On a general approach to dynamic term structures

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Metabief, Jan 2017

Motivation

- Term structure models (risk-free, credit risky, multi-curve) are typically build from a family of fundamental instruments which offer at maturity

$$C_T.$$

- Under a suitable no-arbitrage criterion there exists an equivalent martingale measure with respect to a certain numéraire, say X^0, s.t.

$$P(t,T) = E_Q\left[\frac{X_t^0}{X_T^0} C_T | \mathscr{F}_t\right].$$

- If $C_T = 1$ (interest rates), Heath, Jarrow and Morton proposed

$$P(t,T) = e^{-\int_t^T f(t,u)du} \tag{1}$$

- Many events (ECB-interest rates, earning announcments, Brexit, etc.) occur at **predictable times** (not totally inaccessible) such that (1) may lead to arbitrage possibilities.
- Well-acknowledged in economics literature: e.g. Piazzsesi (2001,2005)

- It is our goal to introduce a general framework which applies to interest-rate modelling, credit risk and multiple yield-curve modelling.
- The key step is to consider term-structures spanned by

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\mu_t(du) \right), \tag{2}$$

where μ is a finite optional random measure $\mu(ds,du)$ on $[0,T^*]^2$, and

$$\mu_t(du) := \mu\big([0,t],du\big).$$

- The processes $f(.,T), 0 \le T \le T^*$ satisfy

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)dA_s + \int_0^t \beta(s,T)\cdot dX_s, \tag{3}$$

A is of finite variation and X is a d-dimensional semimartingale.

Classical examples

Heath-Jarrow-Morton (1992)

consider the interest-rate case, i.e. $C_T = 1$ and

$$P(t,T) = e^{-\int_t^T f(t,u)du}.$$

This leads to arbitrage in the general case.

Björk, Di Masi, Kabanov, Runggaldier (1997) show: if, additionally,

(i) $X^0 = e^{\int_0^\cdot f(s,s)ds}$ is the numéraire, and

(ii) X is stochastically continuous with local characteristics (b, c, K)

then Q is a local martingale measure $\Leftrightarrow$

$$\bar{\alpha}(t,T) = \Psi_t(-\bar{\beta}(t,T)),$$

for all $0 \leq t \leq T \leq T^*$, $dP \otimes dt$-almost surely; where (recall $\mu_t(du) = du$)

$$\bar{\alpha}(t,T) = \int_{(t,T]} \alpha(t,u)\mu_t(du)$$

$$\bar{\beta}(t,T) = \int_{(t,T]} \beta(t,u)\mu_t(du)$$

$$\Psi_t(z) = \langle b_t, z\rangle + \frac{1}{2}\langle z, c_t z\rangle + \int (e^{\langle z,x\rangle} - 1 - \langle z,x\rangle)K_t(dx).$$

Credit risk

In credit risk, one considers $C_T = \mathbb{1}_{\{\tau > T\}}$ with a stopping time τ. The HJM-approach reads

$$P(t,T) = \mathbb{1}_{\{\tau > t\}} e^{-\int_t^T f(t,u)du}.$$

Again, this leads to arbitrage in the general case.

If the compensator of $H = 1 - C$ is of the form $\int_0^\cdot h_s ds$, $X^0 = e^{\int_0^\cdot r_s ds}$ is the numéraire and X is stochastically continuous, then the considered measure is a local martingale measure $\Leftrightarrow$

$$\begin{aligned} f(t,t) &= r_t + h_t \\ \bar{\alpha}(t,T) &= \Psi_t(-\bar{\beta}(t,T)), \end{aligned}$$

on $\{\tau > t\}$, for all $0 \leq t \leq T \leq T^*$, $dP \otimes dt$-almost surely.

Towards general term structure models

Let us begin with the following observations: we may consider w.l.o.g.

$$f(t,T) = f(0,T) + \int_0^t \beta(s,T) \cdot dX_s, \qquad (4)$$

Lemma (Fontana & S. (2016))

Let τ be an $\mathbb{F}$-stopping time and $H_t = \mathbb{1}_{\{\tau \le t\}}$. The default compensator H^p admits the unique decomposition

$$H_t^p = \int_0^t h_s ds + \lambda_t + \sum_{0<s\le t} \Delta H_s^p, \qquad \text{for all } 0 \le t \le T, \qquad (5)$$

where $(h_t)_{0\le t\le T}$ is a non-negative predictable process such that $\int_0^T |h_s| ds < +\infty$ a.s. and $(\lambda_t)_{0\le t\le T}$ is an increasing and continuous process with $\lambda_0 = 0$ such that $d\lambda_s(\omega) \perp ds$, for a.a. $\omega \in \Omega$.

