Expansion of transition distributions of Lévy processes in small time

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We establish for small time t a series expansion of the transition density and the transition distribution function of Lévy processes in terms of the density respectively the spectral function of the Lévy measure. Furthermore, the integrals w.r.t. the distribution function weighted by $1/t$ are proved to converge to the integral w.r.t. the spectral function of the Lévy measure, when the integrated function does not increase too fast and $t \to 0$.

Keywords: Lévy processes, series expansion, transition density, transition distribution function

1 Introduction

In general Lévy processes are determined by their characteristic functions. The transition density or the transition distribution function of Lévy processes typically cannot be calculated explicitly. It is however known that the local behaviour of the transition distribution P_t of the Lévy process determines the Lévy measure $dG(x)$, in the sense that for every fixed $a > 0$ $1/t \; dP_t(x)$ converges vaguely on $\{|x| > a\}$ to $dG(x)$ as $t \to 0$. In this paper we give more precise information on the behaviour of the transition distribution as $t \to 0$.

In the first part we establish a series expansion of the transition density as well as of the transition distribution function in small time. For the proof we split the process in a compound Poisson and a small jump part, which leads to a sum representation, where we can estimate each term separately. For the transition distribution function the important step is finding an estimate for the small jump part. The expansion holds true in general without any regularity conditions on the Lévy process.

In the case where transition densities and densities of the Lévy measure exist we obtain an expansion of the density in small time under some regularity assumptions. For the proof we use some results on first order estimates of pure jump type Markov processes as $t \to 0$ in Léandre (1987) and Ishikawa (1994). Related estimates for more general Lévy measures are also given in Picard (1997). The special properties of Lévy processes allow to simplify their arguments essentially and to obtain an expansion of the transition density in this case. In a recent paper of Barndorff-Nielsen (2000) independently some related results on the local expansion of densities for subordinated processes were obtained by different methods. In this paper also an interesting approach to a complete infinite expansion is given.

From our results we obtain a local expansion of the transition distribution of a Lévy process in terms of the Lévy measure. Conversely, from the local behaviour of the transition distribution we can infer on the Lévy measure. Both directions of this relationship are of principal importance for questions of statistical inference in Lévy processes which is however not dealt with in this paper.

In the second part of the paper we extend the vague convergence as $t \to 0$ of the normalized distribution of the Lévy process to the Lévy measure to the convergence of integrals of functions belonging to a certain class which do not increase too fast. The proof uses an extension of a result of Kruglov (1970) on a characterization of the tail behaviour of infinitely divisible measures in terms of the finiteness of integrals. Further we use the decomposition of the Lévy process in a compound Poisson and a small jump part as well as some approximation technique in Sato (1999). In this case the influence of the big jumps is dominating and we finally obtain approximations of the form $1/t \int f(x)dP_t(x) \to \int f(x)dG(x)$ as $t \to 0$. These results again have applications to the asymptotics of statistical estimators.

2 Expansion of the transition density

In this section we shall derive an approximation of the transition density in terms of the density of the Lévy measure for small time. Léandre (1987) and Ishikawa (1994) gave a first order estimate of the transition density of a pure jump type Markov process, when the time tends to zero. We adopt their method for proving a series expansion of the transition density of Lévy processes, which involves much less technique due to the special properties of Lévy processes.

Let X_t be a Lévy process with transition density $p_t(x)$ and characteristic function

$$\hat{p}_t(u) = \exp\left\{ t\left(i\mu u - \frac{\sigma^2}{2}u^2 + \int \left(e^{iuz} - 1 - \frac{iuz}{1+z^2}\right) g(z)dz \right)\right\}, \qquad (1)$$

where $g(z)$ denotes the density of the Lévy measure, such that $\int 1 \wedge x^2 g(x) dx < \infty$.

Theorem 1 *Assume the following conditions:*

1. *The Lévy process X_t possesses a C^∞ transition density $p_t(x)$.*
2. *The density of the Lévy measure $g(x)$ is C^∞ and for $\varepsilon > 0$*

$$\int_{|z|\geq\varepsilon} \frac{|g'(z)|^2}{g(z)} dz < \infty \tag{2}$$

$\exists h \in C^\infty$ such that $h(z) \leq c|z|^2$, $h(z) > 0$ if $g(z) > 0$ and $z \neq 0$ for which

$$\int_{|z|\leq 1} \left| \frac{d}{dz} h(z) g(z) \right|^2 \frac{dz}{g(z)} < \infty. \tag{3}$$

Then for $N \geq 1$ and $\eta > 0$ there exist $\varepsilon'(N) > 0$ and $t_0 > 0$ such that for all $0 < \varepsilon < \varepsilon'(N)$, $0 < t < t_0$ and $|x| > \eta > 0$

$$p_t(x) = e^{-t \int g_\varepsilon(y) dy} \sum_{i=0}^{N-1} \frac{t^i}{i!} g_\varepsilon(x)^{*i} + O_{\varepsilon,\eta}(t^N), \tag{4}$$

where $g_\varepsilon(x) = 1_{|x|\geq\varepsilon} g(x)$ is the truncated Lévy density.

Remark.

a) Condition 1) may be ensured by conditions on $g(x)$. For example by a result of Orey (1968) an infinitely divisible distribution μ on $\mathbb{R}$ has a density of class C^∞ and all derivatives of the density tend to 0 as $|x| \to \infty$ if the Lévy measure ν satisfies

$$\liminf_{r\to 0} \frac{\int_{[-r,r]} x^2 \nu(dx)}{r^{2-\alpha}} > 0 \tag{5}$$

for some $0 < \alpha < 2$.

b) Condition 2) is used to establish an upper bound for the transition density by Léandre (1987).

