ON THE MULTIDIMENSIONAL ASSIGNMENT PROBLEM

Ludger Rüschendorf, Freiburg

Abstract: The aim of the present paper is to show the importance of the concept of majorization to the solution of multidimensional assignment problems.

1. Introduction.

Let $(z_{i_0,\dots,i_m})_{1\le i_j\le n}$ be a (m+1)-dimensional matrix. As m-dimensional (axial) assignment problem we consider the following problem:

(1) Find $v_1,\dots,v_m \in \mathfrak{S}_n$, such that

$$\sum_{i=1}^{n} z_{i,v_1(i),\dots,v_m(i)} = \min! \text{ , where}$$

$\mathfrak{S}_n$ are the permutations of $\{1,\dots,n\}$ and the minimum is over all permutations.

A related problem is the m-dimensional bottleneck-problem:

(2) Find $v_1,\dots,v_m \in \mathfrak{S}_n$ such that

$$\max_i z_{i,v_1(i),\dots,v_m(i)} = \min!$$

Problems (1), (2) were treated e.g. by Pierskalla (1968), Burkard (1979) and Burkard, Fröhlich (1979). There are a lot of modifications and generalizations of these assignment problems as e.g. the corresponding (hyper-) plane assignmentand bottleneck problems or the quadratic assignment and bottleneck problems, but we shall mainly deal in the following with (1) and (2).

In the ample literature on this subject one can find explicit simple solutions only in certain cases for $m = 1$ (e.g. Gilmore's solution to the assignment problem with cost coefficients $z_{ij} = \alpha_i\beta_j$) and for certain cases of the quadratic assignment problem, which are based on two rearrangement results of Hardy, Littlewood and Polya (cf. e.g. Pratt (1972)). We want to show, that by means of (weak) majorization one can get explicit solutions in several further cases.

Weak majorization is defined for a, $b \in R^n$ by

(3) $$a \prec_w b \quad \text{if} \quad \sum_{j=1}^{i} a_{[j]} \leq \sum_{j=1}^{i} b_{[j]}, \quad 1 \leq i \leq n,$$

where $a_{[j]}$, $b_{[j]}$ are the j-largest components of a,b. Majorization is defined for $a,b \in R^n$ by

(4) $$a \prec b \quad \text{if} \quad a \prec_w b \text{ and } \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i .$$

<u>2. Optimal assignments and majorization.</u>

In the following we shall deal explicitely with the case $m = 2$ (and sometimes with $m = 1$). The results for $m = 2$ are easily extended to the general case $m \geq 2$.

Assume that $a,b,c \in R^n$ such that

(5) $z_{i,j,k} = \varphi \circ h(a_i, b_j, c_k)$, $1 \leq i,j,k \leq n$, where $h : R^3 \to R^1$ and $\varphi : R^1 \to R^1$. Define for $\pi, \mu \in \mathfrak{S}_n$

$h(\pi,\mu) = (h(a_i, b_{\pi(i)}, c_{\mu(i)}))_{1 \leq i \leq n}$ and

(6) $H(h) = \{h(\pi,\mu); \pi,\mu \in \mathfrak{S}_n\}$.

Let, furthermore, $M_w(h)$ $(M(h))$ be the set of all maximal elements of $H(h)$ w.r.t $\prec_w$ $(\prec)$ and $m_w(h)$ $(m(h))$ be the set of all minimal elements of $H(h)$ w.r.t. $\prec_w$ $(\prec)$.

<u>Proposition 1.</u>

a) If φ is (increasing and convex) convex, then $(m_w(h))m(h)$ contains a solution of (1), (2).

b) If φ is (decreasing and concave) concave then $(M_w(h))(M(h)$ contains a solution of (1).

<u>Proof.</u> For all $x \in H(h)$ there exists an element $y \in m(h)$ such that

$y \prec x$. If φ is convex this implies that $(\varphi(y_1),\dots,\varphi(y_n)) \prec_w (\varphi(x_1),\dots,\varphi(x_n))$ (cf. Marshall, Olkin (1979), pg. 115) and, therefore, $\sum_{i=1}^{n} \varphi(y_i) \le \sum_{i=1}^{n} \varphi(x_i)$ and $\max_{i\le n} \varphi(y_i) \le \max_{i\le n} \varphi(x_i)$. The remaining cases of Proposition 1 are treated similarly. □

Proposition 1 shows that it is interesting to determine the sets $M(h)$, $m(h)$ or $M_w(h), m_w(h)$. There are some general results concerning this question of minimal and maximal elements (cf. Marshall, Olkin (1979), pg. 132 - 137). The general idea is that minimal elements have nearly equal components, while maximal elements have a large variation in the components. By Proposition 1 we have already found a solution of (1), (2) if there is even a smallest or a largest element w.r.t. $\prec(\prec_w)$

We shall now discuss some examples to show the applicability of this idea. Let in the following $\bar{\pi}(\bar{\mu})$ resp. $\underline{\pi}(\underline{\mu})$ denote the permutation which arranges $b(c)$ in decreasing resp. increasing order. For a large class of functions the determination of $M_w(h)$ is simple by the following proposition essentially due to Lorentz (1953) and Day (1972).

