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CONSTRUCTION OF MULTIVARIATE DISTRIBUTIONS WITH GIVEN MARGINALS

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Summary

We make some remarks on the problem how to construct probability measures with given marginals. Questions of this kind arise if one wants to build a stochastic model in a situation where one has some idea of the kind of dependence and knows exactly certain marginal distributions.

1. Introduction

For modelling stochastic dependence e.g. for the description of alternatives in problems of testing stochastic independence, a lot of classes of multivariate distributions have been proposed. The most famous are the Farlie-Gumbel-Morgenstern (FGM) distributions and their generalizations (cf. Johnson, Kotz [9], Mardia [18], Kimeldorf and Sampson [13]), the translation families (cf. Mardia [17]) and the Plackett [19] distributions (we clearly cannot mention all particular, parametric families of distributions as e.g. exponential families). A special prominent role, when considering dependence properties, always play the product measure (the independent case) and the 'counterpart' (in dimension two) the Fréchet distributions

$$
\begin{aligned}
&H_+(x, y)=\min\{F(x), G(y)\}\\
&H_-(x, y)=\max\{F(x)+G(y)-1, 0\}\ ,
\end{aligned}
\tag{1}
$$

where F and G are marginal distribution functions.

A basic problem of modelling is, to find parametric families of distributions with high degree of dependence as measured by correlation or other dependence measures (cf. Farlie [6], Johnson and Kotz [9], [10], Schucany, Parr and Boyer [20], Barnett [1], Cook and Johnson [4]). A simple special advice in this direction is to consider convex

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combinations of the product measure and the Fréchet-distributions.

In the following we consider this problem for the case of uniform marginals on [0, 1] only, The reason for this restriction is that our method involves only consideration of densities which generalizes immediately to general product spaces. Also for many dependence aspects of real random variables which are 'representation invariant' it is sufficient to consider the uniform maginals. This aspect is worked out in Kimeldorf and Sampson [14].

2. Multivariate distributions with one dimensional marginals

Let M_n be the set of all signed measures on $[0,1]^n$ with uniform marginals and let M_n^+ denote the probability measures in M_n. We shall concentrate in the following on those elements of M_n which are continuous w.r.t. λ^n, the Lebesgue-measure on $[0,1]^n$. The reason for this restriction is that on one hand these distributions are easier to handle, on the other hand this class is in a strong sense dense in M_n.

For an integrable function $f\in L^1(\lambda^n)$ on $[0,1]^n$ and $T\subset\{1,\cdots,n\}$ define

$$f_T=\int f\prod_{i\in T}dy_i\,; \tag{2}$$

i.e. we integrate out the components in T. We can consider formally f_T as real function on $[0,1]^n$ which is constant in the T-components. The following linear operator $S\colon L^1(\lambda^n)\to L^1(\lambda^n)$ turns out to be important:

$$Sf=f-\sum_{\substack{T\subset\{1,\cdots,n\}\\ |T|=n-1}} f_T+(n-1)f_{\{1,\dots,n\}}\,. \tag{3}$$

$f\lambda^n$ denotes in the following the measure with density f w.r.t. λ^n. For measures μ and ν we define $\mu\ll\nu$ if μ is continuous w.r.t. ν.

THEOREM 1. *All distributions on $[0,1]^n$ with uniform marginals and which are continuous w.r.t. λ^n are of the form $(1+Sf)\lambda^n$ where $f\in L^1(\lambda^n)$; more precisely:*

$$\begin{aligned}&\{P\in M_n;\ P\ll\lambda^n\}=\{(1+Sf)\lambda^n;\ f\in L^1(\lambda^n)\}\,,\\ &\{P\in M_n^+;\ P\ll\lambda^n\}=\{(1+Sf)\lambda^n;\ f\in L^1(\lambda^n),\ Sf\geqq -1\}\,.\end{aligned} \tag{4}$$

PROOF. By definition of M_n and the Radon-Nikodym theorem

$$\begin{aligned}&\{P\in M_n;\ P\ll\lambda^n\}=\{(1+f)\lambda^n;\ f\in L^1(\lambda^n),\ \text{and}\\ &f_T=0 \text{ for all } T\subset\{1,\cdots,n\},\ |T|\geqq n-1\}\,.\end{aligned} \tag{5}$$

If now $P=(1+f)\lambda^n \in M_n$, then $f_T=0$ for all $|T|\geqq n-1$ and, therefore, $Sf=f$, i.e. $P=(1+Sf)\lambda^n$.