Proof. For the proof we split the process X_t in one part X_t^ε with jumps smaller than ε, where the transition density is denoted by $p_t^\varepsilon(x)$. The other part $X_t^{c,\varepsilon}$ is a compound Poisson process, whose transition density for $x \neq 0$

is denoted by $p_t^{\varepsilon,c}(x)$ and the density of the corresponding Lévy measure by $g_\varepsilon(x) = 1_{|x|\geq\varepsilon} g(x)$. This decomposition yields for all $\varepsilon > 0$

$$\begin{aligned} p_t(x) &= p_t^\varepsilon * p_t^{\varepsilon,c}(x) \\ &= e^{-t\int g_\varepsilon(y)dy}\left(p_t^\varepsilon(x) + \sum_{i=1}^{\infty} \frac{t^i}{i!} p_t^\varepsilon * g_\varepsilon^{*i}(x)\right) \\ &= e^{-t\int g_\varepsilon(y)dy}\left(p_t^\varepsilon(x) + \sum_{i=1}^{N-1} \frac{t^i}{i!} p_t^\varepsilon * g_\varepsilon^{*i}(x)\right) + \bar{p}_t^\varepsilon(N,x). \qquad (6) \end{aligned}$$

For the estimate of the first term of the sum we use the following estimate of Léandre (1987). It states that for every $p > 1$ and any $\eta > 0$ there exists $\varepsilon'(p) > 0$ such that for all $t \leq 1$ and every $\varepsilon \in (0, \varepsilon'(p))$

$$\sup_{|x|\geq\eta} e^{-t\int g_\varepsilon(y)dy} p_t^\varepsilon(x) \leq C(\varepsilon,\eta)t^p. \qquad (7)$$

For the estimate of the remainder term we apply a lemma of Ishikawa (1994) which yields that for every $\varepsilon > 0$ and $t > 0$

$$\sup_{x\in\mathbb{R}} \bar{p}_t^\varepsilon(N,x) \leq C(\varepsilon)t^N. \qquad (8)$$

Hence in particular as $t \to 0$ for every $\varepsilon > 0$

$$\bar{p}_t^\varepsilon(N,x) = O_\varepsilon(t^N). \qquad (9)$$

By the following lemma we obtain finally an upper and lower bound for each summand in the convolution (6).

Lemma 1 *Let $\mathcal{K}$ be the class of nonnegative, bounded functions. Then, given $\delta > 0$ and $\eta > 0$, there exists $t_0 > 0$ such that*

$$\begin{aligned} \int f(y-z)p_t^\varepsilon(z)dz &\geq f(y) - \delta \\ \int f(y-z)p_t^\varepsilon(z)dz &\leq f(y) + \delta + C(\varepsilon,\eta)t^p \qquad (10) \end{aligned}$$

for all $f \in \mathcal{K}$, $|y| > \eta$, $y \in C_f$, $t \in (0, t_0)$ and $\varepsilon \in (0, \varepsilon'(p))$, where $\varepsilon'(p)$ is given in (7).

Proof. The main part of the proof is similar to the proof in Ishikawa (1994). Given $\delta > 0$, choose $\chi > 0$ such that $|f(y) - f(z)| < \delta/4$ for $|y - z| < \chi$. We may assume $\chi < \eta$. Then,

$$\begin{aligned}
\int f(y-z)p_t^\varepsilon(z)dz &\geq \int_{|z|\leq\chi} f(y-z)p_t^\varepsilon(z)dz \\
&= \int_{|z|\leq\chi} (f(y-z)-f(y))p_t^\varepsilon(z)dz + f(y)\int_{|z|\leq\chi} p_t^\varepsilon(z)dz \\
&\geq (f(y)-\delta/4)\int_{|z|\leq\chi} p_t^\varepsilon(z)dz \\
&\geq f(y)-\delta
\end{aligned}$$

for $t \in (0,t_1)$ for some appropriate $t_1 > 0$, since

$$\begin{aligned}
\lim_{t\to 0}\int_{|z|\leq\chi} p_t^\varepsilon(z)dz &= \lim_{t\to 0}(P_t^\varepsilon(\chi)-P_t^\varepsilon(-\chi)) \\
&= P_0^\varepsilon(\chi)-P_0^\varepsilon(-\chi) = 1.
\end{aligned}$$

For the inequality in opposite direction we obtain for $t \in (0,t_2)$ by choosing an appropriate t_2

$$\begin{aligned}
&\int_{|z|\leq\eta} f(y-z)p_t^\varepsilon(z)dz \\
&\quad= \int_{|z|\leq\chi} (f(y-z)-f(y))p_t^\varepsilon(z)dz + f(y)\int_{|z|\leq\eta} p_t^\varepsilon(z)dz \\
&\qquad + \int_{\chi<|z|\leq\eta} (f(y-z)-f(y))p_t^\varepsilon(z)dz \\
&\quad\leq (f(y)+\delta/4)\int_{|z|\leq\eta} p_t^\varepsilon(z)dz + \sup_x f(x)\int_{\chi<|z|\leq\eta} p_t^\varepsilon(z)dz \\
&\quad\leq f(y)+\delta
\end{aligned}$$

and for $\varepsilon \in (0,\varepsilon'(p))$ by Léandre's estimate (7)

$$\int_{|z|>\eta} f(y-z)p_t^\varepsilon(z)dz \leq C'(\varepsilon,\eta)t^p \int f(y-z)dz.$$

