. A short FORTRAN 77 program performs the requisite operations. The most natural and direct implementation of the recursion is via a recursive routine (although the FORTRAN 77 standard does not provide for recursion, many FORTRAN 77 compilers permit it). This routine, which is called LRES, is invoked by the main routine. LRES invokes a function called BINOMIAL which returns a binomial random value. A SUN Unix / IBM Aix implementation of LRES is shown below.

```
INTEGER FUNCTION LRES (N,P)
INTEGER N, I, BINOMIAL
AUTOMATIC I
REAL P
EXTERNAL BINOMIAL
IF (N.LE.1) THEN! No further splitting required.
      LRES = 1
ELSE! Further splitting is required.
      I = BINOMIAL (N,P)
      LRES = 1 + LRES(I,P) + LRES(N-I,P)
END IF
END
```

500,000 to 1,000,000 trials were required to obtain accurate histograms. Simulation run times were found to grow roughly linearly with n. Output values of L_n were normalized and binned; bin counts were then plotted against a normal curve having the same mean and standard derivation. The curves in Figures 3 are for $n = 1000$. Simulation results, which are based on over 930,000 trials, agree fairly well with the normal curve. As mentioned in the introduction in this case $(L_n - l_n)/\sigma_n$ is asymptotically normal in spite of the fact that l_n and σ_n are not asymptotically linear.

Figure 3. Simulation curve for $Y_n = (L_n - l_n)/\sigma_n$ for $n = 1000,\ p = 0.5$, $L_0 = L_1 = 1$, based on 936,725 trials, and the fitted normal curve with mean zero and variance 3.3834 as given in Table 1.

Figure 4. Normal fit to empirical *df* with $n = 50, p = 0.5$

For $n = 20$ or $n = 30$ the normal fit is no longer good. Further simulation results indicate stability w.r.t the value of p.

Figure 5. Normal fit with $\sigma^2 = 3.3874$ to the simulated Y_n's; $n = 1000,\ p = 0.49,\ L_0 = L_1 = 1$ based on 697,675 trials.

The Shapiro-Wilks test applied to our simulation results confirmed the above findings. We considered the binomial case with $p = 0.5$, 0.499, 0.498 and $n = 10$, 30, 100, 300, 1000, 40000 and 50000. Let W denote the Shapiro-Wilks statistic, w the observed value of W and $\alpha = P(W \leq w)$ under the assumption of normality, then for $n \leq 30$ we obtained for any p that $\alpha \approx 0$ (up to 4 digits). For the other values of n we did two samples in order to control the sampling error.

Table 3. Shapiro-Wilks test for normality and various values of n, p.

$n = 100$

p	w_1	α_1	w_2	α_2
0.5	0.9810	0.1146	0.9905	0.9564
0.499	0.9815	0.1461	0.9852	0.5191
0.498	0.9838	0.3611	0.9845	0.4392

n=300

p	w_1	α_1	w_2	α_2
0.5	0.9783	0.0233	0.9803	0.0791
0.499	0.9861	0.6308	0.9823	0.2091
0.498	0.9808	0.1029	0.9822	0.2030

n=1000

p	w_1	α_1	w_2	α_2
0.5	0.9814	0.1439	0.9865	0.6701
0.499	0.9832	0.2913	0.9861	0.6339
0.498	0.9825	0.2233	0.9813	0.1354

Similarly for $n = 40000$, 50000 the Shapiro-Wilks statistic did not produce an anomalous result and was consistent with the normal hypothesis.