If, conversely, $f \in L^1(\lambda^n)$, $P=(1+Sf)\lambda^n$, and $T_0 \subset \{1, \cdots, n\}$, $|T_0|=n-1$, then for $T \subset \{1, \cdots, n\}$, $|T|=n-1$, $T \neq T_0$ we have $(f_T)_{T_0}=f_{\{1,\cdots,n\}}$ and, therefore,

$$\begin{aligned}(Sf)_{T_0}&=(f-f_{T_0}-\sum_{T\neq T_0,\,|T|=n-1} f_T+(n-1)f_{\{1,\cdots,n\}})_{T_0}\\ &=f_{T_0}-f_{T_0}-\sum_{T\neq T_0,\,|T|=n-1}(f_T)_{T_0}+(n-1)f_{\{1,\cdots,n\}}=0\,.\end{aligned}$$

Therefore, $(1+Sf)\lambda^n \in M_n$. The second part of relation (4) is immediate.

By Theorem 1 we are led to propose the following method to construct parametric families of distributions with uniform marginals: *Let f_ϑ, $\vartheta \in \Theta$, be a parametric family of functions in $L^1(\lambda^n)$ such that $Sf_\vartheta \geqq -1$, $\vartheta \in \Theta$, and consider $\mathcal{P}=\{P_\vartheta;\, \vartheta \in \Theta\}$, where $P_\vartheta=(1+Sf_\vartheta)\lambda^n$.*

The idea of this method is that the functions f_ϑ describe the dependence structure of an underlying situation and that our fit P_ϑ to the given marginals does not disturb too much this feature. The following examples indicate that this idea works well.

Example 1. a) Let $f \in L^1(\lambda^n)$, $\alpha_0=\inf\{(Sf)(x);\, x \in [0,1]^n\}>-\infty$ and consider $f_\vartheta(x)=\vartheta f(x)$, $x \in [0,1]^n$, $\vartheta \in \Theta=\left[0, \dfrac{1}{|\alpha_0|}\right]$.

1) If $f(x)=\prod_{i=1}^n v_i(x_i)$, where $\int v_i(x_i)dx_i=0$, $1\leqq i\leqq n$, then $Sf=f$ and $\mathcal{P}$ gives a generalized FGM-family (cf. Johnson, Kotz [9], Kimeldorf, Sampson [14]).

2) If $f(x)=\prod_{i=1}^n x_i^{m_i}$, then

$$Sf(x)=\prod_{i=1}^n x_i^{m_i}-\sum_{i=1}^n\Big(\prod_{j\neq i}\frac{1}{m_j+1}\Big)x_i^{m_i}+(n-1)\prod_{i=1}^n\frac{1}{m_i+1}\,,$$

which gives a new family.

3) If $n=2$ and $f(x,y)=\dfrac{1}{|x-y|^{1/2}}$, then

$$Sf(x,y)=\frac{1}{|x-y|^{1/2}}-2(x^{1/2}+(1-x)^{1/2}+y^{1/2}+(1-y)^{1/2})+8/3\,;$$

$\mathcal{P}$ gives a family of distributions which is like f highly concentrated near the diagonal.