This yields $\int f(y-z)p_t^\varepsilon(z)dz \leq f(y)+\delta+C'(\varepsilon,\eta)t^p\int f(y-z)dz$. Since $\delta > 0$ is arbitrary we obtain for $1 \leq n \leq N-1$

$$e^{-t\int g_\varepsilon(y)dy}\frac{t^n}{n!}p_t^\varepsilon * g_\varepsilon^{n*}(x) = e^{-t\int g_\varepsilon(y)dy}\frac{t^n}{n!}g_\varepsilon^{n*}(x) + O_{\varepsilon,\eta}(t^N)$$

for $t < t_0 = \min\{t_1,t_2\}$ and $\varepsilon \in (0,\varepsilon'(N))$.

Lemma 1 and the steps described before yield for any $|x| > \eta > 0$, $\varepsilon \in (0, \varepsilon'(N))$ and $t \in (0, t_0)$,

$$p_t(x) = e^{-t\int g_\varepsilon(x)dx} \sum_{i=0}^{N-1} \frac{t^i}{i!} g_\varepsilon^{*i}(x) + O_{\varepsilon,\eta}(t^N). \tag{11}$$

Corollary 1 *Under the conditions of Theorem 1 we obtain for fixed* $x \neq 0$

$$\lim_{t\to 0} \frac{1}{t} p_t(x) = g(x). \tag{12}$$

Example 1 (Gamma process)
For the Gamma process the transition density is given by

$$\begin{aligned} p_t(x, \alpha, \beta) &= \frac{\beta^{\alpha t} x^{\alpha t - 1} e^{-\beta x}}{\Gamma(\alpha t)} \\ &= \frac{\alpha t \beta^{\alpha t} x^{\alpha t - 1} e^{-\beta x}}{\Gamma(\alpha t + 1)}, \text{ and} \end{aligned}$$

$$\lim_{t\to 0} \frac{1}{t} p_t(x, \alpha, \beta) = \frac{\alpha e^{-\beta x}}{x}$$

which is indeed the density of the Lévy measure.

Example 2 (Normal inverse Gaussian process)
For the Normal inverse Gaussian process the transition density is given by

$$\begin{aligned} & p_t(x, \alpha, \beta, \delta, \mu) \\ &= \frac{\alpha e^{t\delta\sqrt{\alpha^2-\beta^2} - t\beta\mu}}{\pi} \frac{t\delta}{\sqrt{t^2\delta^2 + (x - t\mu)^2}} K_1(\alpha\sqrt{t^2\delta^2 + (x - t\mu)^2}) e^{\beta x}. \end{aligned}$$

For $x \neq 0$ this leads to

$$\lim_{t\to 0} \frac{1}{t} p_t(x, \alpha, \beta, \delta, \mu) = \frac{\delta\alpha K_1(\alpha|x|) e^{\beta x}}{\pi|x|}, \tag{13}$$

which is the density of the Lévy measure.

3 Approximate distribution function

In this section we give an expansion of the transition distribution function $P_t(x)$, without regularity assumptions on the existence of the densities of the Lévy measure or the transition distribution function. Let $G(x)$ denote the spectral function of the Lévy measure, i.e. $G(x) = \nu([x, \infty))$ for $x > 0$ and $G(x) = \nu((-\infty, x])$ for $x < 0$.

The proof of the expansion is similar to the one in the case with transition densities. We have to find analogues to the estimates of the different terms in that proof.

Lemma 2
Let A be the smallest nonnegative number such that $\{x : |x| \leq A\}$ contains the support of the Lévy measure $dG(x)$, which should not be identically zero. Then for all $x > 0$ and $a < 1/A$ there exists a constant $K > 0$ such that as $t \to 0$

$$\begin{aligned} 1 - P_t(x) &\leq K \exp\{ax - ax \log x\} t^{xa} \\ P_t(-x) &\leq K \exp\{ax - ax \log x\} t^{xa}. \end{aligned}$$

Proof. We use essentially ideas of the proof of Sato (1973, Lemma 1) and of Zolotarev (1965, Theorem 1).

For Lévy measures with bounded support we may write

$$\begin{aligned} \hat{P}_t(u) &= \exp\left\{ t \left(i\mu u + \frac{\sigma^2}{2}(iu)^2 + \int (e^{iuz} - 1 - iuz) dG(z) \right) \right\} \\ &= \exp\{t\psi(iu)\} \end{aligned}$$

and

$$\psi(x) = \mu x + \sigma^2/2x^2 + \int (e^{xz} - 1 - xz) dG(z)$$

exists for any real number x. We have

$$\begin{aligned} \psi'(x) &= \mu + \sigma^2 x + \int (e^{xz} - 1) z dG(z) \\ \psi''(x) &= \sigma^2 + \int z^2 e^{xz} dG(z) \end{aligned}$$

Since $\psi''(x) > 0$ the function $\psi'(x)$ is monotone increasing and hence there exists a monotone increasing inverse function τ for $0 \leq x < \infty$, which is given by

$$x = \psi'(\tau(x)) = \mu + \sigma^2 \tau(x) + \int (e^{\tau(x)y} - 1) y dG(y) \tag{14}$$

for $0 \le x < B \le \infty$. We consider $\tau(x)$ as defined on $0 \le x < \infty$, letting $\tau(x) = \infty$ for $x \ge B$ if $B < \infty$.

