ON THE RATE OF CONVERGENCE IN THE CLT WITH RESPECT TO THE KANTOROVICH METRIC

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Abstract

In this paper, the rate of convergence in the CLT is estimated w.r.t. the Kantorovich metric for random variables with values in separable Banach spaces. In the first part, the rate in stable limit theorems for sums of i.i.d. random variables is considered. The method of proof is an extension of the Bergström convolution method. All assumptions regarding the domain of attraction are given in a metric form. In the second part of the paper an extension is given to the martingale case. For the proof we investigate smoothing properties of suitable conditional versions of the Kantorovich metric. As a consequence of the results for the Kantorovich metric one obtains rate of convergence results in stable limit theorems for martingales w.r.t. the Prohorov metric.

1 Introduction

In this paper, rate of convergence results are exhibited for sequences of random variables with values in a separable Banach space U w.r.t. the Kantorovich metric. Smoothing inequalities for the Kantorovich metric are established and the Bergström convolution method (cf. Zolotarev (1977), (1979), (1983), (1986), Senatov (1980), Rachev and Yukich (1989), (1991), Rachev (1991)) is extended to the case of stable limit theorems and to the Kantorovich metric at the same time. All assumptions concerning the domain of attraction and the right order of convergence are described in terms of finiteness conditions for certain convolution type metrics. Our approach is based on the theory of probability metrics developed by V. M. Zolotarev and his students, see Zolotarev (1983), (1986).

In the second part of the paper, we give an extension of the method developed in Section 2 to the martingale case. Here the only results in the literature so far concern the Gaussian case and mainly the one dimensional case w.r.t. the uniform metric. Some results in the literature are formulated more generally but use bounds involving moments of order ≥ 2 and, therefore, are restricted to the Gaussian case. For some recent literature we refer to Bolthausen (1982), Häusler (1988), Bentkus et al. (1990) and Rackauskas (1990). Our method involves various extensions of an idea in Gudynas (1985) on suitable conditioned versions of probability metrics.

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2 Rate of Convergence in the i.i.d. Case

Consider the separable Banach space $(U, \|\cdot\|)$ and the space $\mathcal{X}(U)$ of U-valued r.v.'s defined on a rich enough probability space. The r.v. $\vartheta \in \mathcal{X}(U)$ is said to be symmetric α-stable $(0 < \alpha \leq 2)$, if

$$n^{-1/\alpha} \sum_{i=1}^{n} \vartheta_i \stackrel{d}{=} \vartheta, \ \vartheta \stackrel{d}{=} -\vartheta, \tag{2.1}$$

where ϑ_i's are i.i.d. copies of ϑ. We are interested in the rate of convergence of the normalized sum

$$Z_n = n^{-1/\alpha} \sum_{i=1}^{n} X_i \tag{2.2}$$

of i.i.d. r.v.'s to ϑ with respect to the Kantorovich metric:

$$\begin{aligned} \ell_1(X, Y) : \ &= \ \sup\{|E(f(X) - f(Y))| : f : U \to \mathbb{R} \text{ bounded}. \\ & \quad\ |f(x) - f(y)| \leq \|x - y\|\}. \end{aligned} \tag{2.3}$$

Recall that ℓ_1-convergence is equivalent to convergence in distribution and convergence of the moments $E\|\cdot\|$ (existence assumed, cf. Rachev (1991), Sec. 6.3); in $\mathbb{R}, \ell_1(X, Y) = \int |F_X(x) - F_Y(x)| dx$. For the Prohorov metric π holds $\pi^2 \leq \ell_1$. Therefore, ℓ_1-convergence rates imply convergence rates for π. Recall that ℓ_1 is an ideal metric of order 1 w.r.t. summation, i.e.

1. μ is regular, $\mu(X_1 + Z, X_2 + Z) \leq \mu(X_1, X_2)$ for all Z independent of X_1, X_2 and

2. μ is homogeneous of order $r = 1$, i.e. $\mu(cX_1, cX_2) = |c|^r \mu(X_1, X_2)$ for all $c \in \mathbb{R}^1$.

Theorem 2.1 *For any* $0 < \alpha < 1$,

$$\ell_1(Z_n, \vartheta) \leq n^{1-1/\alpha} \ell_1(X_1, \vartheta). \tag{2.4}$$

Proof: It follows readily from (2.1) - (2.3) and the contraction properties of ℓ_1. This is a general property for every ideal metric of order 1, see e.g. Zolotarev (1979). $\square$

Consider next the rate of convergence in

$$\ell_1(Z_n, \vartheta) \to 0 \tag{2.5}$$

for $1 < \alpha \leq 2$. The main technique to obtain a rate of convergence in (2.5) is to compare the metric ℓ_1 (which is ideal of order 1 only) with a smoothed version ℓ_r of ℓ_1 which is ideal of order $r > 1$. (cf.[9]). Define the following ideal (smoothing) metric of order $r > 1$:

$$\ell_r(X, Y) = \sup_{h>0} h^{r-1} \ell_1(X + h\vartheta, Y + h\vartheta), \ r > 1, \tag{2.6}$$

and

$$\sigma_r(X,Y) = \sup_{h>0} h^r \sigma(X + h\vartheta, Y + h\vartheta),\ r > 0, \tag{2.7}$$

where ϑ in (2.6) and (2.7) is assumed to be independent of X and Y and σ is the total variation metric:

$$\begin{aligned} \sigma(X,Y) &= \sup\{|E(f(X) - f(Y))| : f : U \to [0,1] \text{ continuous}\} \\ &= 2 \sup_{A\in\mathcal{B}(U)} |P(X \in A) - P(Y \in A)|. \end{aligned} \tag{2.8}$$

In what follows C stands for an absolute constant that can be different in different places. Denote $\ell_1 = \ell_1(X_1,\vartheta)$, $\ell_r = \ell_r(X_1,\vartheta)$, $\sigma = \sigma(X_1,\vartheta)$, $\sigma_r = \sigma_r(X_1,\vartheta)$. We always assume $r > 0$.

