SHOT-NOISE DRIVEN MULTIVARIATE DEFAULT MODELS

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ABSTRACT. The recent financial crisis, responsible for massive accumulations of credit events, emphasizes the urgent need for adequate portfolio default models. Due to the high dimensionality of real credit portfolios, balancing flexibility and numerical tractability is of uttermost importance. To acknowledge this, a multivariate default model with interesting stylized properties is introduced in the following way: a non-decreasing shot-noise process serves as common stochastic clock. Individual default times are defined as the first-passage times of the common clock across independent exponentially distributed threshold levels. We obtain a default model which has a dynamic stochastic representation, contagion effects, a positive probability for joint defaults, the ability to separate univariate marginal laws from the dependence structure, and the option for efficient pricing routines under a "large homogeneous groups" assumption. Besides this, the model is well-suited for insurance portfolios which are subject to catastrophe risks and the pricing of catastrophe derivatives.

Keywords: multivariate default model; shot-noise process; default dependence; copula; contagion effect; tail dependence; catastrophe derivatives.

1. INTRODUCTION

The aim of the present investigation is the construction of a new multivariate model for default events or the arrival times of insurance claims. We consider d components whose random vector of extinction times is denoted by $(\tau_1, \dots, \tau_d) \in [0, \infty)^d$. Real portfolios often consist of hundreds of components, providing a truly high-dimensional problem. In this regard, a main challenge is the adequate balance of numerical tractability and sufficient flexibility concerning dependence structure and marginal laws. Combining recent academic research and practical demands, we identified the following list of desirable properties:

(1) Positive lower tail dependence among the extinction times to account for the risk of joint extremes, especially of joint early defaults.
(2) A singular component of the model-implied copula to allow for catastrophes in which several components can simultaneously be destroyed.
(3) Contagion effects in the sense that a major adverse event increases the likelihood of subsequent adverse events.
(4) The possibility to separate the marginal laws from the dependence structure. This property is especially convenient for practical applications such as the calibration of the model.
(5) A flexible spectrum of dependence patterns, interpolating in a parametric way from independence to complete co-monotonicity.
(6) A dynamic representation of the model in the sense that the dependence structure is generated by a stochastic process and not by a static random variable. This allows

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for changing random environments and the possibility to update the loss distribution over time.[1]

(7) Sufficient structure to allow for the derivation of the portfolio loss distribution, which in most applications is the quantity that is ultimately required.

(8) The possibility for efficient simulation schemes to apply the model in situations where closed form solutions are not available.

Concerning the literature on portfolio default models, condition (1) is mostly[2] satisfied, see, e.g., [36, 18, 1]. Models satisfying (2) are often based on a Marshall–Olkin type dependence structure, see, e.g., [17, 22, 24] and [29, 4] for extensions. Various kinds of contagion effects, i.e. (3), have been proposed and are discussed, e.g., by [19, 29, 14, 11]. The copula behind specific models, i.e. property (4), is computed in, e.g., [21, 35, 36, 24]. Examples of dynamic models (6) are, e.g., [9, 24, 15]. Models satisfying (7) are mostly based on some sort of conditionally independence structure in the spirit of [12], a reference for mixture models in credit risk is [13].

Our proposal to meet the above requirements (1)–(8) is as follows: We start by defining the default times by

$$\tau_k := \inf\left\{t \geq 0 : S_{g_k(t)} \geq E_k\right\}, \qquad k = 1, \ldots, d; \tag{1}$$