Now we want to find an upper bound for $1 - P_t(x)$. From Chebyshev's inequality we obtain

$$1 - P_t(x) = P\left(X_t \ge x\right) \le \frac{E\exp\{sX_t\}}{e^{sx}} = \exp\left\{t\psi(s) - sx\right\}.$$

Minimizing the right hand side over s, this implies

$$\inf_s \exp\{t\psi(s) - sx\} = \exp\left\{-t\int_0^{x/t} \tau(z)dz\right\}.$$

For the proof note that

$$\frac{\partial}{\partial s}\exp\{t\psi(s) - sx\} = (t\psi'(s) - x)\exp\{t\psi(s) - sx\} = 0$$

and, therefore, $\psi'(s) = x/t$ and $s = \tau(\psi'(s)) = \tau(x/t)$. Hence

$$\inf_s \exp\{t\psi(s) - sx\} = \exp\{t(\psi(\tau(x/t)) - \tau(x/t)\psi'(\tau(x/t)))\}.$$

This together with

$$\begin{aligned}
&\psi\left(\tau\left(\frac{x}{t}\right)\right) - \tau\left(\frac{x}{t}\right)\psi'\left(\tau\left(\frac{x}{t}\right)\right) \\
&= \int_0^{\tau\left(\frac{x}{t}\right)} \psi'(u)du - \tau\left(\frac{x}{t}\right)\psi'\left(\tau\left(\frac{x}{t}\right)\right) \\
&= \int_{\psi'(0)}^{x/t} \psi'(\tau(w))d\tau(w) - \tau\left(\frac{x}{t}\right)\psi'\left(\tau\left(\frac{x}{t}\right)\right) \\
&= \int_{\psi'(0)}^{x/t} w d\tau(w) - \tau\left(\frac{x}{t}\right)\frac{x}{t} \quad = \quad -\int_{\psi'(0)}^{x/t}\tau(z)dz
\end{aligned}$$

leads to the desired estimate.

The next step is to find an estimate for τ. Since by Taylor expansion $(e^{\tau(x)y} - 1)y \le e^{\tau(x)A}\tau(x)y^2$ for $|y| \le A$, we obtain from (14)

$$x \le \mu + \sigma^2\tau(x) + e^{\tau(x)A}\tau(x)\int y^2 dG(y).$$

and, therefore, letting $a < 1/A$, we get $\log z \le \tau(z)/a$ for large enough z, say $z > x_0$. Hence for t small enough, i.e. such that $x/t > x_0$

$$\begin{aligned}
1 - P_t(x) &\le \exp\left\{-ta\int_{x_0}^{x/t}\log z dz\right\} \\
&= \exp\left\{-ta[-z + z\log z]_{x_0}^{x/t}\right\} \\
&= \exp\left\{-tax_0 + tax_0\log x_0 + ax - ax\log x\right\}t^{ax}. \qquad (15)
\end{aligned}$$

Similar calculations yield the corresponding result for $P_t(-x)$.

Now we can give the analogue to Léandre's estimate (7) of the part with small jumps.

Corollary 2 *Let ε be the smallest nonnegative number such that $\{x : |x| \leq \varepsilon\}$ contains the support of the Lévy measure $dG(x)$, which should not be identically zero. Then, for all $\eta > 0$ and $N \geq 1$ there exists $\varepsilon'(N) > 0$ such that for any $\varepsilon \in (0, \varepsilon'(N))$*

$$\begin{aligned} \sup_{x>\eta}(1 - P_t^\varepsilon(x)) &= O_{\varepsilon,\eta}(t^N) \\ \sup_{-x<-\eta} P_t^\varepsilon(-x) &= O_{\varepsilon,\eta}(t^N) \end{aligned} \tag{16}$$

as $t \to 0$.

Proof. Choose $\varepsilon'(N)$ such that $\eta/\varepsilon'(N) > N$. Then there exists $a < 1/\varepsilon'(N)$ with $a\eta \geq N$. Hence we have $a < 1/\varepsilon \ \forall \varepsilon \in (0, \varepsilon'(N))$ and $ax \geq N \ \forall x > \eta$. Furthermore $\sup_{x>\eta} \exp\{ax - ax \log x\} = c(\varepsilon, \eta)$.

The next lemma is analogous to Lemma 1 and gives estimates for the convolution terms $1 \leq n \leq N-1$.

Lemma 3 *Let $\mathcal{K}$ be the class of nonnegative, bounded functions. Fix $N \geq 1$ and $\eta > 0$; then there exists $\varepsilon'(N)$ and $t_0 > 0$ such that for given $\delta > 0$, $\varepsilon \in (0, \varepsilon'(N))$ and $t \in (0, t_0)$ we obtain*

$$\begin{aligned} \int f(y-z)dP_t^\varepsilon(z) &\geq f(y) - \delta \\ \int f(y-z)dP_t^\varepsilon(z) &\leq f(y) + \delta + C(\varepsilon,\eta)t^N \end{aligned} \tag{17}$$

for all $f \in \mathcal{K}$ and $|y| > \eta$, $y \in C_f$.

