ASSIGNMENT MODELS FOR CONSTRAINED MARGINALS AND RESTRICTED MARKETS

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Abstract:

Duality theorems for assignment models are usually derived assuming countable additivity of the population measures. In this paper, we use finitely additive measures to model assignments of buyers and sellers. This relaxation results in more complete duality theorems and gives greater flexibility concerning the existence of solutions, assumptions on the spaces of agents and on profit functions. We treat two modifications of the nonatomic assignment model. In the first model, upper and lower bounds are imposed on the marginal measures representing the activities of the buyers and sellers where the lower bounds reflect a certain minimum required level of activity on the agents. In the second model, the interaction of the agents is further restricted to a certain specified subset of all matchings of buyers and sellers.

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1 Introduction

The assignment model of Shapley and Shubik(1972) has been extended to the following version with a continuum of buyers and sellers by Gretsky, Ostroy and Zame(1992) called the nonatomic assignment model:

Consider the set X_1 of buyers in the housing market and the set X_2 of sellers each having a distinct house to sell. We associate two probability spaces $(X_i, \mathcal{A}_i, \tau_i),\ i = 1, 2$ with the buyers and the sellers respectively where τ_i represent the population distributions. Each buyer $x_1 \in X_1$ has a price $b(x_1, x_2)$ that he would be willing to pay to buy the house of x_2; on the other hand, each seller x_2 has a reservation value $r(x_2)$ indicating the minimum amount of maney that he would be willing to accept for selling his house. If buyer x_1 and seller x_2 were to transfer ownership of the house that x_2 sells then the monetary value of this transfer between the pair (x_1, x_2) can be represented by $h(x_1, x_2) = b(x_1, x_2) - r(x_2)$. Thus, if buyer x_1 is matched with the seller x_2 then $h(x_1, x_2)$ is the profit available to the pair (x_1, x_2). We suppose that the profit function $h(x_1, x_2)$ is a measurable function on $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$. The goal of the assignment problem then is to match buyers and sellers so as to maximize the total available profit. Define a $\sigma-$additive *assignment* of buyers to sellers as a measure μ on the product space and interpret it as a statistical summary of the activities in the housing market; i.e., $\mu(E \times F)$ is the distribution of buyers in E purchasing from sellers of houses in F. Call an assignment μ *feasible* for (τ_1, τ_2) if

$$\mu(E \times X_2) \leq \tau_1(E) \ \text{ and } \ \mu(X_1 \times F) \leq \tau_2(F) \quad \forall\ E \in \mathcal{A}_1, F \in \mathcal{A}_2;$$

that is, the marginals of μ are bounded by the $\tau_i, i = 1, 2$. Let $\mathcal{M}^{\leq}(\tau_1, \tau_2)$ denote the set of all σ-additive feasible assignments. The assignment problem thus formulated in the context of housing market is related to the transportation problem where h is the associated transportation cost, and equality constraints are imposed on the feasible transportation measures (see Rachev and Rüschendorf (1998)).

The aim of the assignment problem is to maximize the total available profit over all possible feasible assignments to attain

$$S(h) = \sup\left\{\int h(x_1, x_2)\, d\mu : \mu \text{ is feasible for } (\tau_1, \tau_2)\right\}.$$

This gives rise to the corresponding dual problem of finding τ_i-integrable functions $f_i, i = 1, 2$ to attain the dual functional

$$I(h) \ = \ \inf\left\{\sum_{i=1}^{2}\int f_i(x_i)\, d\tau_i : h \leq f_1 \oplus f_2\right\}$$

where $f_1 \oplus f_2 = \sum_i f_i \circ \pi_i$ with $\pi_i : X_1 \times X_2 \to X_i$ being the canonical projections on $X_1 \times X_2$ for $i = 1, 2$. The abbreviation $\oplus_i f_i$ is used for $f_1 \oplus f_2(x_1, x_2) = f_1(x_1) + f_2(x_2)$.

When the underlying spaces of the economic agents (the buyers and the sellers) can be taken to be the closed unit interval along with a nonnegative upper semicontinuous profit function h, Gretsky, Ostroy, and Zame (1992) showed that the duality

$$S(h) = I(h)$$

holds and optimal solutions exist. They also show that the dual solutions are identical to the core of an associated market game and are also equivalent to Walrasian equilibria of an associated exchange economy; thus the solution of the assignment model yields existence and characterization of equilibria. The duality result of Gretsky *et al.* has been extended in Ramachandran and Rüschendorf (1999). These extended results allow us to consider any measurable profit function which need not possess any continuity properties and to consider more general spaces of buyers and sellers. Sellers may be thus included with several commodities whose quality is represented by a function describing the price over the last year or a risk curve of an insurance product (noting that typically such spaces are not compact metric spaces).

In this paper we investigate assignment models with two distinct kinds of restrictions on the market. In the first model we treat the case where additional constraints are imposed on the market by lower bounds on the activities of the economic agents. By these lower bounds a certain minimal level of activity in the market is required of the agents. One might consider some minimum level of supply and consumption of communication networks of public telephone companies, some minimum level of supply and consumption of energy to be supplied by distributors of energy, some minimum level and consumption of public transportation connections, etc.