here S is a non-decreasing *shot-noise process* independent of the i.i.d., exponential(1)-distributed $E_1, \ldots, E_d$. Furthermore, $g_1, \ldots, g_d$ are deterministic and increasing functions from $[0, \infty)$ to $[0, \infty)$.[3] Defined in this way, choosing g_k suitably allows to adjust the marginal laws of the default times to any given continuous distribution on $[0, \infty)$, corresponding to property (4), see Lemma 3.4. Similarly to the model in [24], dependence is introduced by the common factor S that can be interpreted as a stochastic time transformation, providing a dynamic model, i.e. (6). In [24], this transformation relies on a Lévy subordinator. Such processes have a convenient mathematical structure and the resulting model fulfills most of the above-mentioned requirements, but their independent and stationary increments imply precisely a model, respectively dependence structure,[4] with multivariate lack-of-memory property. In this regard, Lévy-frailty models do not support a contagion effect. To account for this property we suitably alter the setting, replacing the Lévy subordinators by a non-decreasing shot-noise process. Jumps of the shot-noise process correspond to major adverse events and the reaction thereafter implies the desired contagion effect, so the stylized fact (3) is met. This improvement of the model comes at the price of losing the convenient Lévy-Khintchine formula. Still, it turns out that we have sufficient analytical structure to derive many interesting quantities concerning the model-implied dependence structure. In particular, we show that the model has a singular component, i.e. (2), and positive lower tail dependence, i.e. (1). We investigate the contagion effect and the implied (implicit) copula. The latter turns out to be flexible enough to interpolate between independence and comonotonicity in a multi-parametric way, i.e. (5) is met. The investigation of the model is accompanied by simulations, requested in (8), that help to illustrate the model's properties. Finally, we illustrate as an application how CDOs can be priced using the Laplace transform of the shot-noise process, rendering the model tractable (7) for this application,

[1] Note that one can construct models that have the very same portfolio loss distribution for any fixed $t > 0$, yet, the dynamic nature of the loss distribution is fundamentally different. One such example is provided in [4].

[2] An exception is the Gaussian copula model by [38, 21] and its variants with the same dependence structure.

[3] The model might be generalized by using multiple shot-noise processes as stochastic factors.

[4] A detailed investigation of the induced copula is presented in [25].

and we draw the relation to insurance problems like the pricing of catastrophe bonds and reinsurance contracts.

Summing up, our model aims at contributing to the existing literature by providing a framework that combines as many interesting stylized facts as possible, while remaining tractable for the usual applications in the field. To the best of our knowledge, a similar model satisfying properties (1)–(8) has not been presented so far.

2. The model

In this section we introduce the model studied in this work. Most notably, the considered vector of default times typically does not admit a default intensity and we start by giving the appropriate definitions for conditionally independent defaults and reviewing related results for credit derivatives. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. At the moment $\mathbb{Q}$ is an arbitrary probability measure, but later on it will take the role of the risk-neutral measure. Our aim is to study a portfolio of defaultable securities. To this end, we consider d different components (e.g. defaultable companies or insurance claims). We associate with each component a *default time* τ_k, $k = 1, \dots, d$, which is simply an $\mathbb{F}$-stopping time.

We define the default times along the lines of the so-called canonical construction of conditionally independent default times (see Section 9.1.2 in [5]). Consider a Poisson process N with intensity $l > 0$ and denote by $T_1, T_2, \dots$ its jump times. Let $V_1, V_2, \dots$ be i.i.d. and independent of N. Then $(\sum_{T_n \leq t} V_n)_{t\geq 0}$ is a compound Poisson process. Finally, consider a constant $\mu \in \mathbb{R}$, a measurable function $h : \mathbb{R}_+ \to \mathbb{R}$, called *responce function*, and define the process S by

$$S_t := \mu t + \sum_{T_n \leq t} V_n h(t - T_n). \tag{2}$$

Then S is a shot-noise process with drift μ. If $\mu = 0$, S is a classical shot-noise process (see [28] or [33] for a detailed analysis).

