On stochastic recursive equations of sum- and max-type

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Abstract

In this paper we consider stochastic recursive equations of sum-type $X \stackrel{d}{=} \sum_{i=1}^{K} A_i X_i + b$ and of max-type $X \stackrel{d}{=} \max(A_i X_i + b_i; 1 \le i \le k)$ where A_i, b_i, b are random and (X_i) are iid copies of X. Equations of this type typically characterize limits in the probabilistic analysis of algorithms, in combinatorial optimization problems as well as in many other problems having a recursive structure. We develop some new contraction properties of minimal L_s-metrics which allow to establish general existence and uniqueness results for solutions without posing any moment conditons. As application we obtain a one to one relationship between the set of solutions of the homogeneous equation and the set of solutions of the inhomogeneous equation for sum- and max-type equations. We also give a stochastic interpretation of a recent transfer principle of Rösler (2003) from nonnegative solutions of sum-type to those of max-type by means of random scaled Weibull distributions.

1 Introduction

Stochastic recursive equations of the sum and max-type arise in a great variety of problems with a recursive stochastic component as in the probabilistic analysis of algorithms or in combinatorial optimization problems. For a list of examples in these areas see the recent survey of Aldous and Bandyopadhyay (2004) on max-recursive equations and Neininger and Rüschendorf (2004a) on additive equations. In particular the limiting distribution of parameters

of algorithms of divide and conquer type leads typically for additive parameters like path length or insertion depth in random trees to equations of the sum-type while parameters like worst case behaviour, height of random trees and others lead typically to equations of max-type. The contraction method is an effective tool for proving limit theorems and existence and uniqueness results for recursive algorithms and in particular for recursive equations. The method was introduced for the analysis of the Quicksort algorithm in Rösler (1991) and then developed further independently in Rösler (1992) and Rachev and Rüschendorf (1995) (this paper was submitted in 1990). It was then used and extended to the analysis of a large variety of algorithms in a series of papers; see in particular Rösler (2001) and Neininger and Rüschendorf (2004a, 2004b, 2005) which give general and easy to apply conditions for convergence results. The contraction method has also been successfully applied to some nonlinear stochastic equations as e.g. for the analysis of iterated function systems, random fractal measures and fractal stochastic processes (see [24, 12, 13, 14]).
There has been an extensive literature on the characterization and existence of additive equations of the sum-type (as for branching type processes) and quite general existence results are known for the homogeneous nonnegative case (see [2, 3, 4, 7, 17] and [18]). Contraction arguments based on suitable probability metrics for this problem are given in [26, 24, 5].
In particular the minimal L_s-metric ℓ_s and the Zolotarev metric ζ_s have been applied to stochastic equations. For sum recursions the metrics ℓ_2 and ζ_s are particularly well suited. They yield good contraction factors for the distributional operator T on the set M_s of distributions with finite s-th moments

$$\begin{aligned} T:\ M_s &\to M_s \quad (s = 2 \text{ in case } \ell_2) \\ TX &\stackrel{d}{=} \sum_{i=1}^K A_i X_i + b \end{aligned} \tag{1.1}$$

where (X_i) are iid copies of X and $(A_i, b)_{,1\le i\le K}$ are independent of (X_i) and $\stackrel{d}{=}$ denotes equality in distribution. One obtains:

$$\ell_2^2(TX, TY) \le E\left(\sum_{i=1}^K A_i^2\right) \ell_2^2(X, Y) \tag{1.2}$$

if $EX = EY$ and for all $s > 0$

$$\zeta_s(TX, TY) \le E\sum_{i=1}^K |A_i|^s \quad \zeta_s(X, Y) \tag{1.3}$$

see [26, 23, 24]. For $0 < s \le 1$, ℓ_s has the same good contraction factor $E\sum_{i=1}^K |A_i|^s$ as the ζ_s-metric but for $1 < s < 2$ one only obtains

$$\ell_s(TX, TY) \le K_s\left(E\sum_{i=1}^K |A_i|^s\right)^{\frac{1}{s}} L_s(X, Y) \tag{1.4}$$

for any coupling (X,Y) with $E(Y-X)=0$ with a constant $K_s>1$ see Rachev and Rüschendorf (1992). Inequality (1.4) is based on Woyczynskis inequality. It is valid in general Banach spaces under a type condition. For real random variables the type is 2 and K_s can be taken as $K_s = 18 s^{\frac{3}{2}}(s-1)^{\frac{1}{2}}$ for $1<s\le 2$.

For max-type recursions it has been established in Rachev and Rüschendorf (1992, 1995) and Neininger and Rüschendorf (2005) that the minimal L_s-metric ℓ_s is also well suited even not being an ideal metric in the sense of Zolotarev. For the max operator

$$TX \stackrel{d}{=} \bigvee_{i=1}^{K} (A_i X_i + b_i) \tag{1.5}$$

where again (X_i) are iid copies of X, independent of $(A_i, b_i)_{,1\le i\le K}$, and $\bigvee$ denotes the maximum one obtains for any $s>0$

$$\ell_s(TX,TY) \le \left(E\sum_{i=1}^{K} |A_i|^s \right)^{\frac{1}{s}\wedge 1} \ell_s(X,Y) \tag{1.6}$$

The contraction properties in (1.3)–(1.6) can be extended to random K or to $K=\infty$ as well as to Banach spaces but in this paper we restrict to the case of distributions and random variables in $\mathbb{R}^1$. If not necessary we will use freely random variables or their distributions as arguments of the metrics.

For the application of contraction arguments to the problem of existence and characterization of solutions in the sum case it is important to be able to apply the ℓ_s-metrics also in domain $1\le s\le 2$ since they allow to obtain much easier upper estimates for the sum-recursive equation in (1.1) compared to the Zolotarev metric ζ_s. In section 2 we prove that in spite of the bad contraction factor K_s in (1.4) one can get existence and uniqueness results for sum recursions w.r.t. ℓ_s for any $1<s\le 2$ under the natural contraction condition $\eta_s = E\sum_{i=1}^K |A_i|^s < 1$. The proof of this result uses a coupling construction based on weighted branching trees. We then extend the existence results without using any moment conditions on the solutions. To this aim we introduce a new variant of the minimal L_s-metric called ℓ_s^0 which allows to apply contraction arguments without involving moment conditions. This extension of the applicability of ℓ_s metrics to the analysis of sum equations is the main contribution of this paper.
As consequence of these developments we obtain an interesting equivalence theorem which establishes a one to one relationship between the set of all solutions of homogeneous and inhomogeneous additive recursive equations. For max-recursive sequences the minimal L_s-metrics ℓ_s have been shown in a recent paper of Neininger and Rüschendorf (2005) to be ideally suited for existence and stability results. In section 4 we establish the corresponding

one to one relationship for max-recursive sequences.

We also give an analogue of Guivarchs transformation method for sum recursions (see [10]) to the case of max-recursive equations. This principle allows to transfer nonnegative solutions of additive stochastic equations to max-recursive equations. A central role in this transformation is taken by the Weibull distribution and the solution set constructed this way can be seen as set of random scaled Weibull distributions. In operator language this transfer was detected recently by Rösler (2003).
A basic source for max equations arises from limits of max recursive sequences. We end the paper by an application of the recent limit theorem for max-recursive algorithms in [21] to the limit for the worst case of FIND, which is characterized by a max-recursive stochastic equation.