Theorem 2.2 *Suppose $r > \alpha$*
(a) $E\|\vartheta\| < \infty$,
(b) $\ell_1 + \ell_r + \sigma_1 + \sigma_r < \infty$.
Then

$$\ell_1(Z_n,\vartheta) \le C(n^{1-r/\alpha}\ell_r + \tau_r n^{-1/\alpha}), \tag{2.9}$$

where

$$\tau_r = \max(\ell_1, \sigma_1, \sigma_r^{1/(r-\alpha)}). \tag{2.10}$$

Remark 2.1 *Zolotarev (1986), §5.4 provides a similar bound for $\ell_1(Z_n,\vartheta)$ in the normal univariate case. Zolotarev's bound contains ζ_r-metrics in the right-hand side of (2.9) (for definition cf.[9]), which can be easily estimated from above in the normal case; in the stable case, however, we need more refined bounds. The problem of finiteness of σ_r was discussed in Rachev and Yukich (1989), for the finiteness of ℓ_r see the next corollary. Further in this section the sum of any random variables $X + Y$ means $\tilde{X} + \tilde{Y}$, where $\tilde{X}$ and $\tilde{Y}$ are independent and $\tilde{X} \overset{d}{=} X$, $\tilde{Y} \overset{d}{=} Y$. ϑ and ϑ_i are defined as in (2.1) and satisfy (a).*

Proof: The proof is similar to that of Theorem 3.6 (see further Section 3) which we shall show in detail. Here we only give a short sketch of the proof. It uses the following two properties of the metrics ℓ_1, ℓ_r, σ_r, see Zolotarev (1986), §5.4.

Smoothing Property 1. For any $X, Y \in \mathcal{X}(U)$

$$\ell_1(X,Y) \le \ell_1(X + \varepsilon\vartheta, Y + \varepsilon\vartheta) + 2\varepsilon E\|\vartheta\|. \tag{2.11}$$

Smoothing Property 2. For any X, Y, Z, W independent,

$$\ell_1(X + Z, Y + Z) \le \ell_1(Z,W)\sigma(X,Y) + \ell_1(X + W, Y + W). \tag{2.12}$$

Next let $m = [\frac{n}{2}]$; then by (2.11)

$$\ell_1(Z_n,\vartheta_1) \ \le\ \ell_1(Z_n + \varepsilon\vartheta, \vartheta_1 + \varepsilon\vartheta) + C\cdot\varepsilon \le \ell_1(Z_n + \varepsilon\vartheta, \frac{\vartheta_1 + X_1 + \ldots + X_n}{n^{1/\alpha}} + \varepsilon\vartheta)$$

$$\begin{aligned}
& \sum_{j=1}^{m} \ell_1(\frac{\vartheta_1+\ldots+\vartheta_j+X_{j+1}+\ldots+X_n}{n^{1/\alpha}}+\varepsilon\vartheta, \\
& \frac{\vartheta_1+\ldots+\vartheta_{j+1}+X_{j+2}+\ldots+X_n}{n^{1/\alpha}}+\varepsilon\vartheta) \\
& +\ell_1(\frac{\vartheta_1+\ldots+\varepsilon_{m+1}+X_{m+2}+\ldots+X_n}{n^{1/\alpha}}+\varepsilon\vartheta,\ \vartheta_1+\varepsilon\vartheta) \\
= \ & I_o+\sum_{j=1}^{m} I_j+I_{m+1}. \text{ By (2.12),} \\
I_o \ \leq \ & \ell_1(\frac{X_2+\ldots+X_n}{n^{1/\alpha}}, \frac{\vartheta_2+\ldots+\vartheta_n}{n^{1/\alpha}})\sigma(n^{-1/\alpha}X_1+\varepsilon\vartheta,\ n^{-1/\alpha}\ \vartheta_1+\varepsilon\vartheta) \\
& +\ell_1(\frac{X_1+\vartheta_2+\ldots+\vartheta_n}{n^{1/\alpha}}+\varepsilon\vartheta,\ \frac{\vartheta_1+\ldots+\vartheta_n}{n^{1/\alpha}}+\varepsilon\vartheta).
\end{aligned}$$

Similar upper bounds are obtained for $I_j,\ 1\leq j\leq m+1$. Some of the terms obtained in this way can be estimated using the ideality properties of the metrics. For example, the term $\Delta_3=(m+1)\ell_1(\frac{X_1+\vartheta_2+\ldots+\vartheta_n}{n^{1/\alpha}},\vartheta)$ can be estimated by

$$\begin{aligned}
\Delta_3 \ = \ & (m+1)\ell_1(n^{-1/\alpha}X_1+(\frac{n-1}{n})^{1/\alpha}\vartheta,\ n^{-1/\alpha}\vartheta_1+(\frac{n-1}{n})^{1/\alpha}\vartheta) \\
\leq \ & (m+1)(\frac{n}{n-1})^{\frac{r-1}{\alpha}}\ell_r(n^{-1/\alpha}X_1,n^{-1/\alpha}\vartheta) \\
\leq \ & (m+1)(\frac{n}{n-1})^{\frac{r-1}{\alpha}}n^{-r/\alpha}\ell_r\leq Cn^{1-r/\alpha}\ell_r,
\end{aligned}$$

where in the first inequality we use the obvious relation: $\ell_r(X,Y)\geq h^{r-1}\ell_1(X+h\vartheta,Y+h\vartheta)$. For the other terms like $B_j=\ell_1(\frac{X_1+\ldots+X_j}{j^{1/\alpha}},\ \frac{\vartheta_1+\ldots+\vartheta_j}{j^{1/\alpha}})$ we use an induction proof to get the bound e.g. $B_j\leq C(\ell_r\,j^{1-r/\alpha}+\tau_r\ j^{-1/\alpha})$. For more details see the proof of Theorem 3.6. □

Corollary 2.3 *Suppose that $U=\mathbb{R}^k$ and that ϑ has a Fréchet differentiable density p_ϑ and let*

$$C(\vartheta)=\sup_{\|z\|\leq 1}\int|p'_\vartheta(y)(z)|dz<\infty. \tag{2.13}$$

Suppose that $E\|\vartheta\|<\infty$ and $\ell_1+\ell_r<\infty$, then

$$\ell_1(Z_n,\vartheta)\leq C(n^{1-r/\alpha}\ell_r+\tau_r^* n^{-1/\alpha}), \tag{2.14}$$

where $\tau_1^*=\max(\ell_1,\ell_r^{1/(r-\alpha)})$.