To serve as a time-transformation, we need S to be pathwise non-decreasing. Hence, we assume that $\mu \geq 0$, $\mathbb{Q}(V_1 > 0) = 1$, and that h is non-negative and increasing. Then S is a non-negative, non-decreasing process. Moreover, let $g_k : \mathbb{R}_+ \to \mathbb{R}_+$, $k = 1, \dots, d$, be strictly increasing functions such that $g_k(0) = 0$ and $\lim_{t\to\infty} g_k(t) = \infty$. Let $E_1, \dots, E_d$ be independent, exponential(1)-distributed random variables which are also independent of S. Then we define the default times by

$$\tau_k := \inf \left\{ t \geq 0 : S_{g_k(t)} \geq E_k \right\}, \quad k = 1, \dots, d. \tag{3}$$

Note that the processes S_k, given by $S_{t,k} := S_{g_k(t)}$, are path-wise non-decreasing, as S and g_k are. Here, g_k allows to adjust the marginal distributions $\mathbb{Q}(\tau_k \leq t)$ to any given continuous distribution functions on $[0, \infty)$, which is an important property of portfolio default models, the so-called *separation property.* We analyse this in detail in Section 3.2. If $g_1 = \dots = g_d$, then all default times have the same marginal distribution and the components are conditionally i.i.d. given the σ-algebra generated by S. In general, the default times are only conditionally independent and we provide some further results in this direction.

Example 2.1 (Parametric families)**.** For a concrete application of the model it is very important to have a repertory of parametric families of the response function h. Below, we provide some specifications which lead to analytically tractable models. To ensure comparability of the approaches we require $\lim_{t\to\infty} h(t) = 1$, i.e. each jump V_n is ultimately absorbed into S.

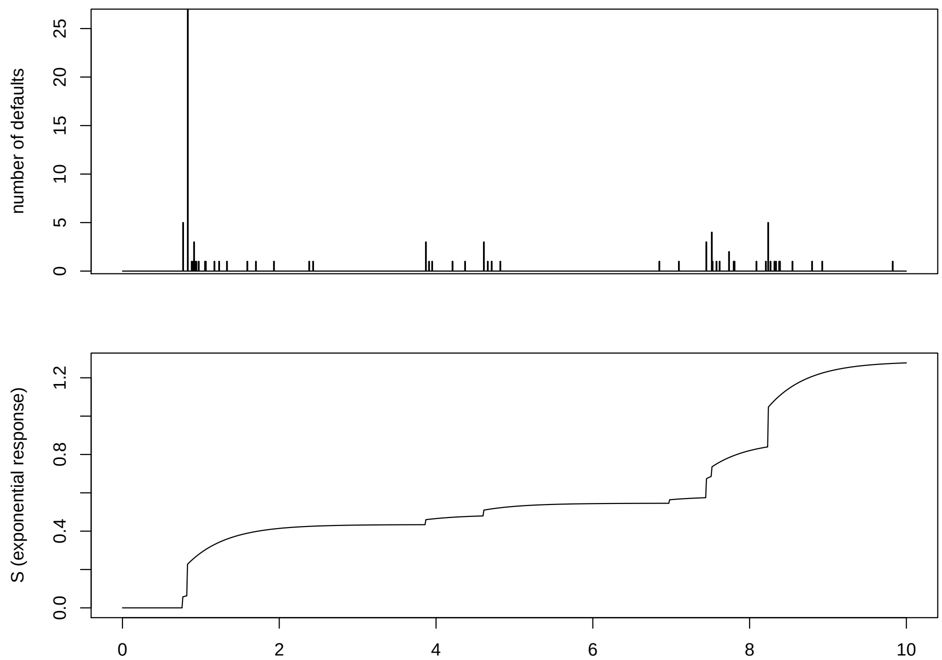

FIGURE 1. Illustration of one realization of the model with exponential structure ($d = 125$, $t \in [0, 10]$ years). One observes that whenever S jumps, accumulations of defaults are possible. Moreover, after such an event, an increasing default activity is to observe.

(1) *Compound Poisson*: the choice $h(t) \equiv 1$ corresponds precisely to the case of a compound Poisson process (with drift, if $\mu > 0$).