2 Additive recursive equations – analysis by ℓ_s-metrics

For probability measures $\mu, \nu \in M = M^1(\mathbb{R}^1, \mathcal{B}^1)$ we denote for $s > 0$ by $\ell_s(\mu, \nu)$ the minimal L_s-metric

$$\ell_s(\mu,\nu) = \inf\left\{(E|X-Y|^s)^{\frac{1}{s}\wedge 1}; X\stackrel{d}{=}\mu, Y\stackrel{d}{=}\nu\right\} \tag{2.1}$$

We use synonymously also the notation $\ell_s(X,Y)$ or $\ell_s(X,\mu)$ for the distance of the corresponding distributions. While $\ell_s(\mu,\nu)$ in (2.1) is defined for all $\mu,\nu \in M$ it will be finite only if ν is in the ℓ_s surrounding $M_s(\mu)$ of μ,

$$M_s(\mu) := \{\nu \in M; \ell_s(\mu,\nu) < \infty\} \tag{2.2}$$

For any $\mu \in M_s$ – the class of all probability measures with finite *s-th* moments – holds

$$M_s(\mu) = M_s\,; \tag{2.3}$$

in particular $M_s = M_s(\varepsilon_0)$. For $s \geq 1$ we will additionally have to consider subsets of $M_s(\mu)$, where the first moment is fixed to have the value c,

$$M_s(\mu, c) = \{\nu \in M_1; \ell_s(\mu,\nu) < \infty, E\nu = c\} \tag{2.4}$$

Let T denote the operator on the set of probability measures corresponding to (1.1), $TX\stackrel{d}{=}\sum_{i=1}^K A_i X_i + b$. The ℓ_s-metrics have the following contraction properties w.r.t. the operator T.

Lemma 2.1 *Let $\mu_0 \in M$ and $\mu,\nu \in M_s(\mu_0)$ then* *a) For any $s > 0$ holds with $r := \min(s,1)$*

$$\ell_s(T\mu, T\nu) \le \sum_{i=1}^{K} E|A_i|^r \quad \ell_s(\mu,\nu) \tag{2.5}$$

b) For $s = 2$ and $\mu,\nu \in M_2(\mu_0, c)$ holds

$$\ell_2(T\mu, T\nu) \le \left(E\sum_{i=1}^{K} A_i^2 \right)^{\frac{1}{2}} \ell_2(\mu,\nu) \tag{2.6}$$

c) For $1 \le s \le 2$ and $\mu,\nu \in M_s(\mu_0, c)$ holds

$$\ell_s(T\mu, T\nu) \le K_s \left(E\sum_{i=1}^{K} |A_i|^s \right)^{\frac{1}{s}} \ell_s(\mu,\nu) \tag{2.7}$$

where $K_1 = 1, K_s = 18 s^{\frac{3}{2}} (s-1)^{\frac{1}{2}}, 1 < s \le 2$.

For the proof of Lemma 2.1 see Rachev and Rüschendorf (1992, resp. 1995, Prop. 2, 3) respectively Rösler (1992) for $s = 2$. The results are stated there for the moment class M_s but can be extended to the generalized classes $M_s(\mu_0)$ resp. $M_s(\mu_0, c)$ considered here. The cases $s = 2$ and $0 < s < 1$ lead to existence and uniqueness results for additive recursive stochastic equations of the type

$$TX \stackrel{d}{=} X \tag{2.8}$$

under the natural contraction condition

$$\eta_s = E\sum_{i=1}^{K} |A_i|^s < 1 \tag{2.9}$$

in $M_s(\mu_0)$ for $0 < s \le 1$ resp. $M_2(\mu_0, c)$ for $s = 2$.

Theorem 2.2 *Assume that $\eta_s < 1$ and consider the stochastic equation*

$$X \stackrel{d}{=} \sum_{i=1}^{K} A_i X_i + b \tag{2.10}$$

a) If $0 < s \le 1$ and $\mu_0 \in M$ satisfies $\ell_s(\mu_0, T\mu_0) < \infty$, then (2.10) has a unique solution in $M_s(\mu_0)$.

b) If $s = 2$ and $b \in L^1$ and if $\ell_2(\mu_0, T\mu_0) < \infty$ for some $\mu_0 \in M$ and

b1)
$$Eb = 0 \quad \text{and} \quad E\sum_{i=1}^{K} A_i = 1 \qquad \text{or} \tag{2.11}$$

b2)
$$E\sum_{i=1}^{K} A_i \neq 1 \quad \text{and let} \quad c^* := \frac{Eb}{1 - E\sum_{i=1}^{K} A_i} \tag{2.12}$$

then (2.10) *has a unique solution in* $M_2(\mu_0, c)$ *for any* $c \in \mathbb{R}^1$ *under b1) and in* $M_2(\mu_0, c^*)$ *under b2).*

Proof:

a) We have to establish that $T : M_s(\mu_0) \to M_s(\mu_0)$.
Let $\mu \in M_s(\mu_0)$, then there exist rv's $X \stackrel{d}{=} \mu_0, Y \stackrel{d}{=} \mu$ with $E|X-Y|^s < \infty$. Let (X_i, Y_i) be iid couplings with $(X_i, Y_i) \stackrel{d}{=} (X, Y)$, then

$$W := \sum_{i=1}^K A_i Y_i + b \stackrel{d}{=} T\mu \quad \text{and} \quad V := \sum_{i=1}^K A_i X_i + b \stackrel{d}{=} T\mu_0$$

are couplings of $T\mu, T\mu_0$ with

$$E|W-V|^s \leq \sum_{i=1}^K E|A_i|^s \, E\,|X-Y|^s \tag{2.13}$$

(cp. (2.5) for $0 < s \leq 1$). By assumption $\ell_s(\mu_0, T\mu_0) < \infty$. Thus there exist couplings $U, \widetilde{U}$ of $\mu_0, T\mu_0$ with $E|U - \widetilde{U}|^s < \infty$. Without loss of generality we may assume that $\widetilde{U} = V$ (otherwise we may use a suitable measure preserving transformation). Thus (U, W) is a coupling of $(\mu_0, T\mu)$ with $E|U-W|^s \leq E|U-V|^s + E|V-W|^s < \infty$, i.e. $T\mu \in M_s(\mu_0)$. Completeness of $(M_s(\mu_0), \ell_s)$ is a consequence of completeness of L^s. If $(\mu_n) \subset M_s(\mu_0)$ is a Cauchy sequence in $M_s(\mu_0)$, then choosing optimal couplings $X_n = F_n^{-1}(U), n \geq 0$, simultaneously for all μ_n, we obtain, that $(X_n - X_0)_{n \geq 1}$ is a Cauchy sequence in L^s and thus has a limit $Z \in L^s$. This implies $\ell_s(\mu_n, \tau) \to 0$ where $\tau \stackrel{d}{=} X_0 + Z$. Now an application of Banachs fixed point theorem using the contraction property in (2.5) yields existence and uniqueness of a fixed point in $M_s(\mu_0)$.