b) Let $\Theta=[0,1]$, $n=2$ and $f_\vartheta(x,y)=-1(|x-y|>\vartheta)$ ($1(A)$ denoting the indicator function of A). Defining $g_\vartheta(x)=(1-2\vartheta)1(\vartheta\leqq x\leqq 1-\vartheta)+(1-x-\vartheta)1(0\leqq x<\vartheta)-(x-\vartheta)1(x>1-\vartheta)$ for $0\leqq\vartheta\leqq\dfrac{1}{2}$, while $g_{\mathcal{J}}(x)=(x-\mathcal{J})1(x>\mathcal{J})+(\mathcal{J}-x)1(x<1-\mathcal{J})$, $\frac{1}{2}<\mathcal{J}\leqq 1$, we obtain $(Sf_{\mathcal{J}})(x, y)=f_{\mathcal{J}}(x, y)+g_{\mathcal{J}}(x)+g_{\mathcal{J}}(y)-(1-\mathcal{J})^2$. $P_{\mathcal{J}}=(1+Sf_{\mathcal{J}})\lambda^2$ approaches for $\mathcal{J}\to 1$ the product measure, while for small $\mathcal{J}$ (neglecting small and large x values) $1+(Sf_{\mathcal{J}})(x, y)\approx 2-2\mathcal{J}$ for $|x-y|<\mathcal{J}$ and $\approx 1-2\mathcal{J}$ for $|x-y|>\mathcal{J}$. In order to introduce stronger dependence one can consider an additional parameter and start with $f_{\mathcal{J},a}(x, y)=af_{\mathcal{J}}(x, y)$, which for $\mathcal{J}$ small and a large centers the distribution near the diagonal.

3. Distributions with given independence structure

Let $T_1, \cdots, T_k\subset\{1, \cdots, n\}$ and $C=\{T_1, \cdots, T_k\}$. The question we consider in this section is to construct distributions on $[0, 1]^n$ with given uniform marginals, such that the T_i-subset of the components is independent, $1\leqq i\leqq k$, more precisely we deal with

$$(6)\qquad M_n(C)=\{P\in M_n;\ P\ll\lambda^n \text{ and } \pi_{T_i}(P)=\lambda^{|T_i|},\ 1\leqq i\leqq k\}\,,$$

where π_{T_i} denotes the projection on the T_i-components.

For this problem we need a second linear operator V: $L^1(\lambda^n)\to L^1(\lambda^n)$. Define $R_i=T_i^c=\{1, \cdots, n\}\backslash T_i$ and define for $f\in L^1(\lambda^n)$ inductively

$$(7)\qquad \begin{aligned} &f_{(1)}=f-f_{R_1} \qquad \text{and for } m<k\,,\\ &f_{(m+1)}=f_{(m)}-(f_{(m)})_{R_{m+1}}\,; \end{aligned}$$

finally define: $V(f)=f_{(k)}$.

THEOREM 2.

$$M_n(C)=\{(1+V\circ Sf)\lambda^n;\ f\in L^1(\lambda^n)\}\,.$$

PROOF. By Theorem 1, $M_n(C)=\{(1+Sf)\lambda^n;\ (Sf)_{R_i}=0,\ 1\leqq i\leqq k\}$. If $(1+Sf)\lambda^n\in M_n(C)$, then by definition $V\circ Sf=Sf$, implying the inclusion $M_n(C)\subset\{(1+V\circ Sf)\lambda^n;\ f\in L^1(\lambda^n)\}$.

For the converse inclusion define $D_1=\{Sf; f\in L^1(\lambda^n)\}$. For $g\in D_1$ we prove by induction that $g_{(m)}\in D_1$ and $(g_{(m)})_{R_i}=0$, $1\leqq i\leqq m$, $1\leqq m\leqq k$. If $m=1$, then clearly $(g_{(1)})_{R_i}=0$ and for $|T|=n-1$, $T\neq T_1^c=R_1$ we have $(g_{(1)})_T=g_T-g_{R_1}=0$ since $g\in D_1$. This implies that $g_{(1)}\in D_1$. For the induction step observe that by definition

$$(g_{(m+1)})_{R_i}=(g_{(m)})_{R_i}-(g_{(m)})_{R_i}-(g_{(m)})_{R_i\cup R_{m+1}}=0 \qquad \text{for } 1\leqq i\leqq m+1$$

and as for the case $m=1$ we see that $g_{(m+1)}\in D_1$. Therefore, for $f\in L^1(\lambda^n)$ we get that $V\circ Sf=(Sf)_{(k)}\in D_1$, i.e. $(1+V\circ Sf)\lambda^n\in M_n(C)$.