Remark 2.2 *For an integer r, ℓ_r can be estimated from above by the ζ_r-metric, see Zolotarev (1983), p. 294, $\ell_r\leq C\zeta_r$, if $\sup_{\|z\|\leq 1}\int|p_\vartheta^{(r)}(y)(z)|dz$ is finite.*

Proof: Claim For any X and $Y\in\mathcal{X}(\mathbb{R}^k)$ and $\delta>0$

$$\sigma(X+\delta\vartheta,Y+\delta\vartheta)\leq C(r)\delta^{-r}\ell_r(X,Y), \tag{2.15}$$

with $C(r) = 2^{(2-r)/\alpha} C(\vartheta)$.

To prove the claim we first use the obvious bound

$$\sigma(X + \delta\vartheta, Y + \delta\vartheta) \le \delta^{-r} \sigma_r(X, Y). \tag{2.16}$$

Next we show that for any $\delta > 0$

$$\sigma(X + \delta\vartheta, Y + \delta\vartheta) \le \delta^{-1} C(\vartheta) \ell_1(X, Y). \tag{2.17}$$

Indeed, by the ideality of σ and ℓ_1 it is enough to show (2.17) for $\delta = 1$. Then

$$\sigma(X + \vartheta, Y + \vartheta) \le \sup_{|f| \le 1} \left| \int \bar{f}(x)(P_X(dx) - P_Y(dx)) \right|,$$

where $\bar{f}(x) = \int f(x + y) p_\vartheta(y) dy$. Since $|f| \le 1$,

$$\|\bar{f}'(x)\| = \sup_{\|z\| \le 1} |\bar{f}'(x)(z)| \le \sup_{\|z\| \le 1} \int |p'_\vartheta(y)(z)| dy =: C(\vartheta),$$

and thus $|\bar{f}(x) - \bar{f}(y)| \le C(\vartheta)\|x - y\|$, which clearly implies (2.17). To show (2.14), we use (2.16), (2.17) and the following bound

$$\begin{aligned} \sigma_r(X, Y) &= \sup_{h>0} h^r \sigma(X + h\vartheta, Y + h\vartheta) \\ &\le \sup_{h>0} h^r \ell_1(X + 2^{-1/\alpha} h\vartheta, Y + 2^{-1/\alpha} h\vartheta) \frac{2^{1/\alpha}}{h} C(\vartheta) \\ &= C(\vartheta) 2^{r/\alpha} \ell_r(X, Y). \end{aligned}$$

This completes the proof of the claim as well as that of (2.14). □

Remark 2.3 *(Rate of Convergence in the CLT for Random Elements With Le Page Representation)*
Consider a symmetric α-stable U-valued random variable U with Le Page representation

$$\vartheta \stackrel{d}{=} \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \eta_j Y_j, \tag{2.18}$$

where

(i) $Y_j's$ are i.i.d. with $E\|Y_1\|^r < \infty$;

(ii) η_j are i.i.d. symmetric real-valued random variables with $\|\eta_1\|_\alpha = (E|\eta_1|^\alpha)^{1/\alpha} < \infty$; and

(iii) (Γ_j) are a sequence of successive times of jump of a standard Poisson process. Assume that the three sequences are independent; see Ledoux and Talagrand (1991), Sect. 5.1, and Samorodnitsky and Taqqu (1992).

Suppose X has a similar representation

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \eta_j^* Y_j^*, \tag{2.19}$$

where (Y_j^*) and (η_j^*) are chosen as in (i) and (ii) with the only difference that they are **not** *identically distributed. Form Z_n, the normalized sum of i.i.d. copies X_i as in (2.2); then Theorem 2.2 yields the following rate of convergence of Z_n to ϑ in $\ell_1$metric.*

Corollary 2.4 *Let $1 \vee \alpha < r < 2$, and $E\|Y_1\|^r + \sup_{j\geq 1} E\|Y_j^*\|^r + E\|\eta_1\|_r + \sup_{j\geq 1} E\|\eta_j^*\|_r < \infty$. Then*

$$\ell_1(Z_n, \vartheta) \leq C(n^{1-r/\alpha}\ell_r^* + \tau_r^* n^{-1/\alpha}), \tag{2.20}$$

where $\ell_r^* := \sup_{j\geq 1}(\ell_r(Y_j^*, Y_1) + \ell_r(\eta_j^* Y_1, \eta_1 Y_1))$ and $\tau_r^* = \max\{\ell_1^*, \sigma_1^*, \sigma_r^{*1/(r-\alpha)}\}$ with

$$\sigma_r^* := \sup_{j\geq 1}(\sigma_r(Y_j^*, Y_1) + \sigma_r(\eta_j^* Y_1, \eta_1 Y_1)).$$

Proof: In view of (2.9), (2.10) we need only to show the finiteness of σ_r and ℓ_r. For $\sigma_r = \sigma_r(X, \vartheta)$ we use the ideality of order r, and the asymptotics $E\Gamma_j^{-r/\alpha} \sim j^{-r/\alpha} (j \to \infty)$, to obtain

$$\begin{aligned}
\sigma_r(X, \vartheta) &= \sum_{j\geq 1} E\Gamma_j^{-r/\alpha} \ell_r(\eta_j Y_j, \eta_j^* Y_j^*) \\
&\leq (\sum_{j\geq 1} E\Gamma_j^{-r/\alpha}) \sup_{j\geq 1}\{E|\eta_j^*|^r\, \sigma_r(Y_j^*, Y_j) + \sigma_r(\eta_j^* Y_j, \eta_j Y_j)\} \\
&\leq C \sup_{j\geq 1}(\sigma_r(Y_j^*, Y_1) + \sigma_r(\eta_j^* Y_1, \eta_1 Y_1)).
\end{aligned}$$

The same type estimate is valid for ℓ_r. □

Since in the Le Page representations Y_j, Y_j^* can have any high enough moment, examples with finite ℓ_r^* and τ_r^* can be readily constructed. Take, e.g., U to be a Hilbert space with basis $(h_m)_{m\geq 1}$, and set $Y_j^* \stackrel{d}{=} \sum_{m\geq 1} \zeta_{j,m}^* h_m$, $Y_1 \stackrel{d}{=} \sum_{m\geq 1} \zeta_m h_m$, where $(\zeta_m)_{m\geq 1}$, $(\zeta_{j,m}^*)_{m\geq 1}$ are sequences of independent random variables. Then, by the ideality of σ_r,

$$\begin{aligned}
\sigma_r(Y_j^*, Y_1) &\leq \sum_{m\geq 1} \sigma_r(\zeta_{j,m}^*, \zeta_m) \\
&\leq C \sum_{m\geq 1} \kappa_r(\zeta_{j,m}^*, \zeta_m),
\end{aligned} \tag{2.21}$$

where κ_r is the r-th pseudomoment,

$$\kappa_r(\zeta^*, \zeta) = r \int |x|^{r-1} |F_{\zeta^*}(x) - F_\zeta(x)| dx,$$

see Zolotarev (1983). Similarly,

$$\ell_r(Y_j^*, Y_1) \le C \sum_{m\ge 1} \kappa_r(\zeta_{j,m}^*, \zeta_m). \tag{2.22}$$

The same example is valid if we relax the independence assumption to "independence in finite blocks" requiring only that $(\zeta_{1+\ell,\dots}, \zeta_{L+\ell}), \ell = 0, L, 2L, \dots$, are independent.