b) As for the proof of a) we have to establish that $T : M_2(\mu_0, c) \to M_2(\mu_0, c)$. This is similar to a) using conditions b1) resp. b2) to establish that $E\mu = c$ implies $ET(\mu) = c$. □

Remark 2.3 *If* $\mu_0 \in M_s$ *and* $b \in L^s, 1 \leq s \leq 2$ *then* $M_s(\mu_0, c) = M_s(c) = \{\mu \in M_s; E\mu = c\} \subset M_s, M_s(\mu_0) = M_s$ *and the condition* $\ell_s(\mu_0, T\mu_0) < \infty$ *is satisfied by the assumptions on* A_i, b*. The contraction and existence uniqueness-result can be found in this case in Rösler (1992) for* $s = 2$ *resp. in Rachev and Rüschendorf (1995). The extension to the classes* $M_s(\mu_0)$ *resp.* $M_2(\mu_0, c) \subset M_1$ *allows to consider more general stochastic equations including e.g. characterizations of the Cauchy distribution by an equation of the form*

$$A_1 X_1 + A_2 X_2 + b \stackrel{d}{=} X \tag{2.14}$$

where $b = A_3\,\mathcal{C}(0,1), \mathcal{C}(0,1)$ *a Cauchy distributed rv random variables - where* $0 \le A_i, A_1 + A_2 + A_3 = 1$ *and* $E(A_1^s + A_2^s) < 1$ *for some* $s \le 1$.
Theorem 2.2 a) then implies that the Cauchy distribution $\mu_0 = \mathcal{C}(\mu, \sigma)$ *is the unique solution of* (2.14) *in* $M_s(\mu_0)$.

For the case of interest that $1 < s < 2$ only the contraction property in (2.7) with an additional contraction factor $K_s > 1$ is available. Our next aim is to establish an existence and uniqueness result in M_1 for this case under the natural contraction condition $\eta_s < 1$ which extends the case $s = 2$ in part b) of Theorem 2.2.

Theorem 2.4 (Existence and uniqueness in $\mathbf{M_1}$) *Consider the stochastic equation* $X \stackrel{d}{=} \sum_{i=1}^K A_i X_i + b$ *(as in* (2.10)*) and let* $1 \le s \le 2$. *Furthermore, let* $\mu_0 \in M_1, b \in L^1$ *and assume that* $\eta_s = E\sum_{i=1}^K |A_i|^s < 1$ *as well as condition* (2.11) *or* (2.12).
If $\ell_s(\mu_0, T\mu_0) < \infty$, *then the stochastic equation* (2.10) *has a unique solution in* $M_s(\mu_0, c)$.

Proof: For the proof we establish in the first step that the $m-th$ iterate T^m of T is for all $m \ge m_0$ a contraction on $M_s(\mu_0, c)$, i.e. $\ell_s(T^m\mu, T^m\tau) \le \kappa_s \ell_s(\mu, \tau)$ for some $0 < \kappa_s < 1$ and all $\tau, \mu \in M_s(\mu_0, c)$.
For the proof we consider the random weighted K-ary branching tree T_m^X of depth m, where each node $\sigma = \sigma_1 \dots \sigma_r$ (including the root $\emptyset$) is supplied with an independent copies X_σ and b_σ of the random variables X, b where $X \stackrel{d}{=} \mu$ and the K edges $e_1, \dots, e_K$ leading from σ to the successor $\sigma\sigma_i$ of σ get an independent copies $(A_{e_1}, \dots, A_{e_K})$ of $(A_1, \dots, A_K)$ such that $(A_{e_1}, \dots, A_{e_K}, b_\sigma) \stackrel{d}{=} (A_1, \dots, A_K, b)$ (see e.g. Rösler and Rüschendorf (2001) for this construction). Further for each node $\nu = \nu_1 \dots \nu_r$ at level r we define the multiplicative weights $L(\nu) = A_{\nu_1} \dots A_{\nu_r}$ along its path $\nu_1 \dots \nu_r$ in the tree and we define the additively weighted size of the branching tree by

$$Z_m := \sum_{|\sigma|=m} L(\sigma) X_\sigma + \sum_{i=1}^{m-1} \sum_{|\nu|=i} L(\nu) b_\nu. \tag{2.15}$$

Let $\tau, \mu \in M_s(\mu_0, c)$ and $X \stackrel{d}{=} \mu, Y \stackrel{d}{=} \tau$ with $EX = EY, E|X-Y|^s < \infty$ and let T_m^Y be the induced random weighted branching tree with iid copies Y_σ in the nodes such that $(X_\sigma, Y_\sigma) \stackrel{d}{=} (X, Y)$ and with the same random weights on the edges (A_σ, b_σ) as in the tree T_m^X. Denote the corresponding additively weighted size by

$$W_m := \sum_{|\sigma|=m} L(\sigma) Y_\sigma + \sum_{i=1}^{m-1} \sum_{|\nu|=i} L(\nu) b_\nu \tag{2.16}$$

Then Z_m, W_m satisfy the recursive structure

$$Z_m \stackrel{d}{=} \sum_{i=1}^{K} A_i Z_{m-1}^{(i)} + b, \quad W_m \stackrel{d}{=} \sum_{i=1}^{K} A_i W_{m-1}^{(i)} + b \tag{2.17}$$

where $\left(Z_{m-1}^{(i)}\right), \left(W_{m-1}^{(i)}\right)$ are iid copies of Z_{m-1} resp. W_{m-1}. This recursive structure is obtained by splitting the tree at the root. Z_m, W_m are versions of the *m-th* iterate of the distributional operator T

$$Z_m \stackrel{d}{=} T^m X, \quad W_m \stackrel{d}{=} T^m Y. \tag{2.18}$$

By the multiplicative structure and using the independence assumptions we obtain from the Woyczynski inequality (see (2.7))

$$\begin{aligned} L_s^s(Z_m, W_m) &= E|\sum_{|\sigma|=m} L(\sigma)(X_\sigma - Y_\sigma)|^s \\ &\le K_s E \sum_{|\sigma|=m} |L(\sigma)|^s E|X-Y|^s \\ &= K_s \left(E \sum_{i=1}^{K} |A_i|^s \right)^m E|X-Y|^s \\ &= K_s \eta_s^m E|X-Y|^s \end{aligned} \tag{2.19}$$