Some special attraction has attained the problem of this section in the case that $C=\{T\subset\{1, \cdots, n\};\ |T|=k\}$ (cf. Joffe [7], [8], Bühler and Mieschke [3]). In this case the following more compact representation of the solutions is possible.

Define for $f\in L^1(\lambda^n)$ and $1\leqq k<n$ inductively linear operators $V_1, \cdots, V_n$ by:

$$V_1f=f, \qquad V_{k+1}f=V_kf-\sum_{|T|=n-k+1}(V_kf)_T\,. \tag{8}$$

Call a signed measure P k-independent if the projection of P on any k components equals the product λ^k, $2\leqq k\leqq n$.

THEOREM 3. *The set of all k-independent elements of M_n, which are continuous w.r.t. λ^n is given by* $\{(1+V_k\circ Sf)\lambda^n;\ f\in L^1(\lambda^n)\}$.

PROOF. If $P=(1+f)\lambda^n\in M_n$ is k-independent, then $f\in D_1=\{Sg;\ g\in L^1(\lambda^n)\}$ and, therefore, $f=V_1f=V_2f=\cdots=V_kf$; i.e. f is a fixpoint of V_k implying that the k-independent λ^n-continuous distributions are of the form $(1+V_k\circ Sf)\lambda^n$, $f\in L^1(\lambda^n)$. For the other inclusion observe that $f\in D_1$ implies that $V_kf\in D_1$. Furthermore, for $|T_0|=n-k$ we have

$$(V_kf)_{T_0}=(V_{k-1}f)_{T_0}-(V_{k-1}f)_{T_0}-\sum_{\substack{|T|=n-k\\ T\neq T_0}}(V_{k-1}f)_{T\cup T_0}=0$$

$$\text{since}\quad |T\cup T_0|\geqq n-k+1\,.$$

For the construction of k-independent distributions we need the additional condition $V_k\circ Sf\geqq -1$; this condition is easily satisfied if one uses bounded functions f. Note that the k-independent distributions not being $k+1$-independent are given by the additional condition that $\lambda^n\{(V_k\circ Sf)_T\neq 0\}>0$ for all $|T|=n-k+1$.

If, especially, $k=n-1$ and $f(x)=\prod_{i=1}^n v_i(x_i)$, $\int v_i(x_i)dx_i=0$, then $f=Sf$ and $(1+f)\lambda^n$ is a generalized FGM-distribution (cf. Section 2). If $\lambda^n\{f=0\}<1$, then $V_1f=V_2f=\cdots=V_{n-1}f=f$, which implies that $(1+f)\lambda^n$ is $(n-1)$-independent but not n-independent. This observation strongly indicates the lack of strong dependence in higher dimensional FGM-families and, simultaneously, gives some very natural examples of $(n-1)$-independent distributions, which are not n-independent. Similarly, $(1+f)\lambda^n$ is k-independent but not $k+1$-independent, where $f(x)=\alpha\sum_{|T|=k}\prod_{j\in T}v_j(x_j)$, α being a factor such that $1+f\geqq 0$.

A different method of construction of k-independent families is the following: Let $X_1, \cdots, X_n$ be k-independent random variables on a small set $\{1, \cdots, p\}$ as e.g. given by the construction of Joffe [8] or by ad hoc methods. Assume that $Y_1, \cdots, Y_n$ are independent random variables such that $Z_i=h_i(X_i, Y_i)$ are uniformly distributed on $[0, 1]$, then $Z_1, \cdots, Z_n$ are k-independent and uniformly distributed.

If e.g. Y_i are uniformly distributed on $\left[0, \frac{1}{p}\right]$, $1 \leqq i \leqq n$, and $h_i(j, y) = y + \frac{j-1}{p}$, $1 \leqq j \leqq p$, $y \in \left[0, \frac{1}{p}\right]$, then we get $Z_i = Y_i + \frac{1}{p}(X_i - 1)$, $1 \leqq i \leqq n$, are k-independent, uniformly distributed and not $k+1$-independent, if $X_1, \cdots, X_n$ are not $k+1$-independent. This method to transfer dependence properties from distributions on small sets to any distributions was introduced in the case of normal distributions by Bühler and Mieschke [3].