In the following we will throughout ignore (for simplicity) the singular continuous parts (here: λ)

- We assume $\mu(ds, du) = \mathbb{1}_{\{s<u\}}\mu(ds, du)$, for all $(s, u) \in [0, T^*]^2$.
- Let $\bar{\mu}$ denote the predictable process

$$\bar{\mu}_t := \mu\big([0,\infty),[0,t]\big) = \mu\big([0,t),[0,t]\big).$$

- As $\bar{\mu}$ is increasing (and we drop the singular part)

$$\bar{\mu}_t = \int_0^t \bar{m}_s ds + \sum_{s \le t} \Delta\bar{\mu}_s.$$

- We need techincal conditions on β to ensure existence of the integrals.

We utilize the stochastic Fubini-theorem from Fontana & Schmidt (2016) to obtain

$$\int_{(t,T]} f(t,u)\mu_t(du) = \int_0^t \bar{\beta}(s,T)dX_s + \int_0^t \int_{(s,T]} f(s,u)\mu(ds,du) - \int_0^t f(s,s)\,d\bar{\mu}_s.$$

Comparison: in the classical HJM-case we have

$$\int_{(t,T]} f(t,u)du = \int_0^t \bar{\beta}(s,T)dX_s - \int_0^t f(s,s)\,ds.$$

As a consequence, we introduce the processes

$$Y(t,T) := \int_0^t \bar{\beta}(s,T)dX_s + \int_0^t \int_{(s,T]} f(s,u)\mu(ds,du)$$

and the associated random measure $\mu^{(Y(.,T),H)}$.

For simplicity we consider $X = \int_0^{\cdot} a_s ds + W$. Define the function

$$W(s,y,z) := e^{f(s-,s)\Delta\bar{\mu}_s}\left(e^{-y} - 1 - y\right)(1+z). \tag{6}$$

Recall

$$\bar{\mu}_t = \int_0^t \bar{m}_s ds + \sum_{s\leq t} \Delta\bar{\mu}_s.$$

Theorem (Fontana & S. (2016))

The probability measure $Q^* \sim P$ is an ELMM if and only if

1. $f(t,t)\bar{m}_t = r_t + h_t$, *$dQ \otimes dt$-a.s. for* $t \in [0,T^*]$;
2. $f(t,t)\Delta\bar{\mu}_t = -\log(1-\Delta H_t^p)$, *for all* $t \in [0,T^*]$;
3. *for all* $0 \leq t \leq T \leq T^*$,

$$0 = \Psi_t(-\bar{\beta}(t,T)) - \Big(\int_0^t \int_{(s,T]} f(s,u)\mu(ds,du)\Big)_t^{ac} + (W \star \mu^{p,(Y(.,T),H)})_t^{ac},$$

where $(\cdot)^{ac}$ *denotes the absolutely continuous part.*

For example, if risky times are $u_1, \dots, u_N$ then

$$\mu_t(du) = du + \sum_{i=1}^{N} \delta_{u_i}(du). \tag{7}$$

Corollary

Assume that (7) holds. Then Q^* is an ELMM if and only if

$$\begin{aligned} f(t,t) &= r_t + h_t \\ f(u_i,u_i) &= -\log(1-\Delta H^p_{u_i}), \quad i = 1,\dots,N \\ 0 &= \Psi_t(-\bar\beta(t,T)), \end{aligned}$$

$0 \le t \le T \le T^*$, *$dQ^* \otimes dt$-almost surely on* $\{t < \tau\}$.

A general consideration of multiple yield curve models

- Central instruments are **forward rate agreements** (FRA): the fixation of a rate on the future interval $[T, S]$. If bond prices are sufficiently liquid (and not risky), one obtains the "classical" FRA rate

$$F(t,T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right). \tag{8}$$

- The argument is easy: we receive 1 at T and pay $1 + F(t,T,S) \cdot (S-T)$ at time S. The replicating strategy is to sell 1 T-bond (and get $P(t,T)$ in cash) and invest this money in S-bonds (hence, we get $P(t,T)/P(t,S)$ S-bonds). The formula (8) follows.
- If this is not the case one considers **multiple yield curves**.
- The literature on multiple yield curve models is huge: **short-rate** type models have been considered, e.g., in Kijima, Tanaka, and Wong (2009), Kenyon (2010), as well as Filipović and Trolle (2013). On the other side, **Heath-Jarrow-Morton (HJM)**-like approaches have been considered in Crépey, Grbac, and Nguyen (2012), Crépey, Grbac, Ngor, and Skovmand (2014), Moreni and Pallavicini (2014) as well as in Cuchiero, Gnoatto and Fontana (2016).