In the final simulations we considered the case with nonzero immigrations X, Y in a symmetric and a nonsymmetric case with masses in 0,1,2

Figure 6. Normal fit for $n = 40/100$ and $X \sim \frac{3}{4}\delta_0 + \frac{1}{8}\delta_1 + \frac{1}{8}\delta_2, Y \sim \delta_0$, (nonsymmetric case)

$n = 40$

$n = 100$

Figure 7. Normal fit for n = 50 / 10000 and $X \sim Y \sim \frac{3}{4}\delta_0 + \frac{3}{16}\delta_1 + \frac{1}{16}\delta_2$ (symmetric case)

$n = 50$

$n = 10000$

These examples confirm our general robustness idea, that asymptotic normality is approximatively valid if the variances behave approximatively linear (which is observed in these examples empirically).

4 Conclusions

The essential findings of this paper are the following.

a) Asymptotic normality is predicted by our theoretical results to hold under a broad range of protocols which are not too much concentrated in the extremes (in technical terms, where the variance behaves approximatively linear). This suggests that a normal approximation is approximatively valid if the variances behave empirically or theoretically approximatively linear. These predictions are verified in some cases with immigration in a simulation study. For other (nonlinear) types of behaviour of the variance nonnormal limit distributions arise.

b) The theoretically predicted instability in the case $p = \frac{1}{2}, X = Y = 0$ is numerically detected but only for an extremely large number n. The order of magnitude of the instability can be neglected from a practical point of view and one can use the simple linear normalizations in the range $n \geq 100$ (or even $n \geq 50$).

c) The simulation studies confirm a good fit for the normal approximation in the range $n \geq 100$. They also show stability of the limiting normal approximation w.r.t. p in the binomial protocol (without immigration).

Acknowledgement. The authors thank to U. Rösler for discussions on contraction properties of L_p-minimal metrics. Some of the simulations in section 3 were done by M. Cramer. They also thank the reviewers for their remarks resulting in some essential improvements of the paper.

References

[1] Aldous, D. J.: Ultimate instability of exponential backoff protocol for acknowledgement-based transmission control of random access communication channels. IEEE Transactions on Information Theory IT 33 (1987), 219 - 223

[2] Aldous, D. J.: Asymptotic fringe distribution for general families of random trees. Annals of Applied Probability 1 (1991), 228 - 266

[3] Bertsekas, D. and Gallager, R.: Data Networks. Prentice-Hall, New Jersey, 1987

[4] Borovkov, A. A.: On the ergodicity and stability of the sequence $w_{n+1} = f(w_n, z_n)$: applications to communication networks. Theory of Probability and its Applications 33 (1988), 595 - 611

[5] Fayolle, G., Flajolet, P., Hofri, M.: On a functional equation arising in the analysis of a protocol for a multi-access broadcast channel. Advances in Applied Probability 18 (1986), 441 - 472

[6] Fayolle, G., Flajolet, P., Hofri, M., Jacquet, P.: Analysis of a stack algorithm for random multiple-access communication. IEEE Transactions on Information Theory IT-31 (1985), 244 - 254

[7] Jacquet, P. and Regnier, M.: Normal limiting distribution of the size of tries. In: Performance' 87, P. J. Courtois and G. Latouche (edt.), 209 - 221. Elsevier Science Publications B. V. (North Holland), 1988

[8] Maejima, M. and Rachev, S. T.: An ideal metric and the rate of convergence to a self-similar process. Annals of Probability 15 (1987), 702 - 727

[9] Massey, J. L.: Collision-resolution algorithms and random-access communications, Multi-User Communication Systems. Longo, Ed. CISM Courses and Lectures, 1981

[10] Rachev, S. T.: Probabiltiy Metrics and the Stability of Stochastic Models. Wiley, Chichester, New York, 1991

[11] Rachev, S. T. and Rüschendorf, L.: Recent results in the theory of probability metrics. Statistics and Decisions 9 (1991), 327 - 373

[12] Rachev, S. T. and Rüschendorf, L.: Probability metrics and recursive algorithms. Technical Report (1991). To appear in: Advances in Applied Probability 1995

[13] Regnier, M. and Jacquet, P.: New results on the size of tries. IEEE Transactions on Information Theory IT-35 (1989), 203 - 205