Remark 2.4 *(Finite Dimensional Approximation)*
An alternative use of the explicit upper bounds of the smoothing metrics in the finite dimensional case is to combine Theorem 2.2 with an approximation step by the finite dimensional case. To be concrete, let X, Y be $C(S)$ valued processes, (S,d) a totally bounded metric space. To $\varepsilon > 0$ let V_ε denote a finite covering ε-net and let $P_\varepsilon X = (X_t)_{t\in V_\varepsilon}$ be the corresponding finite dimensional approximation of $X = (X_t)_{t\in S}$. If

$$\begin{aligned} &E \sup_{d(s,t)\le\varepsilon} |X_s - X_t| \le a(\varepsilon) \quad \textit{and} \\ &E \sup_{d(s,t)\le\varepsilon} |Y_s - Y_t| \le b(\varepsilon), \end{aligned} \tag{2.23}$$

then

$$\ell_1(X,Y) \le \ell_1(P_\varepsilon X, P_\varepsilon Y) + a(\varepsilon) + b(\varepsilon). \tag{2.24}$$

So we can combine fluctuation inequalities (2.23) with the finite dimensional bounds derived in (2.9) for the normalized sum Z_n in order to choose an optimal rate of approximation $\varepsilon = \varepsilon(n) \to 0$. A general and simple useful tool to derive fluctuation inequalities as in (2.23) is Pollard's lemma which applied to 2.23 yields

$$E \sup_{d(s,t)\le\varepsilon} |X_s - X_t| \le \sqrt{N_\varepsilon} \sqrt{\max_{1\le i\le N_\varepsilon} E \sup_{\substack{d(s,t_i)<\varepsilon \\ d(t,t_i)<\varepsilon}} |X_s - X_t|}, \tag{2.25}$$

where $N_\varepsilon = card(V_\varepsilon)$ and $V_\varepsilon = \{t_i,\ 1 \le i \le N_\varepsilon\}$.

The case $\alpha = 1$ needs a special consideration. We shall state a variant of Theorem 2.2 which will cover the case $\alpha = 1$ but requires additional smoothing conditions on the law of the X_i's. The next theorem is based on the following lemma (see Rachev and Yukich (1989), or Rachev (1991), Ch. 14).

Lemma 2.5 *Let $0 < \alpha \le 2$, $r > \alpha$, $a_r = \frac{1}{2^{1+r/\alpha}(2^{r/\alpha}+3^{r/\alpha})}$ and $A_r = A_r(a) = 2^{-r/\alpha} a_r$. Suppose*

$$\delta_o := \delta_o(X_1, \vartheta) := \max(\sigma, \vartheta_r) \le a_r. \tag{2.26}$$

Then for any $n \ge 1$

$$\sigma(Z_n, \vartheta) \le A_r \delta_o n^{1-r/\alpha} \le 2^{-r/\alpha} n^{1-r/\alpha}. \tag{2.27}$$

Theorem 2.6 *Suppose condition (2.26) holds and*

$$\bar{\tau}_r := \max(\ell_1, \ell_r) < \infty. \tag{2.28}$$

Then for $\frac{1}{2} < \alpha \leq 2$ *and* $r > \alpha$,

$$\ell_1(Z_n, \vartheta) \leq B_{r,\alpha} \bar{\tau}_r n^{1-r/\alpha}, \tag{2.29}$$

where $B_{r,\alpha} \geq 8^{(r-1)/\alpha} + 2(2^{r/\alpha} + 3^{r/\alpha})$.

The proof uses the following analogue of (2.12): for any independent X, Y, Z, W

$$\ell_1(X + Z, Y + Z) \leq \ell_1(X, Y)\sigma(Z, W) + \ell_1(X + W, Y + W). \tag{2.30}$$

It is similar to that of the smoothing inequality in Zolotarev (1986) §5.4 (see also Theorem 15.2.2 in Rachev (1991)) and thus is omitted.

The theorem is of interest for $1 \leq \alpha \leq 2$, as for $0 < \alpha < r < 1$ we get from (2.4), $\ell_1(Z_n, \vartheta) \leq n^{1-1/\alpha} \bar{\tau}_r$.

3 The Martingale Case and $\mathcal{G}$-Dependence Metrics

In this part, our objective is the extension of Theorem 2.2 to the martingale case. In order to extend the metric approach to the martingale case we introduce as technical tool a dependence metric which allows to estimate dependent sums from above. A basic result then in Lemma 3.2 allows us to estimate the dependence metric from above by the expectation of the usual l_r metric between conditional distributions. We next establish the necessary smoothing properties and in this way are able to abtain the rate of convergence result by means of a variant of the induction proof.

Let $(\Omega, \mathcal{A}, P)$ be a rich enough probability space, $(\mathcal{F}_i)$ an increasing sequence of sub σ-algebras of $\mathcal{A}$, and let $(X_i, \mathcal{F}_i)$ be an adapted martingale difference sequence with values in a separable Banach space $(U, \| \ \|)$, that is, $E(X_i | \mathcal{F}_{i-1}) = 0$ a.s., $i \in \mathbb{N}$. For a given probability metric μ and a sub σ-algebra $\mathcal{G} \subset \mathcal{A}$ define the $\mathcal{G}$-*dependence metric* $\mu(\cdot \| \mathcal{G})$ by

$$\mu(X, Y \| \mathcal{G}) = \sup_{V \in \mathcal{G}} \mu(X + V, Y + V), \tag{3.1}$$

where $V \in \mathcal{G}$ denotes that V is a $\mathcal{G}$-measurable random variable. This notion generalizes an idea due to Gudynas (1985).