In order to solve this problem we extend in the first step the class of σ-additive assignments to the wider class of finite additive assignments, i.e. we dismiss with continuity properties of assignments. This extension leads to a general duality theorem including existence of solutions without any restriction on the space of agents and on the profit function.

Finitely additive measures have been introduced as portfolio measures in recent papers by Gilles and LeRoy (1992), Werner (1997a, 1997b) and others in order to describe a wider range of diversification (like perfectly diversified portfolios) or to allow for bubbles in the market i.e. to allow for discontinuities of the portfolio measures. In the context of proving the equivalence (and existence) of Walrasian equlibrium with the core, Weiss

(1981) used a finitely additive setting.

Our introduction of finitely additive assignment measures, although motivated primarily for mathematical reasons, fits with the spirit of papers mentioned in the preceding paragraph. Consideration of finitely additive measures gives more freedom concerning the class of assignments and a more satisfying and general duality result. As in the case of portfolio measures, finitely additive assignments can be decomposed uniquely into a σ-additive (fundamental) part and a purely finitely additive (bubble) part. The bubble assignments guarantee general existence of optimal assignments. We keep throughout our model the population measures to be σ-additive and the assignment measures to be finitely additive. An extension to the case of finitely additive population measures follows readily. We then identify in a second step natural conditions which ensure that we can restrict to σ-additive assignments which are easier to interpret in comparison to finitely additive assignments.

In the second model, we consider the case where all activities in the market take place within a certain subset of the pairs of economic agents; that is, only those assignments which are concentrated on a specified subset C of $X_1 \times X_2$ are considered feasible. A restriction on the support appears natural in some applications like in the context of housing markets. Some buyers cannot be matched with some sellers no matter what. We show that this model can be solved by reducing it to a related classical assignment model. Therefore, the available results for the classical model allow us to establish a general solution in this setting.

Gretsky, Ostroy, and Zame (1992) study the duality problem for continuous housing markets and investigate the relation to the Walrasian equilibria of the associated market economy and to the (distributional) core of the associated market game. Here a market economy is given by the endowment, the population measure representing the agents, the preferences of the agents, described by their utility functions and finally a price system p. An allocation y and a price system p is a Walrasian equilibrium if for all agents i the allocation $y(i)$ is utility maximizing at prices p, i.e. $u_i(y(i)) - \langle p, y(i) \rangle = \max\{u_i(w) - \langle p, w \rangle\}$, where u_i is the utility funciton of agent i and the maximum is over all admissible allocations for agent i, equivalently p is in the subgradient of u_i at $y(i)$, $p \in \partial u_i(y(i))$. The market game related to the assignment problem is defined by specifying its game-theoretic characteristic function $w(C)$ defined as the maximum profit available to coalition C which is a subset of the set of all buyers and sellers. The core of this game is the system of all 'distributions' of the profit to the possible coalitions described technically in terms of finitely additive set functions. For details on these notions see Gretsky, Ostroy and Zame (1992, p. 110-112).

The main result in the paper of Gretsky, Ostroy and Zame (1992) is on the existence of Walrasian equilibria in a continuous market model and the equivalence of Walrasian equilibrium with the core and with optimal solutions of the dual of the assignment model (Gretsky, Ostroy, and Zame (1992, Theorems 3, 4, 5, 8)). This result has been proved for the finite assignment problem in Shapley and Skubik (1972) by duality theory. Aumann (1966) and Brown (1977) (see the paper of Weiss (1981)) give a different proof of the equivalence theorem based on a version of Lyapounov's convexity theorem. The arguments in the paper of Gretsky, Ostroy and Zame (1992) are based essentially on the duality theorem where solutions of the dual problem are known to exist. Since we prove this duality theorem in our modified assignment models and since the further arguments of Gretsky, Ostroy and Zame (1992) extend to these modified models we obtain as consequence also the equivalence theorem and the existence of Walrasian equilibria for the modified assignment models. This step does not need further new arguments. Therefore, we refer the reader for the equivalence theorem to the paper of Gretsky, Ostroy and Zame (1992) in order to not just repeat their results in our context.

2 Model with constraints on the agents

Let $(X_i, \mathcal{A}_i)$, $i = 1, 2$ be two measurable spaces representing the buyers and sellers respectively. Let λ_i, τ_i, $i = 1, 2$ be finite measures on $(X_i, \mathcal{A}_i)$ such that (i) $\lambda_i \leq \tau_i$ and (ii) $\|\lambda_1\| = \|\lambda_2\|$, $\|\tau_1\| = \|\tau_2\| = 1$; λ_i's represent the lower bounds requiring at least a certain minimal level of activity on the agents imposed by some external regulations on the market and τ_i's represent the common upper bounds in the nonatomic assignment model. Further motivation of this type of restrictions on the agents is given in the introduction. For a finite measure μ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ define

$$\mu_i = \mu \circ \pi_i = \text{the marginal of } \mu \text{ on } \mathcal{A}_i, \ i = 1, 2.$$

Let $\mathcal{M}_{\lambda,\tau}$ denote the set of all σ-additive assignments with lower and upper bounds on the marginals, i.e.,

$$\mathcal{M}_{\lambda,\tau} = \{\mu \text{ on } \mathcal{A}_1 \otimes \mathcal{A}_2 : \lambda_i \leq \mu_i \leq \tau_i, \ i = 1, 2\} .$$