4. Multivariate marginals

A well known and difficult problem of multivariate distribution theory is to construct a probability measure with prescribed multivariate marginals; i.e. let $C = \{T_1, \cdots, T_k\}$ be a family of subsets of $\{1, \cdots, n\}$ and $\{P_T; T \in C\}$ be a consistent family of distributions, P_T being the distribution of the T-components. The problem is whether there exists a distribution P on $[0, 1]^n$ with marginals P_T, $T \in C$, and how to construct it.

Some aspects of this problem are discussed by Dall'Aglio [5]. The problem of existence was solved by Kellerer [11], Satz 4.2, but the solution is essentially of theoretical kind and does not allow to decide the existence problem in most of the practical situations. The cases, which always allow a simple construction, are classified by Kellerer [12], Satz 3.5. A typical example is the case $C = \{\{i, i+1\}; 1 \leqq i \leqq n-1\}$, where a common distribution can be constructed as a distribution of a Markov chain. The simplest unsolved case is for $n=3$ and $C = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. Some necessary conditions have been given in this case by Bass [2] and Dall'Aglio [5].

An interesting observation in this context is due to Kellerer [11], Satz 1.1, who constructed a signed measure with marginals P_T, $T \in C$; i.e. the consistency is a necessary and sufficient condition for the existence of a signed measure with given multivariate marginals. We shall follow this line of approach and assume that P_T are continuous w.r.t. $\lambda^{|T|}$, $T \in C$, and that $\bigcup_{T \in C} T = \{1, \cdots, n\}$. Define

$$W_n(C) = \{P \in M_n;\ P \ll \lambda^n,\ \pi_T(P) = P_T,\ T \in C\}. \tag{9}$$

Note that by our assumption the one dimensional marginals are uniform and that $P_T = g^T \lambda^{|T|}$, $T \in C$.

If $T \in C$ and $T' \subset T$, define $g^{T'} \lambda^{|T'|}$ to be the projection of P_T on the T'-components. For the subset $J \subset \{1, \cdots, k\}$ define $T_J = \bigcap_{j \in J} T_j$ and for the empty set ϕ, $g^\phi = 1$; we consider as in Section 2 the functions

g^T formally as functions on $[0,1]^n$ and introduce

$$h(x)=\sum_{m=1}^{k}(-1)^{m-1}\sum_{J\subset\{1,\cdots,k\},|J|=m}g^{T_J}(x)\,,\qquad x\in[0,1]^n\,. \tag{10}$$

Remember the linear operator V defined in (7).

THEOREM 4. *The set of all distributions on* $[0,1]^n$ *with marginals* P_T, $T\in C$, *which are continuous w.r.t.* λ^n *is*

$$W_n(C)=\{(h+V\circ Sf)\lambda^n;\ f\in L^1(\lambda^n)\}\,. \tag{11}$$

PROOF. In the first step we prove that $h\lambda^n\in W_n(C)$ or, equivalently, that $h_{R_i}=g^{T_i}$, where $R_i=T_i^c$, $1\leq i\leq k$. Without loss of generality we consider the case $i=1$. By definition of h we get

$$\begin{aligned}
h&=g^{T_1}+\sum_{m=1}^{k}(-1)^{m-1}\sum_{|J|=m,J\neq\{1\}}g^{T_J}\\
&=g^{T_1}+\sum_{m=1}^{k}(-1)^{m-1}\Bigl(\sum_{|J|=m,1\in J,J\neq\{1\}}g^{T_{J\cup\{1\}}}+\sum_{|J|=m,1\notin J}g^{T_J}\Bigr)\\
&=g^{T_1}+\sum_{m=1}^{k}(-1)^{m-1}\Bigl(\sum_{|J|=m-1,1\notin J,J\neq\phi}g^{T_{J\cup\{1\}}}+\sum_{|J|=m,1\notin J}g^{T_J}\Bigr)\\
&=g^{T_1}+\sum_{m=1}^{k-1}(-1)^{m}\Bigl(\sum_{|J|=m,1\notin J}g^{T_{J\cup\{1\}}}-\sum_{|J|=m,1\notin J}g^{T_J}\Bigr);
\end{aligned}$$

from the relation $(g^{T_{J\cup\{1\}}})_{R_1}=(g^{T_J})_{R_1}$ we obtain the assertion $h_{T_1^c}=g^{T_1}$.