Lemma 3.1 *If* μ *is homogeneous of order* r, *that is,*

$$\mu(cX, cY) \leq |c|^r \mu(X, Y), \tag{3.2}$$

then $\mu(\cdot \| \mathcal{G})$ *also is homogeneous of order* r.

We shall use the following metrics: $\ell_r(\cdot \| \mathcal{G})$, $\sigma_r(\cdot \| \mathcal{G})$, where ℓ_r, σ_r are the smoothed Kantorovich metric and total variation metric (cf. (2.6), (2.7)).

Lemma 3.2 *Let the regular conditional distributions $P_{X|\mathcal{G}}, P_{Y|\mathcal{G}}$ exist. Then*

$$\ell_r(X,Y\|\mathcal{G}) \le E\ell_r(P_{X|\mathcal{G}}, P_{Y|\mathcal{G}}) \tag{3.3}$$

and

$$\sigma_r(X,Y\|\mathcal{G}) \le E\sigma_r(P_{X|\mathcal{G}}, P_{Y|\mathcal{G}}). \tag{3.4}$$

Proof: Let ϑ be independent of X,Y and $\mathcal{G}$, then

$$\begin{aligned}
&\ell_r(X,Y\|\mathcal{G}) = \sup_{V\in\mathcal{G}} \ell_r(X+V, Y+V)\\
= &\sup_{\|f\|_L\le 1}\sup_{h>0}\sup_{V\in\mathcal{G}} h^{r-1}|E(f(X+V+h\vartheta)-f(Y+V+h\vartheta))|\\
\le &\; E\sup_{\|f\|_L\le 1}\sup_{h>0}\sup_{V\in\mathcal{G}} h^{r-1}|E(f(X+V+h\vartheta)|\mathcal{G})-E(f(Y+V+h\vartheta)|\mathcal{G})|\\
= &\; E\sup_{\|f\|_L\le 1}\sup_{h>0}\sup_{V\in\mathcal{G}} h^{r-1}|E(f_V(X+h\vartheta)|\mathcal{G})-E(f_V(Y+h\vartheta)|\mathcal{G})|,
\end{aligned}$$

where $f_V(\cdot) = f(\cdot + V)$ is the translation by V, which is again a Lipschitz function, and where $\|f\|_L = \sup_{x\neq y}\frac{|f(x)-f(y)|}{\|x-y\|}$ is the Lipschitz norm. We arrive at

$$\begin{aligned}
\ell_r(X,Y\|\mathcal{G}) &\le E\sup_{\|f\|_L\le 1}\sup_{h>0} h^{r-1}|E(f(X+h\vartheta)|\mathcal{G})-E(f(Y+h\vartheta)|\mathcal{G})|\\
&= E\ell_r(P_{X|\mathcal{G}}P_{Y|\mathcal{G}}).
\end{aligned}$$

The proof for the metric σ_r is similar. □

As a consequence we obtain the following regularity property of ℓ_r and σ_r.

Lemma 3.3 *Let $(X_i,\mathcal{F}_i)$ be a stochastic sequence and $(\mathcal{G}_i)$ a decreasing sequence of sub σ-algebras such that Y_j are $\mathcal{G}_i$-measurable for $j\ge i$. If*

$$\begin{aligned}
&X_i \text{ and } \mathcal{G}_{i+1} \text{ are conditionally independent given } \mathcal{F}_{i-1} \text{ and } Y_i \text{ and}\\
&\mathcal{G}_{i+1} \text{ are conditionally independent given } \mathcal{F}_{i-1}, \text{ then for } c_i\in\mathbb{R}
\end{aligned} \tag{3.5}$$

$$\ell_r\Big(\sum_{i=1}^n c_iX_i, \sum_{i=1}^n c_iY_i\Big) \le \sum_{i=1}^n |c_i|^r\, E\ell_r(P_{X_i|\mathcal{F}_{i-1}}, P_{Y_i|\mathcal{F}_{i-1}}) \tag{3.6}$$

and

$$\sigma_r\Big(\sum_{i=1}^n c_iX_i, \sum_{i=1}^n c_iY_i\Big) \le \sum_{i=1}^n |c_i|^r\, E\sigma_r(P_{X_i|\mathcal{F}_{i-1}}, P_{Y_i|\mathcal{F}_{i-1}}), \tag{3.7}$$

the conditional distributions are assumed to exist.

Proof: By Lemma 3.1,

$$\ell_r\Big(\sum_{i=1}^n c_iX_i, \sum_{i=1}^n c_iY_i\Big) \le \sum_{i=1}^n \ell_r(c_1X_1+\ldots+c_iX_i+c_{i+1}Y_{i+1}+\ldots+c_nY_n,$$

$$\begin{aligned}
& c_1X_1 + \ldots + c_{i-1}X_{i-1} + c_iY_i + \ldots + c_nY_n) \\
\leq & \sum_{i=1}^n \sup_{h>0} h^{r-1} \sup_{V\in\mathcal{F}_{i-1}\vee\mathcal{G}_{i+1}} \ell_1(c_iX_i + V + h\vartheta, c_iY_i + V + h\vartheta) \\
= & \sum_{i=1}^n \ell_r(c_iX_i, c_iY_i\|\mathcal{F}_{i-1}\vee\mathcal{G}_{i+1}) \leq \sum_{i=1}^n |c_i|^r\,\ell_r(X_1, Y_1\|\mathcal{F}_{i-1}\vee\mathcal{G}_{i+1}),
\end{aligned}$$

where $\mathcal{F}_{i-1}\vee\mathcal{G}_{i+1}$ is the σ-algebra generated by $\mathcal{F}_{i-1}$ and $\mathcal{G}_{i+1}$. From Lemma 3.2 and the conditional independence assumption

$$\begin{aligned}
\ell_r(X_i, Y_i\|\mathcal{F}_{i-1}\vee\mathcal{G}_{i+1}) & \leq E\ell_r(P_{X_i|\mathcal{F}_{i-1}\vee\mathcal{G}_{i+1}}, P_{Y_i|\mathcal{F}_{i-1}\vee\mathcal{G}_{i+1}}) \\
& = E\ell_r(P_{X_i|\mathcal{F}_{i-1}}, P_{Y_i|\mathcal{F}_{i-1}}).
\end{aligned}$$