<u>Proposition 2.</u>
If $a_1 \geq a_2 \geq \dots \geq a_n$, if h is montonically nondecreasing (or nonincreasing) and L-superadditive, then

(7) $h(\bar{\pi},\bar{\mu})$ is a greatest element of $H(h)$ w.r.t. $\prec_w$. □

If $m = 1$ (i.e. $h(a_i,b_j,c_k) = h(a_i,b_j)$ is independent of c_k) there is also a smallest element.

<u>Proposition 3.</u>
If $m = 1$, $a_1 \geq \dots \geq a_n$, h is monotonically nondecreasing (nonincreasing) and L-superadditive, then

(8) $h(\underline{\pi})$ is a smallest element of $H(h)$ w.r.t. $\prec_w$. □

For definition of L-superadditive functions cf. Marshall, Olkin (1979), pg. 146. Interesting examples are $f(x_1 + x_2 + x_3)$ for convex f, Πx_j for $x_j \geq 0$ and $\min x_j$. For the solution of (1) the monotonicity assumption on h is by Lorentz's theorem not necessary if $\varphi(x) \equiv x$.

Let $h_1(x_1,x_2,x_3) = \max_{1 \leq j \leq 3} x_j$, then h_1 is L- subadditive (i.e. $- h_1$ is L-superadditive and, therefore, by (7) for $a_1 \geq \ldots \geq a_n$

(9) $h_1(\bar{\pi},\bar{\mu})$ is a least element of $H(h_1)$ w.r.t. $\prec_w$.

To determine a largest element of $H(h_1)$ let π^* be the permutation which rearranges b in opposite order to a and μ^* be the permutation which arranges c in opposite order to the vector $(\max(a_1,b_{\pi^*(1)}),\ldots,\max(a_n,b_{\pi^*(n)}))$. Then it is not difficult to prove:

Proposition 4.
$h_1(\pi^*,\mu^*)$ is a largest element of $H(h_1)$ w.r.t. $\prec_w$. □

Example 1. If $n = 5$, $a_i = b_i = c_i = i$, $1 \leq i \leq 5$, then

a	5	4	3	2	1
b_{π^*}	1	2	3	4	5
c_{μ^*}	1	3	5	4	2

A largest element of $H(h_1)$ w.r.t. $\prec_w$ is (5,4,5,4,5), a smallest element is (1,2,3,4,5). □

Let $h_2(x_1,x_2,x_3) = \sum_{j=1}^{3} x_j$ and assume that the minimal difference of $|a_i-a_\ell|, |b_i-b_\ell|, |c_i-c_\ell|, i \neq \ell$, between different components of a,b,c is $d > 0$. Then the following proposition holds.

Proposition 5.
If there exist $c \in R^1$ and $\pi^*, \mu^* \in \mathfrak{S}_n$ with

(10) $h_2(a_i,b_{\pi^*(i)},c_{\mu^*(i)}) \in \{c,c+d\}$, $1 \leq i \leq n$, then $h_2(\pi^*,\mu^*)$

is a smallest element of $H(h_2)$ w.r.t. $\prec$. □

Note that condition (10) corresponds to our general idea for minimal elements of $H(h_2)$. It is possible to prove similar results for a,b,c which do not satisfy (10) exactly.

Example 2. Let $a_i = b_i = c_i = i$, $1 \leq i \leq n$. If $n = 2k$ take $a_i = i$, $1 \leq i \leq n$,

$$b_{\pi^*(i)} = \begin{cases} k+i, & i \leq k \\ i-k, & i \geq k+1 \end{cases}$$

$$c_{\mu^*(i)} = \begin{cases} n-2(i-1), & i \leq k \\ n-2(i-k)+1, & i \geq k+1 \end{cases}$$

Then $a_i + b_{\pi^*(i)} + c_{\mu^*(i)} = \begin{cases} n+k+2, & i \leq k \\ n+k+1, & i \geq k+1 \end{cases}$.