Let now $P=g\lambda^n\in W_n(C)$, then $g=h+(g-h)=h+V\circ S(g-h)$, since $g-h$ is by the first part of this proof a fixpoint of $V\circ S$. Conversely, for $f\in L^1(\lambda^n)$ and $T\in C$: $(h+V\circ Sf)_{T^c}=h_{T^c}+(V\circ Sf)_{T^c}=h_{T^c}=g^T$ by definition of V, i.e. $(h+V\circ Sf)\lambda^n\in W_n(C)$.

Theorem 6 allows in certain cases even to construct probability measures with given multivariate marginals. The idea is to find a function $f\in L^1(\lambda^n)$, such that $V\circ Sf$ is balancing the negative parts of h. Some natural candidates for f are functions which allow an explicit and simple determination of $V\circ Sf$, such as e.g. linear combinations of functions of the type $\prod_{i=1}^{n}v_i(x_i)$ where $\int v_i(x_i)dx_i=0$, $1\leq i\leq n$.

Example 2. Let $n=3$, $C=\{\{1,2\},\{2,3\},\{1,3\}\}$.

a) When the marginal densities are

$$f_{12}(x_1,x_2)=1\,,$$

$$f_{23}(x_1,x_3)=1+\Bigl(x_2-\frac{1}{2}\Bigr)\Bigl(x_3-\frac{1}{2}\Bigr),$$

$$f_{13}(x_1,x_3)=1+\Bigl(x_1-\frac{1}{2}\Bigr)\Bigl(x_3-\frac{1}{2}\Bigr),$$

then

$$h(x_1, x_2, x_3)=\frac{3}{2}+x_1x_2+x_2x_3-x_3-\frac{x_1+x_2}{2}$$

is already a nonnegative density with the given marginals.

b) If

$$f_{12}(x_1, x_2)=1+3\left(x_1-\frac{1}{2}\right)\left(x_2-\frac{1}{2}\right),$$

$$f_{13}(x_1, x_3)=1-3\left(x_1-\frac{1}{2}\right)\left(x_3-\frac{1}{2}\right)$$

and

$$f_{23}(x_2, x_3)=1,$$

then

$$h(x_1, x_2, x_3)=f_{13}(x_1, x_3)+f_{12}(x_1, x_2)-1$$

and

$$\min\{h(x_1, x_2, x_3)\}=-\frac{1}{2}=h(1, 0, 1)=h(0, 1, 0).$$

A function balancing these negative parts is given by

$$f(x_1, x_2, x_3)=-6\left(x_1-\frac{1}{2}\right)\left(x_2-\frac{1}{2}\right)\left(x_3-\frac{1}{2}\right),$$

so that

$$h(x_1, x_2, x_3)+f(x_1, x_2, x_3)=1-6\left(x_1-\frac{1}{2}\right)\left(x_2-\frac{1}{2}\right)\left(x_3-\frac{1}{2}\right)$$
$$+3\left(x_1-\frac{1}{2}\right)\left(x_2-\frac{1}{2}\right)-3\left(x_1-\frac{1}{2}\right)\left(x_3-\frac{1}{2}\right)$$

gives a nonnegative density with the given marginals as can easily be seen discussing the cases $x_1, x_2 \leqq \frac{1}{2}$, $x_3 \geqq \frac{1}{2}$, etc. Instead of the factor 6 in the balancing funtion, one can use a factor a in an interval around 6, in this way obtaining a parametric class of distributions with given multivariate marginals.

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