Define also for a formulation of the dual problem the class $\mathcal{F}$ of functions of a sum form

$$\mathcal{F} = \{\oplus_i (f_i - g_i) : \ f_i \geq 0, \ g_i \geq 0, \ f_i, g_i \in \mathcal{L}^1(\tau_i), \ i = 1, 2\}$$

where $\mathcal{L}^1(\tau_i)$ is the class of τ_i-integrable functions on $(X_i, \mathcal{A}_i)$, for $i = 1, 2$. As class of profit functions we admit any measurable function up to a weak integrability condition

$$\mathcal{L}_m = \{\phi \in \mathcal{L}(\mathcal{A}_1 \otimes \mathcal{A}_2) : \exists f \in \mathcal{F} \text{ with } f \leq \phi\}.$$

where $\mathcal{L}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ is the class of $\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable functions on $X_1 \times X_2$. In particular $\mathcal{L}_m$ contains all bounded, measurable functions.

Analogous to the case of the nonatomic assignment model we define for $h \in \mathcal{L}_m$ the objective functional

$$S_{\lambda,\tau}(h) = \sup\left\{\int h \, d\mu : \mu \in \mathcal{M}_{\lambda,\tau}\right\}$$

and introduce as in Rüschendorf (1981, 1991) a dual functional

$$I_{\lambda,\tau}(h) = \inf\left\{\sum_{i=1}^{2}\left(\int f_i \, d\tau_i - \int g_i \, d\lambda_i\right) : f_i, g_i \geq 0 \text{ and } \oplus_i (f_i - g_i) \geq h\right\}.$$

Remark: Unlike the I-functional in Section 1, $I_{\lambda,\tau}$ fails to satisfy the important partial linearity property

$$I_{\lambda,\tau}(h + h_0) = I_{\lambda,\tau}(h) + I_{\lambda,\tau}(h_0)$$

for simple functions $h_0 = \oplus_i f_i$, $f_i \in \mathcal{L}(\mathcal{A}_i)$, $i = 1, 2$ when $\|\lambda_1\| < 1$. Indeed this property does not hold in the simple case with $h = -1$, $h_0 = 2$ since, in this case, $I_{\lambda,\tau}(-1+2) = I_{\lambda,\tau}(1) = \|\tau_1\| = 1$ while $I_{\lambda,\tau}(-1) = -\|\lambda_1\|$, $I_{\lambda,\tau}(2) = 2\|\tau_1\| = 2$ whereby $I_{\lambda,\tau}(-1+2) < I_{\lambda,\tau}(-1) + I_{\lambda,\tau}(2)$. This nonlinearity causes more technical problems in comparison to the fixed marginals case as considered in Rachev and Rüschendorf (1998). □

We next enlarge the assignment model by requiring assignments only to be finitely additive and not necessarily σ-additive. Finitely additive assignments give a greater degree of flexibility for assigning economic agents. By the well known Yosida-Hewitt theorem we can decompose each finitely additive assignment uniquely into a σ-additive part and a purely finitely additive part which we might call in analogy to the corresponding use in pricing theory as the fundamental assignment and the bubble assignment. The bubble assignments turn out to be important for a general result on the existence of solutions. It also turns out that this relaxation of σ-additivity leads to a more complete duality result.

Define the finitely additive assignment model $\widetilde{\mathcal{M}}_{\lambda,\tau}$ by

$$\widetilde{\mathcal{M}}_{\lambda,\tau} = \{\widetilde{\mu} \text{ on } \mathcal{A}_1 \otimes \mathcal{A}_2 : \widetilde{\mu} \text{ is a finitely additive measure}$$
$$\text{with } \lambda_i \leq \widetilde{\mu}_i = \widetilde{\mu} \circ \pi_i \leq \tau_i,\ i = 1, 2\}.$$

Note that $\mathcal{M}_{\lambda,\tau} \subset \widetilde{\mathcal{M}}_{\lambda,\tau}$. Let $\widetilde{S}_{\lambda,\tau}$ denote the corresponding modified objective functional defined for $h \in \mathcal{L}_m$ by

$$\widetilde{S}_{\lambda,\tau}(h) = \sup \left\{ \int h \; d\widetilde{\mu} : \widetilde{\mu} \in \widetilde{\mathcal{M}}_{\lambda,\tau} \right\} .$$

The integral $\int h d\widetilde{\mu}$ for finitely additive measures $\widetilde{\mu}$ is developed in Dunford and Schwartz (1958, p.112-125) (see also Bhaskara Rao and Bhaskara Rao(1983)). A short introduction to this integral is also given in Weiss (1981).

We first characterize $\widetilde{\mathcal{M}}_{\lambda,\tau}$ by the dual functional $I_{\lambda,\tau}$.