For this estimate equality of first moments is needed. Passing to the minimal L_s-metric ℓ_s we obtain

$$\ell_s(T^m X, T^m Y) \le K_s\, \eta_s^m\, \ell_s(\mu, \tau) \tag{2.20}$$

For $m \ge m_0, K_s \eta_s^m \le K_s \eta_s^{m_0} =: \kappa_s < 1$, i.e. the iterated operator T^m is a contraction w.r.t. ℓ_s on $M_s(\mu_0, c)$.
By assumption $\ell_s(\mu_0, T\mu_0) < \infty$ and thus as in the proof of part b) of Theorem 2.2 for $s = 2$ we obtain that $T : M_s(\mu_0, c) \to M_s(\mu_0, c)$. This implies by the triangle inequality that $\ell_s(\mu_0, T^m \mu_0) < \infty$. Thus $\mu_0, T^m\mu_0, T^{2m}\mu_0, \dots$ is a Cauchy-sequence in $M_s(\mu_0, c)$ and so converges to some limit $\mu^* \in M_s(\mu_0, c), \ell_s(T^{km}\mu_0, \mu^*) \to 0$. For any $1 \le r \le m$ we obtain

$$\ell_s(T^{km}\mu_0, T^{km+r}\mu_0) \le \kappa_s^k \ell_s(\mu_0, T^r\mu_0) \to 0 \tag{2.21}$$

and thus the triangle inequality implies

$$\ell_s(\mu^*, T^r\mu^*) = 0, \quad 1 \le r \le m,$$

and $T^n\mu_0$ converges to μ^* and μ^* is a fixed point of T in $M_s(\mu_0, c)$.
Uniqueness of the fixed point follows from the estimate in (2.20) if applied to two solutions X, Y of (2.10). $\square$

Remark 2.5 *As in the case $s = 2$ the additional assumption $b \in L^s$ implies the condition $\ell_s(\mu_0, T\mu_0) < \infty$ if $\mu_0 \in M_s$.*

One can state a corresponding existence and uniqueness result with respect to the Zolotarev metric ζ_s for any $s > 0$. $\zeta_s(\mu, \nu)$ is defined for $s = m+\alpha, m \in \mathbb{N}_0, 0 < \alpha \leq 1$ and $X \stackrel{d}{=} \mu, Y \stackrel{d}{=} \nu$ by

$$\zeta_s(\mu, \nu) = \sup\{E(f(X) - f(Y)); f \in \mathcal{F}_s\} \text{ where} \tag{2.22}$$

$$\mathcal{F}_s = \{f \in \mathcal{C}^m(\mathbb{R}, \mathbb{R}); ||f^{(m)}(x) - f^{(m)}(y)|| \leq |x - y|^\alpha\}.$$

Finiteness of $\zeta_s(\mu, \nu)$ implies equality of the first m difference moments $E(X^r - Y^r) = 0,\ 1 \leq r \leq m$.

Proposition 2.6 *Let $s > 0$, and $\mu_0 \in M$ be a probability measure such that $\eta_s = E \sum_{i=1}^K |A_i|^s < 1$ and $\zeta_s(\mu_0, T\mu_0) < \infty$. Then the additive stochastic equation (2.10) has a unique solution in $M_s^\zeta(\mu_0) = \{\mu \in M; \zeta_s(\mu, \mu_0) < \infty\}$.*

Proof: For $\mu, \nu \in M_s^\zeta(\mu_0)$ holds

$$\zeta_s(T\mu, T\nu) \leq \left(\sum_{i=1}^K E|A_i|^s \right) \zeta_s(\mu, \nu). \tag{2.23}$$

(e.g. [24, Prop.1]). The assumption $\zeta_s(T\mu_0, \mu_0) < \infty$ implies by the triangle inequality that $T : M_s^\zeta(\mu_0) \to M_s^\zeta(\mu_0)$. Thus $\{T^n\mu_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $M_s^\zeta(\mu_0)$ which implies the existence of a fixed point by completeness of $(M_s^\zeta(\mu_0), \zeta_s)$. The uniqueness part is as in Theorem 2.4. □

Remark 2.7 *In general the finiteness condition $\zeta_s(\mu_0, T\mu_0) < \infty$ of Proposition 2.6 for the Zolotarev metric ζ_s is not easy to check. For $s \in \mathbb{N}$ there are upper bounds of ζ_s in terms of the pseudo difference moments*

$$\kappa_s(X, Y) = s \int |x|^{s-1} |F_X(x) - F_Y(x)| dx \tag{2.24}$$

if the first $s - 1$ moments coincide but for $s \notin \mathbb{N}$ only estimates for the ζ_s-metric including absolute pseudomoments are available. There have been developed several alternative probability metrics μ_s which allow estimates as in (2.23) and simultaneously allow upper bounds by difference pseudomoments (see e.g. [22]). The estimate of $\ell_s(\mu_0, T\mu_0)$ w.r.t. the ℓ_s-metric is however in comparison particularly simple and will be very useful in the following part.

As consequence of Proposition 2.6 we obtain that given the contraction condition $\eta_s < 1$, the problem of existence of a fixed point is equivalent with finding an element $\mu_0 \in M$ such that $\zeta_s(\mu_0, T\mu_0) < \infty$. W.r.t. the ℓ_s-metrics the same is true for $0 < s \leq 1$ in $M_s(\mu_0)$. For the interesting case $1 \leq s \leq 2$

we get a characterization of fixed points in $M_s(\mu_0, c) \subset M_1$. In the next step we want to remove the first moment condition for the ℓ_s-metrics. To that aim we introduce for any $\mu_0 \in M$

$$M_s^0(\mu_0) = \{\mu \in M; \exists\, rv's\ \ X \stackrel{d}{=} \mu, Y \stackrel{d}{=} \mu_0 \ \text{ such that } E(X-Y)=0, E|X-Y|^s < \infty\}. \tag{2.25}$$

On $M_s^0(\mu_0)$ we define the modified ℓ_s-metric

$$\ell_s^0(\mu,\nu) = \inf\{(E|X-Y|^s)^{\frac{1}{s}\wedge 1}; X \stackrel{d}{=} \mu, Y \stackrel{d}{=} \nu, E(X-Y)=0, E|X-Y|^s < \infty\} \tag{2.26}$$

Since $M_s^0(\mu_0) \subset M_s(\mu_0)$ we obtain $\ell_s(\mu,\nu) \leq \ell_s^0(\mu,\nu)$ and ℓ_s^0 satisfies the triangle inequality. Finiteness of $\ell_s^0(\mu,\nu)$ needs a more stringent coupling than finiteness of $\ell_s(\mu,\nu)$. In the next theorem we will see that this finiteness is sufficient for a general existence result for solutions of the stochastic recursive equation $X \stackrel{d}{=} \sum_{i=1}^K A_i X_i + b$ using ℓ_s-type estimates.

Theorem 2.8 (Existence and uniqueness in M) *Let* $1 \leq s \leq 2$ *let* $\mu_0 \in M$ *satisfy* $\ell_s^0(\mu_0, T\mu_0) < \infty$ *and assume* $\eta_s = E\sum_{i=1}^K |A_i|^s < 1$. *Then the stochastic equation* (2.10) *has a unique solution in* $M_s^0(\mu_0)$.