Proof. The proof is essentially the same as the proof of Lemma 1. At the final estimate we apply Corollary 2 to obtain

$$\begin{aligned} \int_{|z|>\eta} f(y-z)dP_t^\varepsilon(z) &= \int_\eta^\infty f(y-z)dP_t^\varepsilon(z) + \int_{-\infty}^{-\eta} f(y-z)dP_t^\varepsilon(z) \\ &\leq \sup_x f(x)(1 - P_t^\varepsilon(\eta) + P_t^\varepsilon(-\eta)) \\ &\leq \sup_x f(x)C(\varepsilon,\eta)t^N. \end{aligned}$$

Next we need an estimate for the remainders.

Lemma 4 *Let* $\bar{P}_t(\varepsilon, N, x) = \sum_{i=N}^{\infty} \frac{t^i}{i!} P_t^{\varepsilon} * G_{\varepsilon}^{*i}(x)$. *For every* $\varepsilon > 0$ *and* $t > 0$ *we have*

$$\sup_x \bar{P}_t(\varepsilon, N, x) \leq C(\varepsilon) t^N. \tag{18}$$

Proof. The proof is similar to that in Ishikawa (1994, Lemma 4.1).

Theorem 2 *Let* P_t *be the distribution function of* X_t *and* G *the spectral function of the associated Lévy measure, i.e.* $G(x) = \nu([x, \infty))$ *for* $x > 0$ *and* $G(x) = \nu((-\infty, x])$ *for* $x < 0$. *Fix* $N \geq 1$*; then there exists* $\varepsilon'(N) > 0$ *and* $t_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon'(N))$ *and* $t \in (0, t_0)$ *we obtain:*
1) For $x > \eta > 0$ *holds*

$$1 - P_t(x) = \sum_{i=1}^{N-1} \frac{t^i}{i!} G_{\varepsilon}^{*i}(x) + O_{\varepsilon,\eta}(t^N). \tag{19}$$

2) For $x < -\eta < 0$ *holds*

$$P_t(x) = \sum_{i=1}^{N-1} \frac{t^i}{i!} G_{\varepsilon}^{*i}(x) + O_{\varepsilon,\eta}(t^N), \tag{20}$$

where $G_{\varepsilon}(x)$ *denotes the truncated spectral function* $1_{|x|\geq\varepsilon} G(x)$.

Proof. We split the process X_t into two components, one $X_t^{c,\varepsilon}$ a compound Poisson process with Lévy measure $1_{|x|\geq\varepsilon} dG(x) = dG_{\varepsilon}(x)$ and the other X_t^{ε} with Lévy measure $1_{|x|<\varepsilon} dG(x)$, which is a process of the type as in Lemma 2. Hence

$$\begin{aligned} P^{X_t} &= P^{X_t^{\varepsilon}} * P^{X_t^{\varepsilon,c}} \\ &= e^{-t|G_{\varepsilon}|} \left(\sum_{i=0}^{\infty} \frac{t^i}{i!} P^{X_t^{\varepsilon}} * \nu_{\varepsilon}^{*i} \right), \end{aligned}$$

$$\begin{aligned} 1 - P_t(x) &= P^{X_t}([x, \infty)) \\ &= e^{-t|G_{\varepsilon}|} \left(1 - P_t^{\varepsilon}(x) + \sum_{i=1}^{\infty} \frac{t^i}{i!} P^{X_t^{\varepsilon}} * \nu_{\varepsilon}^{*i}([x, \infty)) \right) \\ &= e^{-t|G_{\varepsilon}|} \left(1 - P_t^{\varepsilon}(x) + \sum_{i=1}^{N-1} \frac{t^i}{i!} P_t^{\varepsilon} * G_{\varepsilon}^{*i}(x) + \overline{P}_t(\varepsilon, N, x) \right), \end{aligned}$$

and

$$\begin{aligned}
P_t(x) &= P^{X_t}((-\infty, x]) \\
&= e^{-t|G_\varepsilon|}\left(P_t^\varepsilon(x) + \sum_{i=1}^{\infty} \frac{t^i}{i!} P^{X_t^\varepsilon} * \nu_\varepsilon^{*i}((-\infty, x])\right) \\
&= e^{-t|G_\varepsilon|}\left(P_t^\varepsilon(x) + \sum_{i=1}^{N-1} \frac{t^i}{i!} P_t^\varepsilon * G_\varepsilon^{*i}(x) + \bar{P}_t(\varepsilon, N, x)\right).
\end{aligned}$$

Now the proof is similar to the one of Theorem 1, using Corollary 2, Lemma 3 and Lemma 4.

As a corollary we get that the Lévy measure can be obtained from the distribution function of the transition probability measure.

Corollary 3 *Let P_t be the distribution function of X_t and G the associated spectral function, then we obtain:*

1) For each fixed $x > 0$ holds

$$\lim_{t \to 0} \frac{1}{t}(1 - P_t(x)) = G(x). \tag{21}$$

2) For each fixed $x < 0$ holds

$$\lim_{t \to 0} \frac{1}{t} P_t(x) = G(x). \tag{22}$$

If P_t and G possess densities p_t and g, then for $x \neq 0$

$$\lim_{t \to 0} \frac{1}{t} p_t(x) = \frac{\partial}{\partial t} p_t(x)|_{t=0} = g(x), \tag{23}$$

when we additionally assume that $P_t(x)$ is continuous in a neighbourhood of $(t = 0, x)$ and furthermore that $\frac{\partial}{\partial t} P_t(x), \frac{\partial}{\partial x} P_t(x), \frac{\partial}{\partial t}\frac{\partial}{\partial x} P_t(x)$ exist and are continuous in $(t = 0, x)$.