Lemma 1 *Let $\widetilde{\mu}$ be a finite signed measure on $\mathcal{A}_1 \otimes \mathcal{A}_2$. Then $\widetilde{\mu} \in \widetilde{\mathcal{M}}_{\lambda,\tau}$ if and only if $\widetilde{\mu}$ is majorized by $I_{\lambda,\tau}$ i.e., $\widetilde{\mu} \leq I_{\lambda,\tau}$.*

<u>Proof:</u> Suppose $\widetilde{\mu} \in \widetilde{\mathcal{M}}_{\lambda,\tau}$. Then for $f_i, g_i \geq 0$, $\sum_i (f_i - g_i) \geq h$ we have

$$\begin{aligned} \sum_i \left(\int f_i \; d\tau_i - \int g_i \; d\lambda_i \right) \;&\geq\; \sum_i \left(\int f_i \; d\widetilde{\mu_i} - \int g_i \; d\widetilde{\mu_i} \right) \\ &=\; \int \left(\oplus_i (f_i - g_i) \right) \; d\widetilde{\mu} \\ &\geq\; \int h \; d\widetilde{\mu} \end{aligned}$$

giving the "$\Rightarrow$" part.

Conversely, if $\widetilde{\mu} \leq I_{\lambda,\tau}$ then $\widetilde{\mu}$ is positive,

$$\int 1_{A_1} \; d\tau_1 \;=\; I_{\lambda,\tau}(1_{A_1 \times X_2}) \;\geq\; \int 1_{A_1 \times X_2} \; d\widetilde{\mu} \;=\; \widetilde{\mu}_1(A_1)$$

and

$$-\int 1_{A_1} \; d\lambda_1 \;=\; I_{\lambda,\tau}(-1_{A_1 \times X_2}) \;\geq\; -\int 1_{A_1 \times X_2} \; d\widetilde{\mu} \;=\; -\widetilde{\mu}_1(A_1)$$

and so $\lambda_1 \leq \widetilde{\mu}_1 \leq \tau_1$. Similarly $\lambda_2 \leq \widetilde{\mu}_2 \leq \tau_2$ giving the "$\Leftarrow$" part. □

With the aid of Lemma 1 we obtain a general finitely additive duality result.

Theorem 1 (Optimal finitely additive assignments)
(a) For all $h \in \mathcal{L}_m$

$$\widetilde{S}_{\lambda,\tau}(h) \;=\; I_{\lambda,\tau}(h)$$

(b) There exists an optimal assignment $\widetilde{\mu}_h \in \widetilde{\mathcal{M}}_{\lambda,\tau}$ *with*

$$I_{\lambda,\tau}(h) = \int h \; d\widetilde{\mu}_h.$$

<u>Proof:</u> Using subadditivity of the funcitonal $I_{\lambda,\tau}$ we conclude from the Hahn Banach Theorem the existence of a linear functional T on $\mathcal{L}_m$ such that $T \leq I_{\lambda,\tau}$ and such that for a fixed $h \in \mathcal{L}_m \quad T(h) = I_{\lambda,\tau}(h)$ holds. Next we use the Riesz representation theorem for the space of bounded measurable funtions $B(X, \mathcal{A})$ (see Dunford and Schwartz (1958, p. 258)). It states that the dual of $B(X, \mathcal{A})$ is given by $ba(X, \mathcal{A})$ the set of finitely additive bounded measures on $(X, \mathcal{A})$ with $x^*(f) = \int f d\mu$ for $x^* \in B(X, \mathcal{A})^*$ for a finitely additive measure μ representing x^*. This implies the existence of a finitely additive measure $\widetilde{\mu}_h$ representing T using that for $h \in \mathcal{L}_m$ we have $I_{\lambda,\tau}(h) > -\infty$. By Lemma 1 it holds that $\widetilde{\mu}_h \in \widetilde{\mathcal{M}}_{\lambda,\tau}$ and $I_{\lambda,\tau}(h) = \int h d\widetilde{\mu}_h$. This implies the result. For more details of this proof in a related contex of transportation problems see Rüschendorf (1981) resp. Rachev and Rüschendorf (1998). $\square$

So for the finitely additive duality result where we require that the assignments need only be finitely additive measures, we do not need any topological assumptions on the underlying spaces of agents (or commodities) as well as any continuity assumptions on the profit functions. Thus finite additivity gives greater flexibility of assigning agents and results in a possibly larger value in the assignment problem. A similar consequence of finitely additive measures has also been observed in recent papers on valuation theory of asset prices (see Gilles and LeRoy (1991), Werner (1997a, 1997b)). We note that Theorem 1 is also valid in the context of finitely additive population measures.

Next, we single out conditions where the class $\widetilde{\mathcal{M}}_{\lambda,\tau}$ of finitely additive assignments can be replaced by its subclass $\mathcal{M}_{\lambda,\tau}$ consisting of the usual σ-additive assignments in $\widetilde{\mathcal{M}}_{\lambda,\tau}$ and where we can replace $\widetilde{S}_{\lambda,\tau}$ by the corresponding $S_{\lambda,\tau}$.

Recall that a probability space $(\Omega, \mathcal{A}, P)$ is called *perfect* (equivalently, the probability P on $(\Omega, \mathcal{A})$ is called *perfect*) if for every $\mathcal{A}$-measurable, real-valued function f on Ω we can find a Borel set $B_f \subset f(\Omega)$ such that $P(f^{-1}B_f) = 1$; all probabilities on the Borel subsets of a complete, separable metric space are perfect and perfect probabilities have many desirable features for probabilistic applications (see Ramachandran (1979)).