Proof: We first establish that $T : M_s^0(\mu_0) \to M_s^0(\mu_0)$. Let $\mu \in M_s^0(\mu_0)$ and denote for rv's X, Y by $X \approx Y$ that $E(X-Y)=0$ and $E|X-Y|^s < \infty$. By assumption there exist rv's $X \stackrel{d}{=} \mu_0, Y \stackrel{d}{=} T\mu_0$ such that $X \approx Y$. Let (X_i, Y_i) be iid copies of (X, Y). Furthermore, define couplings of $T\mu, T\mu_0$ by

$$Z := \sum_{i=1}^K A_i Y_i + b \stackrel{d}{=} T\mu \text{ and } W := \sum_{i=1}^K A_i X_i + b \stackrel{d}{=} T\mu_0. \text{ Then}$$

$$E(Z-W) = \sum_{i=1}^K EA_i E(Y-X) = 0 \text{ and}$$
$$E|Z-W|^s \leq K_s^s \eta_s E|Y-X|^s < \infty \text{ i.e. } Z \approx W.$$

Since $\ell_s^0(\mu_0, T\mu_0) < \infty$ we obtain by the triangle inequality for ℓ_s^0

$$\ell_s^0(\mu_0, T\mu) \leq \ell_s^0(\mu_0, T\mu_0) + \ell_s^0(T\mu_0, T\mu) \leq E|X-Y|^s + E|Z-W|^s < \infty$$

i.e. $T\mu \in M_s^0(\mu_0)$. Now we can follow the proof of Theorem 2.4 using the branching tree contruction with $Z_{km} \stackrel{d}{=} T^{km}\mu_0, W_{km+r} \stackrel{d}{=} T^{km+r}\mu_0$ (cp. (2.15), (2.16), (2.18)) with iid couplings (Y_σ, X_σ) of $(\mu_0, T^r\mu_0)$ such that $Y_\sigma \approx X_\sigma$. Then we obtain as in (2.19) for any $r \geq 1$

$$L_s^s(Z_{km}, W_{km+r}) \leq K_s^s (\eta_s^m)^k E|Y-X|^s \to 0 \text{ as k} \to \infty \tag{2.27}$$

where $(X,Y)\stackrel{d}{=}(X_\sigma,Y_\sigma)\stackrel{d}{=}(T^r\mu_0,\mu_0)$. In particular, $\mu_0, T^m\mu_0, T^{2m}\mu_0,\dots$ is a Cauchy sequence in $M_s^0(\mu_0)$ with corresponding couplings $X_0, X_1 = Z_m, X_2 = Z_{2m},\dots$
The related differences $X_k - X_0$ are a Cauchy sequence in $L^s(0)$ (i.e. they have mean zero) and thus converge to some limit $Z \in L^s(0)$,
$X_k - X_0 \stackrel{L^s}{\longrightarrow} Z$. This implies that $X_k \stackrel{L^s}{\longrightarrow} Z + X_0$ and thus with $\mu^*\stackrel{d}{=}Z + X_0$ we obtain

$$\ell_s^0(T^{km}\mu_0,\mu^*) \le E|X_k - (Z+X_0)|^s \to 0. \tag{2.28}$$

This argument yields completeness of $(M_s^0(\mu_0),\ell_s^0)$. From (2.27) we conclude, that

$$\ell_s^0(\mu^*, T^r\mu^*) = 0, \ \ 1 \le r \le m \tag{2.29}$$

and thus μ^* is a fix point of T in $M_s^0(\mu_0)$. Uniqueness follows from the estimate (2.27) applied to two solutions μ,ν of the stochastic equation and using the corresponding weighted branching tree construction Z_n, W_n with couplings $X\stackrel{d}{=}\mu, Y\stackrel{d}{=}\nu$ such that $X \approx Y$. □

As consequence of Theorems 2.8 and 2.2 we obtain the following characterization of the existence of solutions.

Corollary 2.9 *Let $0 < s \le 2$ and $\eta_s = E\sum_{i=1}^K |A_i|^s < 1$. Then the stochastic equation*

$$X\stackrel{d}{=}\sum_{i=1}^K A_iX_i + b \tag{2.30}$$

has a solution if and only if there exists some $\mu_0 \in M$ such that

$$\begin{aligned} &\ell_s\ (\mu_0, T\mu_0) < \infty \quad \text{if} \quad 0 < s \le 1\\ \text{resp.}\quad &\ell_s^0\ (\mu_0, T\mu_0) < \infty \quad \text{if} \quad 1 < s \le 2. \end{aligned} \tag{2.31}$$

Remark 2.10 *In particular the extended contraction results for the ℓ_s-metrics allow to characterize stable distributions as unique solutions of the associated stochastic equations in $M_s(\mu_0)$ resp. $M_s^0(\mu_0)$. Let $0 < \alpha < 2$, let U be uniformly distributed on [0,1] and let $X^*\stackrel{d}{=}\mu_0 = \mathcal{S}(\alpha)$ be a symmetric stable distribution with index α and scale factor c with characteristic function*

$$\ln\ \varphi_{\mathcal{S}(\alpha)}(t) = -c|t|^\alpha. \tag{2.32}$$

Then X^* is the unique solution of the stochastic recursion

$$X\stackrel{d}{=}U^{1/\alpha}X_1 + (1-U)^{1/\alpha}X_2 \tag{2.33}$$

in $M_s(\mu_0)$ if $0 < \alpha < s \le 1$, resp. in $M_s^0(\mu_0)$ if $1 \le \alpha < s \le 2$.

3 Homogeneous and inhomogeneous additive recursive equations

In this section we obtain as application of the contraction results established in section 2 a one to one relationship between solutions of homogeneous and inhomogeneous linear stochastic equations. Consider the inhomogeneous equation

$$X \stackrel{d}{=} \sum_{i=1}^{K} A_i X_i + b \tag{3.1}$$

with induced operator T on M and the corresponding homogeneous equation

$$X \stackrel{d}{=} \sum_{i=1}^{K} A_i X_i \tag{3.2}$$

with induced operator T_0. To establish a one to one relationship we assume that $b \in L^s$ and further the natural contraction condition $\eta_s = E\sum_{i=1}^K |A_i|^s < 1$. In section 2 we have obtained various conditions on existence and uniqueness of solutions of (3.1), (3.2).

Theorem 3.1 (homogeneous and inhomogeneous equations) *Let* $0 < s \leq 2$ *and* $A_i, b \in L^s$ *such that* $\eta_s = E\sum_{i=1}^K |A_i|^s < 1$ *and* $Eb = 0$ *in case* $1 < s \leq 2$. *Then the following equivalence holds:*

a) *For any solution* μ_0 *of the homogeneous equation* $T_0\,\mu \stackrel{d}{=} \mu$ *there exists exactly one solution* μ^* *of the inhomogeneous equation* $T\mu \stackrel{d}{=} \mu$ *such that*

$$\mu^* \in \begin{cases} M_s(\mu_0) & \text{if } \; 0 < s \leq 1 \\ M_s^0(\mu_0) & \text{if } \; 1 < s \leq 2 \end{cases} \tag{3.3}$$

b) *For any solution* μ^* *of the inhomogeneous equation* $T\mu \stackrel{d}{=} \mu$ *there exists exactly one solution* μ_0 *of the homogeneous equation* $T_0\mu \stackrel{d}{=} \mu$ *such that*

$$\mu_0 \in \begin{cases} M_s(\mu^*) & \text{if } \; 0 < s \leq 1 \\ M_s^0(\mu_0) & \text{if } \; 1 < s \leq 2. \end{cases}$$