As for the metric σ_r the proof is similar. □

Remark 3.1 *If Y_i are independent of $\mathcal{F}_{i-1}$, $EY_i = 0$, then:*

$$\begin{aligned}
\ell_r(P_{X_i|\mathcal{F}_{i-1}}, P_{Y_i}) & \leq C_r\zeta_r(P_{X_i|\mathcal{F}_{i-1}}, P_{Y_i}) \\
& \leq C_r\kappa_r(P_{X_i|\mathcal{F}_{i-1}}, P_{Y_i}),
\end{aligned} \tag{3.8}$$

where ζ_r is the Zolotarev metric and κ_r is the pseudo-difference moment (cf. Rachev (1991), p. 377). In the α-stable case $1 < \alpha < 2$ and $r = 2$ the finiteness of κ_r implies that

$$E(X_i|\mathcal{F}_{i-1}) = EY_i = 0, \tag{3.9}$$

which is fulfilled in the martingale case. In the normal case $\alpha = 2$ and $r = 3$, the finiteness of ζ_r implies in the Euclidean case that the conditional covariance

$$Cov(X_i|\mathcal{F}_{i-1}) = Cov(Y_i) \tag{3.10}$$

is almost surely constant. This and related conditions have been assumed in several papers on martingale convergence theorem (cf. Basu (1976), Dvoretzky (1970), Bolthausen (1982), Butzer et al. (1983), Häusler (1988), Rackauskas (1990)).

Lemma 3.4

$$\ell_r(X, Y) \geq (\ell_1(X, Y))^r(\frac{r-1}{r})^{r-1}\frac{1}{r}2E\|\vartheta\|. \tag{3.11}$$

Proof: By the triangle inequality and from the definition of ℓ_r

$$\ell_r(X, Y) \geq \ell_1(X + \varepsilon\vartheta, Y + \varepsilon\vartheta)\varepsilon^{r-1} \geq (\ell_1(X, Y) - 2\varepsilon E\|\vartheta\|)\varepsilon^{r-1} := \varphi(\varepsilon).$$

Maximizing $\varphi(\varepsilon)$ w.r.t. ε we obtain (3.11). □

In the next step we extend the smoothing inequality (2.12) to the present situation.

Lemma 3.5 *Let X, Z, Y, W be random variables with values in U such that (X, Z) is independent of (Y, W) and Y, W are independent, then*

$$\begin{aligned} \ell_1(X+Z, Y+Z) &\leq \ell_1(Z, W)\sigma(X, Y) + \ell_1(X+W, Y+W) \\ &+ \ell_1(Z+X, \tilde{Z}+X), \end{aligned} \tag{3.12}$$

where $\tilde{Z} \stackrel{d}{=} Z$ and $\tilde{Z}$ is independent of X, and

$$\begin{aligned} \sigma(X+Z, Y+Z) &\leq \sigma(Z, W)\sigma(X, Y) + \sigma(X+W, Y+W) \\ &+ \sigma(Z+X, \tilde{Z}+X). \end{aligned} \tag{3.13}$$

Proof: By the triangle inequality,

$$\begin{aligned} \ell_1(X+Z, Y+Z) &= \sup_{\|f\|_L \leq 1} |E[(f(X+Z) - f(X+W)) - (f(Y+Z) - f(Y+W))]| \\ &+ \sup_{\|f\|_L \leq 1} |E(f(X+W) - f(Y+W))|. \end{aligned}$$

Furthermore,

$$\begin{aligned} & |E[f(X+Z) - f(X+W) - (f(Y+Z) - f(Y+W))]| \\ = & \left| \int (E(f(Z+x)|X=x) - Ef(W+x))dP_X(x) \right. \\ & \left. - \int (Ef(Z+x) - Ef(W+x))dP_Y(x) \right| \\ \leq & \left| \int (E(f(Z+x)|X=x) - Ef(Z+x))dP_X(x) \right| \\ & + \left| \int Ef(Z+x)(dP_X(x) - dP_Y(x)) - \int Ef(W+x)(dP_X(x) - dP_Y(x)) \right| \\ \leq & \ \ell_1(Z+X, \tilde{Z}+X) + \ell_1(Z, W)\sigma(X, Y). \end{aligned}$$

The proof of (3.13) is similar. □

The last term in (3.12) is a measure of dependence of Z, X which disappears if Z, X are independent.

Making use of the smoothing properties we next prove a generalization of Theorem 2.2 to the martingale case. Let $(X_i, \mathcal{F}_i)$ be a martingale difference sequence, $Z_n = n^{1/\alpha} \sum_{j=1}^n X_j$, and as in (2.6) let ϑ, ϑ_i be independent, symmetric α-stable distributed r.v.'s. We also define for $r > \alpha$

$$\begin{aligned} \ell_r &= \sup_j \ell_r(X_j, \vartheta_j),\ \tau_r = \sup_j E\ell_r(P_{X_j|\mathcal{F}_{j-1}}, P_{\vartheta_j}),\ \tilde{\ell}_r = \ell_r \vee \tau_r, \\ \tilde{\tau}_r &= \sup_j E\ell_r(P_{X_j|\mathcal{F}_{j-1}}, P_{X_j}),\ \hat{\tau}_r = \sup_j E\ell_r(P_{X_j|\hat{\mathcal{G}}_{j+1}}, P_{X_j}), \end{aligned} \tag{3.14}$$

where $\hat{\mathcal{G}}_{j+1} = \sigma(X_{j+1}, X_{j+2}, \ldots)$, and $\sigma_r = \sup_j \sigma_r(X_j, \vartheta_j)$, the conditional distributions are assumed to exist.

Theorem 3.6 *Suppose that $E\|\vartheta\| < \infty$, then*

$$\ell_1(Z_n, \vartheta) \le C(n^{1-r/\alpha}\tilde{\ell}_r + n^{-1/\alpha} t_r), \tag{3.15}$$

where $t_r = \max(\ell_1, \sigma_1, \sigma_r^{\frac{1}{r-\alpha}}, \hat{\tau}_r^{\frac{1}{r-\alpha}}, \tilde{\tau}_1)$.