We obtain the following σ-additive duality theorem. Remind that a function h is called upper semicontinuous if for all α the set $\{h \geq \alpha\}$ is closed.

Theorem 2 (σ-additive assignments)
A1) Let one of the measures $\tau_i,\ i = 1, 2$ be perfect and let $U(\mathcal{R})$ denote the class of all functions h which can be approximated uniformly by $\mathcal{A}_1 \otimes \mathcal{A}_2-$measurable functions of the form $\sum \alpha_j\ 1_{A_j \times B_j}$. Then

$$S_{\lambda,\tau}(h) = I_{\lambda,\tau}(h)$$

for all functions h in $U(\mathcal{R})$ and there exists an optimal assignment measure.

A2) If $(X_i, \mathcal{A}_i)$ are Hausdorff spaces and $\tau_i,\ i = 1, 2$ are Radon measures then

$$S_{\lambda,\tau}(h) = I_{\lambda,\tau}(h)$$

for all upper semicontinuous functions h bounded below and there exists an optimal assignment measure.

A3) If h is bounded then there exist $f_i^*, g_i^* \geq 0$ in $\mathcal{L}^1(\tau_i)$ such that

$$I_{\lambda,\tau}(h) \ = \ \sum_i \left(\int f_i^*\ d\tau_i - \int g_i^*\ d\lambda_i \right).$$

<u>Proof:</u> A1 and A2 (for bounded continuous functions) follow from a result of Marczewski and Ryll-Nardzewski (1953) and using similar techniques as in Rüschendorf (1991). Essentially we can infer under the above conditions on the functions considered that for any finitely additive measures $\tilde{\mu}$ we can find a corresponding σ-additive μ such that $\int h\ d\tilde{\mu} \ = \ \int h\ d\mu$. For A3 note that

$$\begin{aligned} I_{\lambda,\tau}(h) \ &= \ \inf \left\{ \sum_i \left(\int f_i\ d\tau_i - \int g_i\ d\lambda_i \right) : f_i, g_i \geq 0,\ \sum_i (f_i - g_i) \geq h \right\} \\ &= \ \inf \left\{ \sum_i \left(\int f_i^+\ d\tau_i - \int f_i^-\ d\lambda_i \right) : f_i \in \mathcal{L}^1(\tau_i),\ \sum f_i \geq h \right\}. \end{aligned}$$

Then one can assume the functions f_i to be bounded by a universal bound k depending only on h and argue as in Gaffke and Rüschendorf (1981).

As in Proposition 1.28 of Kellerer (1984) this property implies that $I_{\lambda,\tau}$ is σ-continuous upwards and so we get $S_{\lambda,\tau}(h) = I_{\lambda,\tau}(h)$ for all upper semi-continuous functions bounded below. Note that the existence of solutions in the dual functional holds without any topological assumptions. $\square$

Remarks: (a) The proof of the equivalence theorem in the paper of Gretsky, Ostroy and Zame (1992) is based essentially on the duality theorem for the assignments problem (their Theorems 1,2). We would like to point out that the proof of this duality theorem of Gretsky, Ostroy and Zame (their Theorem 1, p.118) is incomplete for the σ-additive case. Their argument on the existence of a norm one projection on p. 119 does not imply that integrals remain the same and one needs additional arguments to establish the σ-additive duality result. So A2 of our Theorem 2 fills this gap in their paper for the standard assignment model.

(b) There are simple examples where $\widetilde{S}_{\lambda,\tau}(h) = S_{\lambda,\tau}(h)$ and where the optimal assignment $\widetilde{\mu} \in \widetilde{\mathcal{M}}_{\lambda,\tau}$ is a purely finitely additive (bubble) assignment. So the increased flexibility of the class of finitely additive assignments does not lead to an increased value of the objective functional but results in the existence of an optimal assignment. Take for example $X_1 = X_2 = [0,1], \lambda_i = \tau_i$ is the Lebesgue measure λ on $[0,1]$ and $h = 1_{\{(x,y):0\leq x<y\leq 1\}}$. Then the optimal assignment is given by the purely finitely additive measure with marginals λ concentrated on the 'subdiagonal' which gives mass zero to any rectangle below the diagonal and also does not charge the diagonal whereas no σ-additive assignment exists. $\square$

For a measure μ we denote by μ^+ the positive part $(\mu \vee 0)$ and for a function h we denote by h^+, h^- its positive and negative parts respectively. To distinguish the $S-$ and $I-$functionals in the case of fixed marginal measures, say $\alpha_i, i = 1,2$ we introduce the notation

$$S_{(\alpha_i)}(h) = \sup\{\int h \, d\mu : \mu \in \mathcal{M}_{\lambda,\tau} \text{ with } \mu \circ \pi_i = \alpha_i\}$$

$$I_{(\alpha_i)}(h) \; = \; \inf\left\{ \sum_{i=1}^{2} \int f_i(x_i)\, d\alpha_i : h \leq f_1 \oplus f_2 \right\} .$$

For a general $h \in \mathcal{L}_m$ we point out a reduction formula to calculate $S_{\lambda,\tau}(h)$ using the simpler S-functionals for the case of fixed marginals.