Proof:

a) If μ_0 is a solution of the homogeneous equation $T_0\mu_0 \stackrel{d}{=} \mu$ then we obtain a coupling of $\mu_0, T\mu_0$ by $Y_0 := \sum_{i=1}^K A_i Y_i$, where (Y_i) are iid, $Y_i \stackrel{d}{=} \mu_0$, and $X^* := \sum_{i=1}^K A_i Y_i + b = Y_0 + b$. This implies $E|X^* - Y_0|^s = E|b|^s < \infty$ and thus $\ell_s(\mu_0, T\mu_0) < \infty$ for $0 < s \leq 1$ and $\ell_s^0(\mu_0, T\mu_0) < \infty$ for $1 < s \leq 2$

using the additional assumption $Eb = 0$ in this case. From Theorem 2.2 a) we obtain a unique solution $\mu^* \in M_s(\mu_0)$ of $T\mu \stackrel{d}{=} \mu$ in case $0 < s \leq 1$, while Theorem 2.8 implies the existence of a unique solution $\mu^* \in M_s^0(\mu_0)$ of $T\mu \stackrel{d}{=} \mu$ in case $1 < s \leq 2$.

b) The converse direction is similar. If μ^* is a solution of the inhomogeneous equation $T\mu \stackrel{d}{=} \mu$, then let(X_i) be iid random variables with $X_i \stackrel{d}{=} \mu^*$. $X^* := \sum_{i=1}^K A_i X_i + b$ and $Y_0 := \sum_{i=1}^K A_i X_i$ define a coupling of μ^* and $T_0\mu^*$ such that $E|Y_0 - X^*|^s = E|b|^s < \infty, \ell_s(\mu^*, T_0\mu^*) < \infty$ for $0 < s \leq 1$, and $\ell_s^0(\mu^*, T_0\mu^*) \leq E|Y_0 - X^*|^s < \infty$ for $1 < s \leq 2$ and further $T_0\mu^* \in M_s^0(\mu^*)$. Thus again by Theorems 2.2 a) and Theorem 2.8 we obtain a unique solution μ_0 of the homogeneous equation $T_0\mu = \mu$ with

$$\begin{aligned} &\mu_0 \in M_s(\mu^*) && \text{if } 0 < s \leq 1 \quad \text{and} \\ &\mu_0 \in M_s^0(\mu^*) && \text{if } 1 < s \leq 2. \end{aligned}$$ □

Remark 3.2 *a) Let $\mathcal{L}_0$, $\mathcal{L}$ denote the solution sets of the homogeneous resp. inhomogeneous equations (either in terms of distributions or in terms of random variables). If $Y \in \mathcal{L}_0 \cap L^1$, $EY = c$ and $1 \leq s \leq 2$, then with $\mu_0 \stackrel{d}{=} Y$ it holds that $M_s(\mu_0) = M_s(\mu_0, c)$. We obtain as consequence from Theorem 2.8 and Theorem 3.1 existence and uniqueness of solutions in $M_s(\mu_0, c)$ as in Theorem 2.4, where we have to specify the first moment of solutions. The remarkable point of Theorem 3.1 is to establish a one to one relationship between $\mathcal{L}_0$ and $\mathcal{L}$ without any assumptions on the moments of the solutions μ_0, μ^* in $\mathcal{L}_0$ resp. $\mathcal{L}$. The existence result based on the Zolotarev metric (Proposition 2.6) would not allow to draw a conclusion as in Theorem 3.1 since the finiteness condition $\zeta_s(\mu_0, T\mu_0) < \infty$ for a fixed point μ_0 of the homogeneous equation would in general need further moment assumptions on μ_0.*

b) For the Quicksort recursion

$$X \stackrel{d}{=} UX_1 + (1-U)X_2 + C(U), \tag{3.4}$$

with $C(U) = 2U \log U + 2(1-U)\log(1-U) + 1$ Fill and Janson (2000) characterized the set of all solutions of (3.4) by ($\oplus$ denoting independent sums)

$$\mathcal{L} = X^* \oplus \mathcal{C}, \tag{3.5}$$

where X^* is the unique solution of the inhomogeneous equation - the Quicksort distribution - with finite 2. moment and $\mathcal{C} = \{C(\mu, \sigma^2), \mu \in \mathbb{R}^1, \sigma^2 \geq 0\}$ is the set of Cauchy distributions $C(\mu, \sigma^2)$ with location parameter μ and scale parameter σ, $C(\mu, 0) = \varepsilon_\mu$. The method of proof [8] applied to the homogeneous equation yields also that

$$\mathcal{C} = \mathcal{L}_0 \tag{3.6}$$

is the set of all solutions of the homogeneous equation. In this case the relation between homogeneous solutions and inhomogeneous solutions is explicit and simple. By the explicit relationship in (3.5) *the knowledge of the solution set in the homogeneous case would yield by our equivalence result in Theorem 3.1 the equality in* (3.6) *for the inhomogeneous case directly.*

Corollary 3.3 (Quicksort type equation) *Let U be* unif$[0,1]$, *$b \in L^s$ with conditions* (2.11) *resp.* (2.12) *if $1 \le s \le 2$ and consider the Quicksort type equation*

$$X \stackrel{d}{=} UX_1 + (1-U)X_2 + b. \tag{3.7}$$

Then (3.7) *has a unique solution $\widetilde{X}$ in $M_s(c)$ (resp. $M_s(c^*)$, see* (2.12)*) and the set of all solutions of* (3.7) *is given by*

$$\mathcal{L} = \widetilde{X} \oplus \mathcal{C}$$

Proof: The proof follows using (3.6) from Theorems 2.2, 2.4, and Theorem 3.1. □

4 Max-recursive sequences

In this section we consider max-recursive equations of the kind

$$X \stackrel{d}{=} \bigvee_{r=1}^{K} (A_r X_r + b_r) \tag{4.1}$$

where (X_i) are iid copies of X and A_i, b_i are random coefficients independent of (X_r). The r.h.s. of (4.1) induces an operator $T : M \to M$ defined for $Q \in M$ and $X \stackrel{d}{=} Q$ by

$$TQ = TX \stackrel{d}{=} \mathcal{L}\left(\bigvee_{r=1}^{K} (A_r X_r + b_r)\right). \tag{4.2}$$

If A_r, $b_r \in L^s$ and $\mathcal{L}(X) \in M_s$, then also $TX \in M_s$ and then T can be considered as operator $M_s \to M_s$. The following existence and uniqueness result for max recursive equations was stated in Neininger and Rüschendorf (2005) based on the contraction property in (1.6). Note that in the max case any $s > 0$ is allowed.

Theorem 4.1 (Existence and uniqueness for max-recursive equations, see [21]) *Let for some $s > 0$ the coefficients A_i, $b_i \in L^s$ and let $\mu_0 \in M$ be such that $\eta_s := E\sum_{r=1}^K |A_r|^s < 1$ and $\ell_s(\mu_0, T\mu_0) < \infty$. Then the stochastic max-recursive equation $X \stackrel{d}{=} \bigvee_{r=1}^K (A_r X_r + b_r)$ has a unique solution in $M_s(\mu_0)$.*

Similarly to the sum case we obtain as consequence a one to one relation between the set $\mathcal{L}$ of the solutions of the inhomogeneous equation

$$X \stackrel{d}{=} \bigvee_{r=1}^{K} (A_r X_r + b_r) \tag{4.3}$$

and the set $\mathcal{L}_0$ of solutions of the homogeneous equation

$$Y \stackrel{d}{=} \bigvee_{r=1}^{K} A_r X_r \tag{4.4}$$

We denote the corresponding distributional operators by T and T_0.