Central instruments: FRA

- In a FRA, a discretely compounded rate is exchanged with payments based on a fixed rate K. Denote its price by $\Pi^{\mathrm{FRA}}(t,T,\delta,K)$.
- Denote the (spot) Libor rate (at T) for $[T,T+\delta)$ by $L(T,T,\delta)$.
- The **forward Libor rate** $L(t,T,\delta)$ is the unique K, such that

$$\Pi^{\mathrm{FRA}}(t,T,\delta,K)=0. \tag{9}$$

- At maturity T,

$$\Pi^{\mathrm{FRA}}(T,T,\delta,K)=(1+\delta L(T,T,\delta))-(1+\delta K),$$

- Discounting, we arrive at

$$\Pi^{\mathrm{FRA}}(t,T,\delta,K)=\underbrace{(1+\delta L(t,T,\delta))P(t,T+\delta)}_{=:S_t(\delta)P(t,T,\delta)}-\bar{K}(\delta)P(t,T+\delta), \tag{10}$$

Dynamic multiple term-structures

- Consider (for simplicity) the continuous case, i.e. $\mathbb{F}$ is generated by a Brownian motion W.
- We assume that forward rates are given by

$$f(t,T,\delta) = f(0,T,\delta) + \int_0^t \beta(s,T,\delta)dX_s.$$

where X is a continuous semimartingale and set for all $0 \leq t \leq T \leq T^*$ and $\delta \in \{0, \delta_1, \dots, \delta_N\}$:

$$\bar\beta(t,T,\delta) := \int_{(t,T]} \beta(t,u,\delta)\mu(du).$$

Here μ is a finite (deterministic) measure.
- We consider an absolutely continuous numeraire

$$X^0 = \exp\left(\int_0^\cdot r_s ds\right).$$

We assume that the multiplicative spread process satisfies

$$S_t(\delta) = S_0(\delta) \exp\left(A_t(\delta) - \frac{1}{2} \int_0^t \|b_s(\delta)\|^2 ds + \int_0^t b_s(\delta) dW_s \right),$$

where

$$A_t(\delta) = \int_0^t a_s(\delta) ds + \sum_{0<s\le t} \Delta A_s(\delta),$$

for all $0 \le t \le T^*$.

Theorem

The equivalent measure $\mathbb{Q}$ is an ELMM with respect to the numéraire X^0 if and only if:

1. $r_t = f(t,t,0)$, *for a.e.* $0 \le t \le T^*$
2. *for every* $T \in [0,T^*]$ *and a.e.* $0 \le t \le T$ *the* **drift condition** *for the risk-free curve,*
$$0 = \Psi_t(-\bar{b}(t,T,0)) \tag{11}$$
holds
3. $\{\Delta A(\delta) \neq 0\} \subseteq \bigcup_{n=1}^N [\![T_n]\!]$ *and* $\Delta A_{T_n}(\delta) = f(T_n,T_n,0) - f(T_n,T_n,\delta)$, *for all* $n = 1,\dots,N$
4. $f(t,t,\delta) = f(t,t,0) - \alpha_t(\delta)$, *for a.e.* $0 \le t \le T^*$
5. *for every* $T \in [0,T^*]$ *and a.e.* $0 \le t \le T$ *the* **drift condition** *for the tenor* δ,
$$0 = \Psi_t(-\bar{b}(t,T,\delta)) - \bar{b}(t,T,\delta)^\top \beta_t(\delta), \tag{12}$$
holds.

Market Models

- Starting from a slightly different fundamental representation we are able to study market models in general:

$$\Pi^{\mathrm{FRA}}(t,T,T+\delta,K) = \delta\big(L(t,T,\delta)-K\big)P(t,T+\delta); \tag{13}$$

  here $0 \le t \le T \le T^*$, $\delta \in \mathscr{D}$ and, in contrast to the HJM-approach, we only consider maturities $T \in \mathscr{T} = \{T_1,\dots,T_N\}$.
- We assume that Libor rate satisfy

$$\begin{aligned} L(t,T,\delta) = L(0,T,\delta) &+ \int_0^t a^L(s,T,\delta)ds + \sum_{0<s\le t} \Delta L(s,T,\delta) \\ &+ \int_0^t b^L(s,T,\delta)dW_s, \end{aligned} \tag{14}$$

Theorem

The measure $\mathbb{Q}$ is an ELMM if and only if the following conditions hold for all $T \in \mathscr{T}$ and $\delta \in \mathscr{D}$:

1. $\Delta L(., T, T+\delta) = 0$ *$\mathbb{Q}$-almost surely and*
2. *for $dt \otimes d\mathbb{Q}$-almost all $t \leq T$*

$$a^L(t, T, \delta) = \bar{b}(t, T+\delta, 0)^\top b^L(t, T, \delta). \tag{15}$$

The drift condition suggests a change of measure and we indeed obtain the following local-martingale condition. Define the density

$$Z_t^{T+\delta} := \frac{1}{X_t^0} \frac{P(t, T+\delta, 0)}{P(0, T+\delta, 0)}, \quad 0 \leq t \leq T+\delta.$$

If this is a true martingale we define $d\mathbb{Q}^{T+\delta} := Z_{T+\delta}^{T+\delta} d\mathbb{Q}$.

Proposition

Assume that for each $(\delta, T) \in \mathscr{D} \times \mathscr{T}$ the processes $(Z_t^{T+\delta})_{0 \leq t \leq T+\delta}$ are true martingales. Then $\mathbb{Q}$ is an ELMM if and only if for each $(\delta, T) \in \mathscr{D} \times \mathscr{T}$, the process $(L(t, T, \delta))_{0 \leq t \leq T}$ is a $\mathbb{Q}^{T+\delta}$-local martingale.

Many thanks for your attention !