Proposition 1 *For $h \in \mathcal{L}_m$*

$$S_{\lambda,\tau}(h) = \sup_{\alpha_i \leq \tau_i} \left(S_{(\alpha_i)}(h^+) - s_{(\lambda_i - \alpha_i)^+}(h^-)\right)$$

where $S_{(\alpha_i)}(h^+) = \sup\{\int h^+ \, d\mu : \mu\circ\pi_i = \alpha_i\}$ *and* $s_{(\lambda_i-\alpha_i)^+}(h^-) = \inf\{\int h^- \, d\mu : \mu \circ \pi_i = (\lambda_i - \alpha_i)^+\}$.

Proof: Let $\mu \in \mathcal{M}_{\lambda,\tau}$ and let $H = \{h \geq 0\}$. Let $\alpha_i = (\mu|_H) \circ \pi_i \leq \tau_i$ and let $\beta_i = (\mu|_{H^c}) \circ \pi_i$. Then $\lambda_i \leq \mu \circ \pi_i = \alpha_i + \beta_i \leq \tau_i$ and

$$\begin{aligned}\int h \, d\mu &\leq \int h^+ \, d\mu - \int h^- \, d\mu \\ &\leq S_{(\alpha_i)}(h^+) - s_{(\beta_i)}(h^-) \\ &\leq S_{(\alpha_i)}(h^+) - s_{(\lambda_i-\alpha_i)^+}(h^-).\end{aligned}$$

Taking the supremum over $\mu \in \mathcal{M}_{\lambda,\tau}$ we get

$$S_{\lambda,\tau}(h) \leq \sup_{\alpha_i \leq \tau_i} (S_{(\alpha_i)}(h^+) - s_{(\lambda_i-\alpha_i)^+}(h^-)).$$

The other inequality is established by similar arguments and noting that for positive functions h^+, we have $S^{\leq}_{(\alpha_i)}(h^+) = \sup\{\int h^+ \, d\mu : \mu \circ \pi_i \leq \alpha_i\} = S_{(\alpha_i)}(h^+)$ as established in Ramachandran and Rüschendorf (1999). $\Box$

Remark: Note that for nonnegative h in $\mathcal{L}_m$, by Proposition 1, we get

$$\begin{aligned} S_{\lambda,\tau}(h) &= S_{(\tau_i)}(h) && (\text{since } h^- \equiv 0) \\ &= I_{(\tau_i)}(h) && (\text{by the duality theorem} \\ & && \quad \text{for fixed marginals}) \\ &= I_{\lambda,\tau}(h) && (\text{since } h \geq 0).\end{aligned}$$

For a general $h \in \mathcal{L}_m$ one might now think that if μ^* is a solution for $S_{(\tau_i)}(h^+)$ and ν^* is a solution for $\inf_{\{(\lambda_i-\alpha_i)^+ \leq \mu\circ\pi_i\}} \int h^- \, d\mu$ where $\alpha_i = (\mu^*|_{\{h\geq 0\}}) \circ \pi_i$ then

$$\eta = \mu^*|_{\{h \geq 0\}} + \nu^*|_{\{h<0\}} \in \mathcal{M}_{\lambda,\tau}$$

is a solution for $S_{\lambda,\tau}(h)$. This is not true however as shown by the following example:

Example: Let $X_1 = X_2 = [0,1], \mathcal{A}_1 = \mathcal{A}_2 =$ the Borel σ-algebra, $\lambda_1 = \lambda_2 = \tau_1 = \tau_2 = \lambda$ the Lebesgue measure and define

$$
h(x_1,x_2)=\begin{cases} -1 & \text{if } 0\le x_1\le \frac12,\quad 0\le x_2<\frac12 \\ -12 & \text{if } 0\le x_1<\frac12,\quad \frac12\le x_2\le 1 \\ +1 & \text{if } \frac12\le x_1\le 1,\quad 0\le x_2<\frac12 \\ -1 & \text{if } \frac12\le x_1\le 1,\quad \frac12\le x_2\le 1. \end{cases}
$$

By choosing μ concentrated on the diagonal with marginals λ, we get $S_{\lambda,\tau}(h)=-2$. But the intended procedure will give $\int h\,d\mu=\frac14(1+(-12))=-\frac{11}{4}$!

It is a rather surprising fact that in general the same measure with given marginals τ_1,τ_2 cannot simultaneously maximize the probability of two sets B_0,B_1 such that $B_0\subset B_1$.

Proposition 2 *Let $B_0,B_1\in\mathcal{A}_1\otimes\mathcal{A}_2$ with $B_0\subset B_1$. Then, in general, there does not exist μ_0 with marginals τ_1,τ_2 such that $\mu_0(B_0)=S_{(\tau_i)}(1_{B_0})$ and $\mu_0(B_1)=S_{(\tau_i)}(1_{B_1})$.*

Proof: To see this, consider the unit square with the Lebesgue measure λ as the marginals and let $B_0=[\frac12,1]\times[0,\frac12]$, $B_1=([0,1]\times[0,1])-([0,\frac12)\times(\frac12,1])$. The measure that maximizes $S_{(\tau_i)}(B_0)=\frac12$ is concentrated in $B_0\cup[0,\frac12)\times(\frac12,1]$ while the measure that maximizes $S_{(\tau_i)}(B_1)$ lives in $[0,\frac12)\times[0,\frac12)\cup[\frac12,1]\times[\frac12,1]$ with $S_{(\tau_i)}(B_1)=1$. □