Proof: Involving (2.11) we estimate $\ell_1(Z_n + \varepsilon\vartheta,\ \vartheta_1 + \varepsilon\vartheta)$: with $m = [\frac{n}{2}]$,

$$\begin{aligned}
&\ell_1(Z_n + \varepsilon\vartheta, n^{-1/\alpha}\sum_{i=1}^n \vartheta_i + \varepsilon\vartheta) \le \ell_1(Z_n + \varepsilon\vartheta, n^{-1/\alpha}(\vartheta_1 + X_2 + \ldots + X_n) + \varepsilon\vartheta) \\
&+ \sum_{j=1}^m \ell_1(n^{-1/\alpha}(\vartheta_1 + \ldots + \vartheta_j + X_{j+1} + \ldots + X_n) + \varepsilon\vartheta, \\
&n^{-1/\alpha}(\vartheta_1 + \ldots + \vartheta_{j+1} + X_{j+2} + \ldots + X_n) + \varepsilon\vartheta) \\
&+ \ell_1(n^{-1/\alpha}(\vartheta_1 + \ldots + \vartheta_{m+1} + X_{m+2} + \ldots + X_n) + \varepsilon\vartheta, n^{-1/\alpha}\sum_{j=1}^n \vartheta_j + \varepsilon\vartheta)
\end{aligned}$$

$$= \ : I_o + \sum_{j=1}^m I_j + I_{m+1}.$$

From the generalized smoothing inequality (3.12)

$$\begin{aligned}
I_o &= \ell_1(\frac{X_2 + \ldots + X_n}{n^{1/\alpha}} + (\frac{X_1}{n^{1/\alpha}} + \varepsilon\vartheta), \frac{X_2 + \ldots + X_n}{n^{1/\alpha}} + (\frac{\vartheta_1}{n^{1/\alpha}} + \varepsilon\vartheta)) \\
&\le \ell_1(\frac{X_2 + \ldots + X_n}{n^{1/\alpha}}, \frac{\vartheta_2 + \ldots + \vartheta_n}{n^{1/\alpha}})\sigma(n^{-1/\alpha}X_1 + \varepsilon\vartheta, n^{-1/\alpha}\vartheta_1 + \varepsilon\vartheta) \\
&+ \ell_1(\frac{X_1 + \vartheta_2 + \ldots + \vartheta_n}{n^{1/\alpha}} + \varepsilon\vartheta, \frac{\vartheta_1 + \ldots + \vartheta_n}{n^{1/\alpha}} + \varepsilon\vartheta) \\
&+ \ell_1(\frac{X_1 + \ldots + X_n}{n^{1/\alpha}} + \varepsilon\vartheta, \frac{X_1 + \ldots + X_{n-1} + \tilde{X}_n}{n^{1/\alpha}} + \varepsilon\vartheta) \\
&= \ : \Delta_1 + \Delta_2 + \Delta_3,
\end{aligned}$$

where $\tilde{X}_n \stackrel{d}{=} X_n$ and $\tilde{X}_n$ is independent of $X_1, \ldots, X_{n-1}, \vartheta$. Similarly,

$$\begin{aligned}
\sum_{j=1}^m I_j &\le \sum_{j=1}^m \ell_1(\frac{X_{j+2} + \ldots + X_n}{n^{1/\alpha}}, \frac{\vartheta_{j+2} + \ldots + \vartheta_n}{n^{1/\alpha}}) \\
&\quad \sigma(\frac{\vartheta_1 + \ldots + \vartheta_j + X_{j+1}}{n^{1/\alpha}} + \varepsilon\vartheta, \frac{\vartheta_1 + \ldots + \vartheta_{j+1}}{n^{1/\alpha}} + \varepsilon\vartheta) \\
&+ \sum_{j=1}^m \ell_1(\frac{\vartheta_1 + \ldots + \vartheta_j + X_{j+1} + \vartheta_{j+2} + \ldots + \vartheta_n}{n^{1/\alpha}} + \varepsilon\vartheta, \frac{\sum_{j=1}^n \vartheta_j}{n^{1/\alpha}} + \varepsilon\vartheta) \\
&+ \sum_{j=1}^m \ell_1(\frac{\vartheta_1 + \ldots + \vartheta_j + X_{j+1} + \ldots + X_n}{n^{1/\alpha}} + \varepsilon\vartheta, \\
&\quad \frac{\vartheta_1 + \ldots + \vartheta_j + \tilde{X}_{j+1} + X_{j+2} + \ldots + X_n}{n^{1/\alpha}} + \varepsilon\vartheta) \\
&= \ : \Delta_4 + \Delta_5 + \Delta_6.
\end{aligned}$$

We first estimate Δ_5 by the ideality of ℓ_r,

$$\begin{aligned}
\Delta_5 &= \sum_{j=1}^{m} \ell_1(\frac{X_{j+1}}{n^{1/\alpha}} + (\frac{n-1}{n})^{1/\alpha}\vartheta_1 + \varepsilon\vartheta,\ \frac{\vartheta_{j+1}}{n^{1/\alpha}} + (\frac{n-1}{n})^{1/\alpha}\vartheta + \varepsilon\vartheta) \quad (3.16)\\
&\leq \sum_{j=1}^{m} \ell_1(\frac{X_{j+1}}{n^{1/\alpha}} + (\frac{n-1}{n})^{1/\alpha}\vartheta, \frac{\vartheta_{j+1}}{n^{1/\alpha}} + (\frac{n-1}{n})^{1/\alpha}\vartheta)\\
&\leq Cn^{1-r/\alpha}\ell_r.
\end{aligned}$$

Similarly, by Lemma 3.3,

$$\begin{aligned}
\Delta_7 := I_{m+1} &\leq \ell_1((\frac{m+1}{n})^{1/\alpha}\vartheta + \frac{X_{m+1}+\ldots+X_n}{n^{1/\alpha}}, \quad (3.17)\\
&\qquad (\frac{m+1}{n})^{1/\alpha}\vartheta + \frac{\vartheta_{m+1}+\ldots+\vartheta_n}{n^{1/\alpha}})\\
&\leq (\frac{m+1}{n})^{(1-r)/\alpha} n^{-r/\alpha} \sum_{j=m+1}^{n} E\ell_r(P_{X_i|\mathcal{F}_{i-1}}, P_{\vartheta_i})\\
&\leq Cn^{1-r/\alpha}\tau_r,
\end{aligned}$$

and in the same way as for Δ_5

$$\Delta_2 \leq Cn^{1-r/\alpha}\ell_r. \quad (3.18)$$

The remaining terms are dealt with by induction. Assume next that for $j < n$:

$$\ell_1(\frac{X_1+\ldots+X_j}{j^{1/\alpha}}, \frac{\vartheta_1+\ldots+\vartheta_j}{j^{1/\alpha}}) \leq B(\tilde{\ell}_r j^{1-r/\alpha} + t_r j^{-1/\alpha}) \quad (3.19)$$

and let

$$\varepsilon = A\max(\sigma_1, \sigma_r^{1/(r-\alpha)}, \ell_r^{1/(r-\alpha)}, \hat{\tau}_r^{1/(r-\alpha)})n^{-1/\alpha} \quad (3.20)$$

with a constant $A \geq 0$ which we shall fix later in the proof. Then

$$\begin{aligned}
\Delta_1 &\leq BC(n^{1-r/\alpha}\tilde{\ell}_r + n^{-1/\alpha}t_r)\varepsilon^{-1}n^{-1/\alpha}\sigma_1(X_1,\vartheta) \quad (3.21)\\
&\leq \frac{1}{A}BC(n^{1-r/\alpha}\tilde{\ell}_r + n^{-1/\alpha}t_r).
\end{aligned}$$