Remark.

a) Since G is a spectral function, density in this case means $g \geq 0$ and $G'(x) = g(x)$ for $x < 0$ and $G'(x) = -g(x)$ for $x > 0$.

b) This theorem sharpens the well-known property, that for every fixed $a > 0$ $1/t\, P(X_t \in dx)$ converges vaguely on $\{|x| > a\}$ to $dG(x)$ as $t \to 0$ (cf. Bertoin (1996, p. 39)).

Proof. The estimates for the distribution functions are an obvious consequence of Theorem 2, by choosing $\varepsilon < |x|$.

Note that $P_0(x) = 0$ as $x < 0$ and 1 as $x > 0$ and define $p_0(x) = 0$ for $x \neq 0$. Hence for $x > 0$, and interchanging the differentiations

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{t} p_t(x) &= \frac{\partial}{\partial t} p_t(x)|_{t=0} &&= \frac{\partial}{\partial t}\frac{\partial}{\partial x}(P_t(x))|_{t=0} \\
&= -\frac{\partial}{\partial t}\frac{\partial}{\partial x}(1 - P_t(x))|_{t=0} &&= -\frac{\partial}{\partial x}\frac{\partial}{\partial t}(1 - P_t(x))|_{t=0} \\
&= -\frac{\partial}{\partial x} G(x) &&= g(x).
\end{aligned}$$

Similar calculations for $x < 0$ yield the desired result.

4 Asymptotics for integrals with respect to dP_t

Since the results in sections 2 and 3 are only valid for fixed $x \neq 0$, they do not provide enough information about the asymptotic behaviour of integrals w.r.t. dP_t. In the following we shall derive such an asymptotic result for functions which basically do not increase too fast. The idea is again splitting the integral in two parts, one with the small jumps and one with big jumps.

In the following section P denotes an infinitely divisible probability measure with Lévy measure dG. The tail behaviour of the distribution for Lévy measures with bounded support is estimated by the following result of Sato (1973) which is used throughout this section and for the ease of reference restated.

Theorem 3 (Sato (1973)) *Let A be the smallest nonnegative number (possibly $+\infty$) such that the set $\{x : |x| \leq A\}$ contains the support of $G(x)$. Then, for every positive $\alpha < 1/A$,*

$$\int_{|x|>r} dP(x) = o(r^{-\alpha r}) \tag{24}$$

as $r \to \infty$ and P has finite $e^{\alpha|x|\log|x|}$-moment. Whereas for every $\alpha > 1/A$

$$\int_{|x|>r} dP(x)/r^{-\alpha r} \to \infty \tag{25}$$

as $r \to \infty$ and P has infinite $e^{\alpha|x|\log|x|}$-moments.

Denote by $\mathcal{S}$ the class of functions f that can be decomposed into a submultiplicative and subadditive component.

$$\begin{aligned}\mathcal{S} \;=\; & \{f(x)|\exists H, K such that\; \forall x, y \in I\!R : \\ & f(x+y) = h(x+y)k(x+y) \leq HK(h(x)+h(y))k(x)k(y)\}\end{aligned} \tag{26}$$

Functions belonging to $\mathcal{S}$ cannot increase faster than $x^k e^{cx}$ (Kruglov (1970)).

The tail behaviour of the transition distribution function is formulated in terms of integrability conditions w.r.t. the Lévy measure next.

Theorem 4 *For all functions ℓ, with $|\ell| \leq f, f \in \mathcal{S}$ and $\int_{|x|\geq 1} f(x)dG(x) < \infty$ and $\int_{|x|\geq 1} |k(x)|dG(x) < \infty$, we obtain $\int \ell(x)dP(x) < \infty$.*

Proof. Theorem 4 is an extension of a theorem in Kruglov (1970). In this paper it has been formulated for functions which are either submultiplicative or subadditive. The proof of Kruglov can be extended to the more general class. In the first step P is decomposed $P(x) = F_1 * F_2(x)$ where F_1 is a compound Poisson distribution and F_2 corresponds to the small jumps. We use the refined estimate in Sato's theorem (Theorem 3) for the decay of F_2 with Lévy measure with bounded support. Kruglov's induction proof for the decay of F_1 can be extended to the more general class $\mathcal{S}$ to yield the result.

Lemma 5 *Weak convergence of $P^{X_t^\varepsilon}(dx)$ to $\varepsilon_{\{0\}}(dx)$ as $t \to 0$ may be extended to functions $f \in C^1$ such that $|f| \leq g_1$, and $|f'| \leq g_2$ for $g_1, g_2 \in \mathcal{S}$, i.e.*

$$\lim_{t\to 0} \int f(x)dP_t^\varepsilon(x) = f(0). \tag{27}$$

Remark. Actually the proof only needs that f and $|f'|$ do not increase faster than $x^k e^{cx}$ for some k and c.