In the topological setting as under assumption A2 of Theorem 1, we next extend the duality for $S_{\lambda,\tau}$ to a large class of functions. Using results in Ramachandran and Rüschendorf (1997) for the fixed marginal case we have for upper semicontinuous functions h bounded from below

$$
\begin{aligned}
S_{\lambda,\tau}(h) &= \sup_{\mu\in\mathcal{M}_{\lambda,\tau}} S_{(\mu\circ\pi_i)}(h)\\
&= \sup_{\mu\in\mathcal{M}_{\lambda,\tau}} I_{(\mu\circ\pi_i)}(h) \quad =: \quad \overline{I}_{\lambda,\tau}(h)
\end{aligned}
$$

Based on this result we next extend the duality result in Theorem 2, A2) using the dual functional $\overline{I}_{\lambda,\tau}$. Let $\mathcal{F}_m$ denote the collection of upper semi-continuous functions in $\mathcal{L}_m$ and let $\mathcal{H}_m$ denote the collection of functions in $\mathcal{L}_m$ which are increasing limits of functions in $U(\mathcal{R})$.

Theorem 3 *The duality result*

$$S_{\lambda,\tau}(h) = \overline{I}_{\lambda,\tau}(h)$$

holds for $h \in \mathcal{H}_m$ under Condition A1) of Theorem 2 and for $h \in \mathcal{F}_m$ under Condition A2) of Theorem 2.

Proof: Generally it holds by definition that $S_{\lambda,\tau}(h) \leq \overline{I}_{\lambda,\tau}(h) \leq I_{\lambda,\tau}(h)$. Therefore, we obtain from Theorem 2

$$\overline{I}_{\lambda,\tau}(h) = S_{\lambda,\tau}(h) = I_{\lambda,\tau}(h)$$

for all bounded, continuous h under A2) and for h in $U(\mathcal{R})$ under A1). Consider $h_n \uparrow h, h_n$ bounded and continuous under A2) or in $U(\mathcal{R})$ under A1) where $h \in \mathcal{F}_m$ resp. $h \in \mathcal{H}_m$. Note that

$$\begin{aligned} S_{\lambda,\tau}(h) &= \sup_{\mu \in \mathcal{M}_{\lambda,\tau}} S_{(\mu\circ\pi_i)}(h) \\ &= \sup_{\mu \in \mathcal{M}_{\lambda,\tau}} \sup_n S_{(\mu\circ\pi_i)}(h_n) \\ &= \sup_n S_{\lambda,\tau}(h_n). \end{aligned}$$

In the same way, we obtain from the continuity of $I_{(\mu_i)}(h)$ (see Kellerer (1984)) and from the definition of $\overline{I}_{\lambda,\tau}$ the similar relation for $\overline{I}_{\lambda,\tau}$. Together this implies

$$S_{\lambda,\tau}(h) = \lim_n \uparrow S_{\lambda,\tau}(h_n) = \lim_n \uparrow \overline{I}_{\lambda,\tau}(h_n) = \overline{I}_{\lambda,\tau}(h). \qquad \Box$$

3 Model for restricted markets

In this section we treat the assignment problem where restrictions are imposed on the admissible interactions among the economic agents; that is, only those feasible assignments that are concentrated on a specified subset C of all matchings of buyers and sellers are considered to be admissible. These restrictions could be imposed by regulations on the market or by market conditions. This model describes the case where each buyer has only a certain subset (depending on the buyer) of sellers available to trade with.

Let $(X_i, \mathcal{A}_i, \tau_i)$, $i = 1, 2$ and $h(x_1, x_2)$ be as in the preceding section where, without loss of generality, the τ_i are assumed to be probabilities. Let $C \in \mathcal{A}_1 \otimes \mathcal{A}_2$ denote the set of admissible interactions and let

$$\mathcal{M}_C^{\leq} = \{\mu \in \mathcal{M}^{\leq}(\tau_1, \tau_2) : \mu(C^c) = 0\}$$

be the set of all feasible assignments whose marginals are bounded by the $\tau_i, i = 1, 2$ and concentrated on C. Our aim in this setting is to find an admissible assignment that maximizes the expected profit; that is, we want to solve the following optimization problem:

$$S_C^{\leq}(h) \;=\; \sup\left\{\int h\; d\mu : \mu \in \mathcal{M}_C^{\leq}\right\}$$

for all $h \in \mathcal{L}_m$. We obtain the following general duality theorem.

Theorem 4 *Let one of the measures $\tau_i, i = 1, 2$ be perfect. Then for any $h \in (\mathcal{A}_1 \otimes \mathcal{A}_2)_m$ the following duality holds:*

$$S_C^{\leq}(h) \;=\; I(h^+ 1_C)$$

where $I(.)$ is as defined in Section 1.

Proof: We will prove this duality result by reducing it to a duality problem with fixed marginals and then use the main result of Ramachandran and Rüschendorf (1995).