Theorem 4.2 (Equivalence theorem) *Let A_r, $b_r \in L^s$ for some $s > 0$ and assume that $\eta_s = E \sum_{r=1}^K |A_r|^s < 1$. Then*

a) The inhomogeneous max-recursive equation (4.3) *has a unique solution in M_s.*

b) For any solution $Y_0 \in \mathcal{L}_0$ of the homogeneous max-recursive equation (4.4)*, there exists exactly one solution X^* of the inhomogeneous max-recursive equation* (4.3) *such that $\ell_s(X^*, Y_0) < \infty$, i.e. $\mathcal{L}(X^*) \in M_s(\mu_0)$, where $\mu_0 = \mathcal{L}(Y_0)$.*

c) For any solution X^* of the inhomogeneous equation (4.3) *there exists exactly one solution Y_0 of the homogeneous equation* (4.4) *with $\ell_s(X^*, Y_0) < \infty$.*

Proof:

a) The proof follows from Theorem 4.1. Let $\mu_0 = \varepsilon_0$; then

$$\ell_s(\mu_0, T\mu_0) \le E \max_r |b_r|^s \le \sum_{r=1}^K E|b_r|^s < \infty.$$

Furthermore, $M_s(\mu_0) = M_s$ and thus a) follows from Theorem 4.1.

b) If $Y_0 \in \mathcal{L}_0$ and $\mu_0 = \mathcal{L}(Y_0)$ then let (X_i) be independent r.v.s with $X_i \stackrel{d}{=} \mu_0$ and consider the coupling $X := \bigvee_{r=1}^K A_r X_r$ and $W := \bigvee_{r=1}^K (A_r X_r + b_r)$ of μ_0 and $T\mu_0$. Then we obtain from Lemma 3.1 in [21] $\ell_s(\mu_0, T\mu_0) \le (E|X - W|^s)^{1/s \wedge 1} \le (E \sum_{r=1}^K |b_r|^s)^{1/s \wedge 1} < \infty$. Theorem 4.1 implies the existence and uniqueness of a solution μ^* of the inhomogeneous equation (4.3) in $M_s(\mu_0)$.

c) Let conversely X^* be a solution of (4.3) with $X^* \stackrel{d}{=} \mu^*$. Further, let (X_i) be iid, $X_i \stackrel{d}{=} \mu^*$ and consider the coupling $X := \bigvee_{r=1}^K A_r X_r$ and $W := \bigvee_{r=1}^K (A_r X_r + b_r)$ of $T_0\mu^*$ and μ^* (as $W \stackrel{d}{=} \mu^*$). Then

$$\begin{aligned}\ell_s(\mu^*, T_0\mu^*) &\leq (E|X - W|^s)^{1/s\wedge 1}\\ &\leq (E\sum_{r=1}^K |b_r|^s)^{1/s\wedge 1} < \infty\end{aligned}$$

and Theorem 4.1 implies the result. □

By Theorem 4.2 there is a one-to-one relationship between the set $\mathcal{L}$ of solutions of the inhomogeneous max-recursive equation and the set $\mathcal{L}_0$ of solutions of the homogeneous max-recursive equation. In the case of non-negative coefficients $A_j \geq 0$, the homogeneous max-recursive equation

$$X \stackrel{d}{=} \bigvee_{j=1}^K A_j X_j \tag{4.5}$$

can be related to the homogeneous additive recursive equation

$$W^{(\alpha)} \stackrel{d}{=} \sum A_j^\alpha W_j^{(\alpha)}, \tag{4.6}$$

where α is chosen such that $E\sum_{j=1}^K A_j^\alpha = 1$. This equation has been studied in detail in the literature. By a result of Biggins (1977) equation (4.6) has a solution if and only if

$$E\sum_{j=1}^K A_j^\alpha \ln A_j \leq 0. \tag{4.7}$$

Let

$$KF(t) = E\prod_j F\left(\frac{t}{A_j}\right) \tag{4.8}$$

denote the operator on the distribution functions corresponding to (4.5). Then Rösler (2003) showed that for any nonnegative solution $W^{(\alpha)}$ of the additive equation (4.6),

$$F_0(t) := Ee^{-W^{(\alpha)}t^{-\alpha}} \tag{4.9}$$

is a d.f. and

$$KF_0 = F_0, \tag{4.10}$$

i.e. F_0 is a solution of the max-recursive equation (4.6). We can interpret Rösler's analytic construction stochastically as an analogue of a transformation of Guivarch (1990) given there for the sum case.

Proposition 4.3 (Additive and max-recursive equations) *Let* $A_j \geq 0$, $E\sum_{j=1}^K A_j^{(\alpha)} = 1$ *and let* $W^{(\alpha)}$ *be a nonnegatiave solution of the additive stochastic equation* (4.6). *Further let* $Z^{(\alpha)}$ *be Weibull-distributed with parameter* α *i.e.* $F_{Z^{(\alpha)}}(x) = e^{-1/x^\alpha}, x > 0$. *Then the random scale Weibull variable*

$$X := \left(W^{(\alpha)}\right)^{1/\alpha} Z^{(\alpha)} \tag{4.11}$$

is a solution of the max-recursive equation (4.5).

Proof: We verify that the d.f. of X is identical to Rösler's distribution F_0 in (4.9).

$$\begin{aligned} F_X(t) &= P(X \leq t) = P((W^{(\alpha)})^{1/\alpha} Z^{(\alpha)} \leq t) \\ &= EF_{Z^{(\alpha)}}\left(\frac{t}{(W^{(\alpha)})^{1/\alpha}}\right) = Ee^{-W^{(\alpha)}/t^\alpha} = F_0(t). \end{aligned}$$

Alternatively, we may use the max-stability of the Weibull distribution. Let $X_j = (W_j^{(\alpha)})^{1/\alpha} Z_j^{(\alpha)}$, be iid copies of X, then

$$\begin{aligned} \bigvee_{j=1}^K A_j X_j &= \bigvee_{j=1}^K A_j (W_j^{(\alpha)})^{1/\alpha} Z_j^{(\alpha)} \\ &\stackrel{d}{=} \left(\sum_{j=1}^K A_j^\alpha W_j^{(\alpha)}\right)^{1/\alpha} Z^{(\alpha)} \\ &\stackrel{d}{=} (W^{(\alpha)})^{1/\alpha} Z^{(\alpha)} = X, \end{aligned}$$

i.e. X is a solution of (4.5). □

Remark 4.4 *The construction in* (4.11) *can also be extended to nonnegative solutions* $W \geq 0$ *of the additive inhomogeneous equation*

$$\sum_{j=1}^K A_j^\alpha (W_j^{(\alpha)} + b_j) \stackrel{d}{=} W^{(\alpha)} \tag{4.12}$$

where (b_j) *are nonnegative iid r.v.s. independent of* (A_j). *Then with* $X := (W^{(\alpha)} + b)^{1/\alpha} Z^{(\alpha)}$ *holds*

$$\begin{aligned} \bigvee_{j=1}^K A_j X_j &= \bigvee_{j=1}^K A_j (W_j^{(\alpha)} + b_j)^{1/\alpha} Z_j^{(\alpha)} \\ &\stackrel{d}{=} (\sum_{j=1}^K A_j^\alpha (W_j^{(\alpha)} + b_j))^{1/\alpha} Z^{(\alpha)} \stackrel{d}{=} (W^{(\alpha)})^{1/\alpha} Z^{(\alpha)} \end{aligned} \tag{4.13}$$