Proof. Integration by parts leads to

$$\int_0^b f'(x)\int_x^\infty dP_t^\varepsilon(y)dx = \left[f(x)\int_x^\infty dP_t^\varepsilon(y)\right]_0^b + \int_0^b f(x)dP_t^\varepsilon(x)$$

which yields

$$\int_0^\infty f'(x)\int_x^\infty dP_t^\varepsilon(y)dx = -f(0)\int_0^\infty dP_t^\varepsilon(y) + \int_0^\infty f(x)dP_t^\varepsilon(x)$$

since $f(b)\int_b^\infty dP_t^\varepsilon(x) \to 0$ for $b \to \infty$ as $|f| \leq g_1$ with $g_1 \in \mathcal{S}$ by Lemma 2. Analogous calculations for the negative part yield

$$\int_{-\infty}^0 f'(x)\int_{-\infty}^x dP_t^\varepsilon(y)dx = f(0)\int_{-\infty}^0 dP_t^\varepsilon(y) - \int_{-\infty}^0 f(x)dP_t^\varepsilon(x)$$

and hence together

$$\int f(x)dP_t^\varepsilon(x) = f(0) + \int_0^\infty f'(x)\int_x^\infty dP_t^\varepsilon(y)dx - \int_{-\infty}^0 f'(x)\int_{-\infty}^x dP_t^\varepsilon(y)dx.$$

Looking at the second term we obtain by using Lemma 2

$$\begin{aligned}
&\left|\int_0^\infty f'(x)\int_x^\infty dP_t^\varepsilon(y)dx\right| \\
&\le \int_0^\infty |f'(x)|(1-P_t^\varepsilon(x))dx \\
&\le \int_0^\infty |f'(x)|\exp\{-tax_0 + tax_0\log x_0 + ax - ax\log x\}t^{ax}dx \\
&\le \exp\{-tax_0 + tax_0\log x_0\}\int_0^\infty |f'(x)|\exp\{ax - ax\log x\}dx < \infty
\end{aligned}$$

as $t \le 1$ and $x/t \ge x_0$. Hence by dominated convergence

$$\lim_{t\to 0}\int_0^\infty f'(x)\int_x^\infty dP_t^\varepsilon(y)dx = 0$$

Analogous calculations for the third term yield $\lim_{t\to 0}\int_{-\infty}^0 f'(x)\int_{-\infty}^x dP_t^\varepsilon(y)dx = 0$ and together we obtain

$$\lim_{t\to 0}\int f(x)dP_t^\varepsilon(x) = f(0).$$

Lemma 6 *If* $\int_{|x|\ge 1} f(x)dG(x) < \infty$, $f(0) = 0$, *and* $f(x)(|x|^2 \wedge 1)^{-1} \in C_b$, *then*

$$\lim_{t\to 0}\frac{1}{t}\int f(x)dP_t(x) = \int f(x)dG(x). \tag{28}$$

Proof. The proof is analogous to the proofs in Sato (1999, Thm. 8.7, 8.1, 8.9). Let $\{t_n\}$ be any sequence such that $t_n \downarrow 0$ and $\mu = P_1$ a given infinitely divisible probability measure. Define μ_n by

$$\begin{aligned}
\hat\mu_n(z) &= \exp[t_n^{-1}(\hat\mu(z)^{t_n} - 1)] \\
&= \exp[t_n^{-1}\int_{\mathbb{R}\setminus\{0\}}(e^{ixz} - 1)dP_{t_n}(x)].
\end{aligned}$$

Hence μ_n is compound Poisson and for each z as $n \to \infty$

$$\begin{aligned}
\hat\mu_n(z) &= \exp[t_n^{-1}(e^{t_n\log\hat\mu(z)} - 1)] \\
&= \exp[t_n^{-1}(t_n\log\hat\mu(z) + O(t_n^2))].
\end{aligned}$$

This implies $\hat{\mu}_n(z) \to \hat{\mu}(z)$ and thus $\mu_n \stackrel{\mathcal{D}}{\to} \mu$.

Now define $\rho_n(dx) = (|x|^2 \wedge 1)\nu_n(dx)$, where $\nu_n(dx)$ denotes the Lévy measure of $\mu_n(dx)$. Then from Sato (1999, Theorem 8.7) we obtain that $\{\rho_n\}$ is tight, i.e. $\sup_n \rho_n(\mathbb{R}^d) < \infty$ and $\lim_{\ell\to\infty} \sup_n \int_{|x|>\ell} \rho_n(dx) = 0$. So there exists a subsequence $\{\rho_{n_k}\}$ such that $\rho_{n_k} \stackrel{\mathcal{D}}{\to} \rho$, where ρ is a finite measure. Define $\nu(dx) = (|x|^2 \wedge 1)^{-1}\rho(dx)$ on $\{|x| > 0\}$ and $\nu(\{0\}) = 0$. As $f(x)(|x|^2 \wedge 1)^{-1} \in C_b$ we obtain

$$\begin{aligned} \lim_{k\to\infty} \int f(x)\nu_{n_k}(dx) &= \lim_{k\to\infty} \int f(x)(|x|^2 \wedge 1)^{-1}\rho_{n_k}(dx) \\ &= \int f(x)(|x|^2 \wedge 1)^{-1}\rho(dx) \\ &= \int f(x)\nu(dx). \end{aligned}$$

This implies the result since $\nu(dx)$ is unique by Sato (1999, p. 43) and noting that $\nu_n(dx) = t_n^{-1} dP_{t_n}(x)$.

Theorem 5 *For all functions f with $f(0) = 0$ and*

1) $|f| \le g_1$ *and* $|f'| \le g_2$ *for* $g_1, g_2 \in \mathcal{S}$

2) $1_{|x|\le 1} f(x)(|x|^2 \wedge 1)^{-1} \in C_b$

3) $\left|\int f(x+y)dG_\varepsilon(y)\right| \le g_3, \left|\frac{\partial}{\partial x}\int f(x+y)dG_\varepsilon(y)\right| \le g_4$ *for* $g_3, g_4 \in \mathcal{S}$

we have

$$\lim_{t\to 0} \frac{1}{t} \int f(x)dP_t(x) = \int f(x)dG(x). \tag{29}$$

Remark. For the case when $\int(|x| \wedge 1)dG(x) < \infty$ it is sufficient to postulate $1_{|x|\le 1} f(x)\ (|x| \wedge 1)^{-1} \in C_b$ and when $\int dG(x) < \infty$ to assume that $1_{|x|\le 1} f \in C_b$.