First note that the objective functional satisfies

$$S_C^{\leq}(h) \;=\; S_C^{\leq}(h 1_C) \;=\; S_C^{\leq}(h^+ 1_C)\;.$$

The first equality is obvious. For the second equality the right side is clearly greater than or equal to the left side. Conversely to any $\mu \in \mathcal{M}_C^{\leq}$ define μ_h to be $\mu|_{\{h\geq 0\}}$. Then $\mu_h \in \mathcal{M}_C^{\leq}$ and $\int h 1_C d\mu_h = \int h^+ 1_C d\mu_h = \int h^+ 1_C d\mu$, giving the second equality.

We next establish that $S_C^{\leq}(h^+ 1_C) = S(h^+ 1_C)$, where $S(.)$ is as defined in Section 1. Obviously $S_C^{\leq}(h^+ 1_C) \leq S(h^+ 1_C)$. On the other hand, for any $\mu \in \mathcal{M}^{\leq}(\tau_1, \tau_2)$ define $\mu_C = \mu|_C$. Then $\mu_C \in \mathcal{M}_C^{\leq}(\tau_1, \tau_2)$ and $\int h 1_C d\mu_C = \int h 1_C d\mu$. This implies that equality holds.

Next, for any nonnegative function g we have that $S(g) = S_{(\tau_i)}(g)$ where $S_{(\tau_i)}(.)$ are defined in the last section and corresponds to the duality problem with fixed marginals. Obviously, $S(g) \geq S_{(\tau_i)}(g)$. On the other hand, for any $\mu \in \mathcal{M}^{\leq}(\tau_1, \tau_2)$ with marginals λ_1, λ_2 where $\lambda_i \leq \tau_i$ let λ be any measure on the product with marginals $\tau_i - \lambda_i$ (e.g., we could take $\lambda = \frac{1}{\|\tau_1 - \lambda_1\|}((\tau_1 - \lambda_1) \otimes (\tau_2 - \lambda_2)))$. Then $\tau := \mu + \lambda$ has fixed marginals τ_i and $\int g d\tau = \int g d\mu + \int g d\lambda \geq \int g d\mu$. This implies $S(g) \leq S_{(\tau_i)}(g)$ and so we have equality. Hence, $S(h^+ 1_C) = S_{(\tau_i)}(h^+ 1_C)$.

Finally, we apply the general duality theorem of Ramachandran and Rüschendorf (1995) for the fixed marginals case to obtain $S_{(\tau_i)}(h^+1_C) = I(h^+1_C)$. Combining the steps, the result follows. $\square$

Remarks: (a) The difference between the model with assignments restricted to a subset C and the model without such a restriction is given by $I(h_+) - I(h_+1_C)$. In specific circumstances we get more information on the loss due to the restriction. Let, for instance, $h = 1_B$ be the indicator funtion of a subset $B \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Then by a well known representation of $I(.)$ (see Ramachandran and Rüschendorf (1995))

$$I(1_B) \;=\; \inf\{\tau_1(A_1) + \tau_2(A_2) \;:\; B \subset (A_1 \times X_2) \cup (X_1 \times A_2)\}$$

while

$$\begin{aligned} I(1_B \cdot 1_C) &= I(1_{B\cap C}) \\ &= \inf\{\tau_1(A_1) + \tau_2(A_2) \;:\; B\cap C \subset (A_1 \times X_2) \cup (X_1 \times A_2)\} \end{aligned}$$

For concrete sets B, C one gets from these representations sharp bounds for the difference.
(b) Note that for proving Theorem 4, if we begin with a version of the duality theorem which is valid only for upper semicontinuous functions in the classical assignment problem then we will be severely limited in the choice of the restriction set C which might not be natural from the point of view of possible applications.
(c) The dual problem in the paper of Gretsky *et al.*(1992) as well as in our Theorems 1, 2, and 3 do not use positive functions. In Theorem 4, we need the positive part of the profit function. This comes from the fact as explained in the proof that on one hand side $S_C^{\leq}(h) = S_C^{\leq}(h^+1_C)$. On the other hand generally $I(h^+1_C)$ is not identical to $I(h1_C)$. Take for example $C = X_1 \times X_2$ and $h \equiv -2$, then $I(h1_C) = -2$ and $I(h^+1_C) = 0$. The result would also be wrong with an alternative dual functional involving only positive functions as $\inf\Big\{ \sum_{i=1}^2 \int f_i(x_i)\, d\tau_i : h \leq f_1 \oplus f_2,\; f_i \geq 0\Big\}$.
(d) As in Section 2 for the model $\widetilde{\mathcal{M}}_{\lambda,\tau}$ we obtain a finitely additive version of the duality result in Theorem 4 for the case of restricted markets without any assumption on the underlying spaces and profit functions if we consider in the definition of the objective functional finitely additive assignment measures which are restricted to C. For the proof we use a reduction similar to the one in the fixed marginal case as in the proof of Theorem 4 and then apply the corresponding finitely additive duality theorem in Rüschendorf (1981).

As in Theorem 1, we have in general existence of a finitely additive optimal assignment.
(e) As mentioned in the introduction, the equivalence of the Walrasian equilibrium with the core and the optimal solutions of the assignment model follows from the arguments of Gretsky *et al.*(1992). □

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