In the same way

$$\begin{aligned}
\Delta_4 &\leq CB(\ell_r(n-m-2)^{1-r/\alpha} \quad (3.22)\\
&\qquad + t_r(n-m-2)^{-1/\alpha}) \sum_{j=1}^{\infty} (\frac{j}{n} + \varepsilon^\alpha)^{-r/\alpha} \sigma_r(\frac{X_{j+1}}{n^{1/\alpha}}, \frac{\vartheta_{j+1}}{n^{1/\alpha}})\\
&\leq CB(\tilde{\ell}_r n^{1-r/\alpha} + t_r n^{-1/\alpha}) \sum_{j=1}^{\infty} \frac{\sigma_r(X_{j+1},\vartheta_{j+1})}{(j + A^\alpha\sigma_r(X_{j+1},\vartheta_{j+1})^{\alpha/(r-\alpha)})^{r/\alpha}}\\
&\leq CB(\tilde{\ell}_r n^{1-r/\alpha} + t_r n^{-1/\alpha})/A^{r-\alpha}
\end{aligned}$$

using that $\varepsilon^\alpha \geq A^\alpha \sigma_r^{\alpha/(r-\alpha)} n^{-1}$. For Δ_3 we use the $\mathcal{G}$-dependence metric $\ell_1(\cdot \| \mathcal{F}_{n-1})$

$$\begin{aligned}\Delta_3 &\leq \ell_1\Big(\frac{X_n}{n^{1/\alpha}} + \varepsilon\vartheta, \frac{\tilde{X}_n}{n^{1/\alpha}} + \varepsilon\vartheta \| \mathcal{F}_{n-1}\Big) \leq \ell_1\Big(\frac{X_n}{n^{1/\alpha}}, \frac{\tilde{X}_n}{n^{1/\alpha}} \| \mathcal{F}_{n-1}\Big) \qquad (3.23)\\ &\leq n^{-1/\alpha} E\ell_1(P_{X_n|\mathcal{F}_{n-1}}, P_{X_n}) \leq n^{-1/\alpha}\tilde{\tau}_1.\end{aligned}$$

It finally remains to estimate Δ_6

$$\begin{aligned}\Delta_6 &= \sum_{j=1}^m \ell_1\Big(\frac{X_{j+1}}{n^{1/\alpha}} + \frac{X_{j+2} + \ldots + X_n}{n^{1/\alpha}} + \Big(\frac{j}{n} + \varepsilon^\alpha\Big)^{1/\alpha}\vartheta,\\ &\qquad \frac{\tilde{X}_{j+1}}{n^{1/\alpha}} + \frac{X_{j+2} + \ldots + X_n}{n^{1/\alpha}} + \Big(\frac{j}{n} + \varepsilon^\alpha\Big)^{1/\alpha}\vartheta\Big)\\ &\leq \sum_{j=1}^m \Big(\frac{j}{n} + \varepsilon^\alpha\Big)^{(1-r)/\alpha} \ell_r\Big(\frac{X_{j+1}}{n^{1/\alpha}} + \frac{X_{j+2} + \ldots + X_n}{n^{1/\alpha}}, \frac{\tilde{X}_{j+1}}{n^{1/\alpha}} + \frac{X_{j+2} + \ldots + X_n}{n^{1/\alpha}}\Big)\\ &\leq \sum_{j=1}^m \frac{(j + n\varepsilon^\alpha)^{(1-r)/\alpha}}{n^{(1-r)/\alpha}} n^{-r/\alpha} \ell_r(X_{j+1}, \tilde{X}_{j+1} \| \hat{\mathcal{G}}_{j+2})\\ &\leq \tilde{\tau}_r n^{-1/\alpha} \sum_{j=1}^m (j + nA^\alpha \hat{\tau}_r^{\alpha/(r-\alpha)} n^{-1})^{(1-r)/\alpha}\\ &\leq C n^{-1/\alpha} \frac{1}{A^{r-1-\alpha}} \hat{\tau}_r^{1/(r-\alpha)}.\end{aligned}$$

Gathering all inequalities, we obtain

$$\begin{aligned}\ell_1(Z_n, \vartheta) &\leq C_1 \frac{B}{A}(n^{1-r/\alpha}\tilde{\ell}_r + n^{-1/\alpha} t_r) + C_2 n^{1-r/\alpha}\ell_r + C_3 n^{-1/\alpha}\tilde{\tau}_1\\ &+ C_4 \frac{B}{A^{r-\alpha}}(n^{1-r/\alpha}\tilde{\ell}_r + n^{-1/\alpha} t_r) + C_5 n^{1-r/\alpha}\ell_r + C_6 \frac{1}{A^{r-1-\alpha}} \hat{\tau}_r^{1/(r-\alpha)} n^{-1/\alpha}\\ &+ C_7 n^{1-r/\alpha}\tau_r + 2E\|\vartheta\| n^{-1/\alpha} \max(\sigma_1, \sigma_r^{1/(r-\alpha)}, \ell_r^{1/(r-\alpha)}, \hat{\tau}_r^{1/(r-\alpha)}).\end{aligned}$$

Choose A large enough such that $\frac{C_1}{A} + \frac{C_4}{A^{r-\alpha}} \leq \frac{1}{2}$ and then choose B large enough such that $C_2 + C_3 + C_5 + C_6 A^{1+\alpha-r} + C_7 + 2E\|\vartheta\| \leq \frac{B}{2}$. Thus one obtains (3.15). □

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