Proof. For the proof we use the technique splitting the process in a compound Poisson process and a process with small jumps, $P_t(x) = \exp\{-t \int dG_\varepsilon(x)\} \left(P_t^\varepsilon(x) + \sum_{i=1}^\infty \frac{t^i}{i!} G_\varepsilon^{i*} * P_t^\varepsilon(x)\right)$ where $G_\varepsilon = 1_{|x|>\varepsilon} G$ and P_t^ε denotes the distribution function belonging to the jumps less than or equal ε, i.e with Lévy measure $1_{|x|\le\varepsilon} dG(x)$. For the limit we then only have to consider the first two terms of the sum. The second leads to

$$\begin{aligned}\lim_{t\to 0}\int f(x)dG_\varepsilon * P_t^\varepsilon(x) &= \lim_{t\to 0}\int\int f(x+y)dG_\varepsilon(y)dP_t^\varepsilon(x)\\ &= \int f(x)dG_\varepsilon(x)\\ &= \int_{|x|>\varepsilon} f(x)dG(x),\end{aligned}$$

which follows immediately from Lemma 5 and assumption 3).

For the first summand we split the integral into two parts, one pertaining to the integral over $|x|\le\varepsilon$ and one to the integral over $|x|>\varepsilon$. From the preceding lemma we already know that

$$\lim_{t\to 0}\frac{1}{t}\int_{|x|\le\varepsilon} f(x)dP_t(x) = \int_{|x|\le\varepsilon} f(x)dG(x).$$

For the other part of the first summand we use the same technique as in Lemma 5. By assumption 1) we have

$$\begin{aligned}\int f(x)dP_t^\varepsilon(x) &= f(0)+\int_0^\infty f'(x)\int_x^\infty dP_t^\varepsilon(y)dx\\ &\quad -\int_{-\infty}^0 f'(x)\int_{-\infty}^x dP_t^\varepsilon(y)dx\end{aligned}$$

hence

$$\int_{|x|>\varepsilon} f(x)dP_t^\varepsilon(x) = \int_{x>\varepsilon} f'(x)\int_x^\infty dP_t^\varepsilon(y)dx - \int_{x<-\varepsilon} f'(x)\int_{-\infty}^x dP_t^\varepsilon(y)dx.$$

For the first term on the right hand side we then obtain by Lemma 2

$$\begin{aligned}&\left|\frac{1}{t}\int_{x>\varepsilon} f'(x)\int_x^\infty dP_t^\varepsilon(y)dx\right|\\ &\le \frac{1}{t}\int_{x>\varepsilon}|f'(x)|(1-P_t^\varepsilon(x))dx\\ &\le \int_{x>\varepsilon}|f'(x)|\exp\{-tax_0+tax_0\log x_0+ax-ax\log x\}t^{ax-1}dx\\ &\le \exp\{-tax_0+tax_0\log x_0\}\int_{x>\varepsilon}|f'(x)|\exp\{ax-ax\log x\}dx<\infty\end{aligned}$$

for $t\le 1$, $x/t\ge x_0$ and $a<1/\varepsilon$. Hence by dominated convergence we may interchange limit and integration and obtain

$$\lim_{t\to 0}\frac{1}{t}\int_{x>\varepsilon} f'(x)\int_x^\infty dP_t^\varepsilon(y)dx = 0.$$

Analogous calculations for the second term yield

$$\lim_{t\to 0}\frac{1}{t}\int_{x<-\varepsilon} f'(x)\int_{-\infty}^{x} dP_t^{\varepsilon}(y)dx = 0,$$

and hence

$$\lim_{t\to 0}\frac{1}{t}\int_{|x|>\varepsilon} f(x)dP_t^{\varepsilon}(x) = 0.$$

Piecing together the three estimates we then obtain the desired result.

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References

Barndorff-Nielsen, O. E. (2000). Probability densities and Lévy densities. MPS-RR 2000-18, MaPhySto, Aarhus.

Bertoin, J. (1996). *Lévy Processes.* Cambridge: Cambridge University Press.

Ishikawa, Y. (1994). Asymptotic behavior of the transition density for jump type processes in small time. *Tôhoku Math. J. 46*, 443–456.

Kruglov, V. M. (1970). A note on infinitely divisible distributions. *Theory of Probability and Applications 15*, 319–324.

Léandre, R. (1987). Densité en temps petit d'un processus de sauts. In *Séminaire de Probabilités XXI*, Volume 1247 of *Lect. N. in Math.* Springer.

Orey, S. (1968). On continuity properties of infinitely divisible functions. *Ann. Math. Stat. 39*, 936–937.

Picard, J. (1997). Density in small time for Lévy processes. *ESAIM: Probability and Statistics 1*, 357–389.

Sato, K. (1973). A note on infinitely divisible distributions and their Lévy measures. *Sci. Rep. Tokyo Kyoiku Daigaku, Sec. A 12*, 101–109.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge: Cambridge University Press.

Zolotarev, V. M. (1965). Asymptotic behavior of the distributions of processes with independent increments. *Theory of Probability and Applications 10*, 28–44.