If $n = 2k+1$ take $a_i = i$, $1 \leqq i \leqq n$,

$$b_{\pi^*(i)} = \begin{cases} i+k, & i \leqq k+1 \\ i-k-1, & k+2 \leqq i \leqq n \end{cases}$$

$$c_{\mu^*(i)} = \begin{cases} n-2(i-1), & i \leqq k+1 \\ n+1+2(k-i+1), & k+2 \leqq i \leqq n. \end{cases}$$

Then $a_i + b_{\pi^*(i)} + c_{\mu^*(i)} = n+k+2$, $1 \leqq i \leqq n$.
Therefore, by Proposition 5 in both cases we have found smallest elements of $H(h)$ w.r.t. $\prec$. □

Note that Proposition 5 can also be applied to find a smallest element of $H(h_3)$, where $h_3(x_1,x_2,x_3) = \prod_{i=1}^{3} x_i$, $x_i \geqq 0$, w.r.t. $\prec_w$, observing that $h_3(x_1,x_2,x_3) = \exp\{h_2(\ell n\, x_1, \ell n\, x_2, \ell n\, x_3)\}$

3. A procedure for the determination of minimal elements.

We consider the special case of (1), (2), where $h(x_1,x_2,x_3) = \prod_{i=1}^{3} x_i$, $x_i \geqq 0$. For $x,y \in R^n$ define $x \circ y = (x_1 y_1, \ldots, x_n y_n)$ and $x \perp y$ to mean that x,y are oppositely ordered. Then with $O(h) = \{h(\pi,\mu);\ \pi,\mu \in \mathfrak{S}_n$ such that $a \perp (b_\pi \circ c_\mu)$, $b_\pi \perp (a \circ c_\mu)$ and $c_\mu \perp (a \circ b_\pi)\}$, where $b_\pi = (b_{\pi(1)}, \ldots, b_{\pi(n)})$, the following result can be derived from the Hardy, Littlewood, Polya rearrangement theorem.

Proposition 6 $m_w(h) \subset O(h)$. □

Therefore, by Proposition 1, for the solution of (1), (2) with increasing and convex φ we can find solutions in O(h) ((O(h) contains even all solutions of (1)). To find elements of O(h) the following procedure is useful.

Start with any rearrangement of a,b,c. Define $\nu \in \mathfrak{S}_n$ such that $a_\nu \perp (b \circ c)$, then let $\pi \in \mathfrak{S}_n$ be such that $b_\pi \perp (a_\nu \circ c)$ and let $\mu \in \mathfrak{S}_n$ be such that $c_\mu \perp (a_\nu \circ b_\pi)$. Now repeat these operations with new initial values a_ν, b_π, c_μ. In each step we obtain new rearrangements such that $a_\nu \circ b_\pi \circ c_\mu$ gets strictly smaller w.r.t. $\prec_w$ until we have found an element of O(h).

<u>Example 3.</u> Let $a_i = b_i = c_i = i$, $1 \leq i \leq n$, then for k = 4

a	1	2	3	4
b	4	3	1	2
c	3	2	4	1

for k = 6

a	1	2	3	4	5	6
b	5	4	3	6	1	2
c	5	4	3	1	6	2

are optimal (i.e. minimal w.r.t. $\prec_w$) rearrangements obtained by our procedure. For k = 8 an optimal rearrangement is

a	1	2	3	4	5	6	7	8
b	8	5	3	7	4	2	1	6
c	7	5	6	2	3	4	8	1

with vectors of products h_1 = (56, 50, 54, 56, 60, 48, 56, 48).

The following rearrangement is also in O(h)

a	1	2	3	4	5	6	7	8
b	7	4	5	3	6	8	1	2
c	7	6	4	5	2	1	8	2

with vector of products h_2 = (49, 48, 60, 60, 60, 48, 56, 48) and h_2 is strictly larger than h_1 w.r.t. $\prec_w$. Therefore, the inclusion in Proposition 6 can be strict. Concerning problem (1) the values of h_1, h_2 are 428, 429; concerning problem (2) the values are both 60 (for $\varphi(x) = x$). □

The experience of many examples shows this behaviour which indicates that in many cases elements of O(h) give good approximations to the optimal solution. We want to remark that a similar procedure works for several similar problems as e.g. $h(x,y,z) = f(x+y+z)$ for convex f (in this case arrange $a \perp (b+c)$ etc.)

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Address:
Institut für Mathematische Stochastik
Universität Freiburg
Hebelstr. 27
D-7800 Freiburg

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