Example 4.5 (Worst case of FIND) *The limiting distribution of the worst case of the FIND algorithm was characterized in Grübel and Rösler (1996) as the unique solution S_0 in M_2 of the fixed point equation*

$$S \stackrel{d}{=} US_1 \vee (1-U)S_2 + 1, \tag{4.14}$$

where U is uniform on $[0,1]$ and S_1, S_2 are iid copies of S. S_0 moreover has finite moments of any order and exponentially decreasing tail. In order to study the solution set $\mathcal{L}$ of (4.14) we first note that the class $\mathcal{W} = \{Q_\lambda, \lambda \geq 0\}$ of Weibull distributions with parameter $\alpha = 1$, with d.f.s $F_\lambda(x) = e^{-\lambda/x}, x > 0$ and with $Q_0 = \varepsilon_0$ are solutions of the homogeneous equation

$$S \stackrel{d}{=} US_1 \vee (1-U)S_2. \tag{4.15}$$

Let $X_\lambda \stackrel{d}{=} Q_\lambda$, then for $\lambda > 0$, X_λ has no finite moments of any order > 1 and for $s > 1$ holds $\ell_s(X_\lambda, X_{\lambda'}) = \infty$ for all $\lambda \neq \lambda'$. The existence Theorem 4.2 implies that $\forall \lambda \geq 0$ there exists exactly one solution S_λ of the worst case FIND equation (4.14) such that $\ell_s(X_\lambda, S_\lambda) < \infty$, i.e.

$$\mathcal{L} \supset \{S_\lambda; \lambda \geq 0\}. \tag{4.16}$$

Since there are no nonnegative solutions of the related homogenous additive equation $W \stackrel{d}{=} UW_1 + (1-U)W_2$ Proposition 4.3 does not add to the set of solutions in this case. It is an open problem whether there are further solutions.

Max-recursive stochastic equations arise under quite general conditions as limits of max-recursive algorithms as was shown recently in Neininger and Rüschendorf (2005). We finish this paper by restating this limit result as a interesting source of max-recursive equations. We then give an application of this limit theorem to the worst case behaviour of FIND where the limiting fixed point equation was stated in (4.14) and discussed above.

Consider a max-recursive algorithm (Y_n) of divide and conquer type,

$$Y_n \stackrel{d}{=} \bigvee_{r=1}^{K} (A_r(n) Y^{(r)}_{I_r^{(n)}} + b_r(n)), \quad n \geq n_0, \tag{4.17}$$

where $I_r^{(n)}$ are subgroup sizes, $b_r(n)$ are random toll terms, $A_r(n)$ are random weights and $(Y_n^{(r)})$ are iid copies of (Y_n), independent also of $(A_r(n), b_r(n), I^{(n)})$. For a limiting result (after normalization) the following conditions were given in [21]. Assume that the coefficients converge in L^s:

$$(A_1^{(n)}, \ldots, A_K^{(n)}, b_1^{(n)}, \ldots, b_k^{(n)}) \xrightarrow{L^s} (A_1^*, \ldots, a_K^*, b_1^*, \ldots, b_K^*). \tag{4.18}$$

Then as a formal limit of equation (4.17) one obtains as limiting equation

$$X \stackrel{d}{=} \bigvee_{r=1}^{K} (A_r^* X_r + b_r^*). \tag{4.19}$$

We need the contraction condition for the limit equation

$$E \sum_{r=1}^{K} |A_r^*|^s < 1 \text{ for some } s > 0 \tag{4.20}$$

as well as a nondegeneracy condition: For any fixed ℓ holds:

$$E[1_{\{I_r^{(n)} \leq \ell\} \cup \{I_r^{(n)} = n\}} |A_r^{(n)}|^s] \to 0. \tag{4.21}$$

Theorem 4.6 (max-recursive limit theorem, see [21]) *Let* (Y_n) *be a max-recursive algorithm of divide and conquer type as in* (4.17) *and assume conditions* (4.18), (4.20), (4.21). *Then* $\ell_s(Y_n, Y^*) \to 0$, *where* Y^* *is the unique solution of the limit equation* (4.19) *in* M_s.

As application of Theorem 4.6 we next give a direct proof of the limiting worst case behaviour of FIND. For an alternative stochastic process approach see Grübel and Rösler (1996).

Example 4.7 *Let* $Y_{n,\ell}$ *denote the number of comparisons of the FIND algorithm for finding the* ℓ*th order statistic. Then*

$$\frac{Y_{n,\ell}}{n} \stackrel{d}{=} \frac{V-1}{n} 1_{\{V>\ell\}} \frac{Y_{V-1,\ell}}{V-1} + \frac{n-V}{n} 1_{\{V<\ell\}} \frac{\overline{Y}_{n-V,\ell-V}}{n-V-1} + \frac{n-1}{n} \tag{4.22}$$

where V *is uniform on* $\{1, \ldots, n\}$ *distributed. With* $V = \lceil U \rceil$, U *uniform on* $[0,1]$ *and the normalization* $X_{n,\ell} := \frac{Y_{n,\ell}}{n}$ *we obtain*

$$X_{n,\ell} \stackrel{d}{=} \frac{\lceil nU \rceil - 1}{n} 1_{\left(\frac{\lceil nu \rceil}{n} > \frac{\ell}{n}\right)} X_{\lceil nu \rceil - 1, \ell} + \frac{n - \lceil nU \rceil}{n} 1_{\left(\frac{\lceil nU \rceil}{n} < \frac{\ell}{n}\right)} \overline{X}_{n - \lceil nU \rceil, \ell - \lceil nU \rceil} + \frac{n-1}{n}.$$

Defining the worst case

$$M_n := \max_{1 \leq \ell \leq n} X_{n,\ell} \tag{4.23}$$

we obtain the recursive equation

$$M_n \stackrel{d}{=} \frac{n-1}{n} + \frac{\lceil nU \rceil - 1}{n} M_{\lceil nU \rceil - 1} \vee \frac{n - \lceil nU \rceil}{n} \overline{M}_{n - \lceil nU \rceil}. \tag{4.24}$$

This leads to the limit equation

$$S \stackrel{d}{=} 1 + US \vee (1-U)\overline{S}, \tag{4.25}$$

the worst case FIND *equation. All conditions of Theorem 4.6 are satisfied for any* $s > 1$. *We therefore obtain that for any* $s > 1$

$$\ell_s(M_n, S) \to 0 \tag{4.26}$$

where S *is the unique solution of* (4.25) *in* M_s. *Thus uniqueness holds true in* $\bigcup_{s>1} M_s$.

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