(2) *Linear structure*: for $\alpha \in [0, 1]$, $\beta > 0$, let

$$h(t) := \alpha + (1 - \alpha)\frac{t}{\beta}\mathbb{1}_{\{t \le \beta\}} + (1 - \alpha)\mathbb{1}_{\{t > \beta\}}.$$

This response function starts at $h(0) = \alpha$ and increases linearly over the interval $[0, \beta]$ until it reaches $h(\beta) = 1$.

(3) *Exponential structure*: for $\alpha \in [0, 1]$, $\beta > 0$, let

$$h(t) := \alpha + (1 - \alpha)\left(1 - e^{-\beta t}\right).$$

Here, h starts at $h(0) = \alpha$ and increases exponentially with limit $h(\infty) = 1$. The parameter α controls the impact of the jump size on S. The parameter β controls the speed of the growth. The limit for $\beta \to \infty$ is $h(t) \equiv 1$.

(4) *Rational structure*: for $\alpha \in [0, 1]$, $\beta > 0$, let

$$h(t) := \alpha + (1 - \alpha)\frac{t}{t + \beta}.$$

This provides an alternative specification to the exponential structure. The limit for $\beta \to 0$ is $h(t) \equiv 1$.

For the description of the statistical properties of the model, the Laplace transform of the shot-noise process is a central quantity.

Proposition 2.1. *Define* $\varphi(\theta) := \mathbb{E}\big(\exp(-\theta V_1)\big)$, *for* $\theta \geq 0$. *Then,*

$$\mathbb{E}\left(e^{-\theta S_t}\right) = \exp\left(-\theta\mu t - l\int_0^t \big(1-\varphi(\theta h(s))\big)\, ds\right), \tag{4}$$

for any $t \geq 0$ *and* $\theta \geq 0$ *such that the integral exists. Moreover, one can generalize this classical result to the case of a response-funtion depending on* t*, i.e.*

$$\mathbb{E}\left(e^{-\theta\sum_{T_n\leq t} V_n h(t,t-T_n)}\right) = \exp\left(-lt + l\int_0^t \varphi\big(\theta h(t,t-s)\big)\, ds\right). \tag{5}$$

Existence of the expectations (4) and (5) in our setup is guaranteed for all $t \geq 0$ and $\theta \geq 0$ as S is non-negative. For the reader's convenience, we give a proof of this classical result and its generalization.

Proof. The fact that N has stationary and independent increments together with the lack-of-memory property of the interarrival times gives the well-known fact, that conditional on $\{N_t = k\}$

$$(T_1,\dots,T_k) \overset{\mathscr{L}}{=} (tU_{1:k},\dots,tU_{k:k});$$

here $U_1,\dots,U_k$ are i.i.d. random variables which have a uniform distribution on $[0,1]$ and are independent of $\{N_t = k\}$, see [31, p. 502]. By $U_{1:k} \leq \cdots \leq U_{k:k}$ we denote their order statistics. Hence,

$$\begin{aligned}
(5) &= \sum_{k\geq 0} e^{-lt}\frac{(lt)^k}{k!}\mathbb{E}\Big(e^{-\theta\sum_{n=1}^k V_n h(t,t-T_n)}\big| N_t = k\Big)\\
&= \sum_{k\geq 0} e^{-lt}\frac{(lt)^k}{k!}\mathbb{E}\Big(e^{-\theta\sum_{n=1}^k V_n h(t,t-tUn:k)}\Big)\\
&= \sum_{k\geq 0} e^{-lt}\frac{(lt)^k}{k!}\mathbb{E}\Big(e^{-\theta\sum_{n=1}^k V_n h(t,t-tUn)}\Big)\\
&=: e^{-lt+lt\Pi}
\end{aligned}$$

where

$$\Pi = \mathbb{E}\Big(e^{-\theta V_1 h(t,t-tU_1)}\Big) = \frac{1}{t}\int_0^t \varphi\big(\theta h(t,t-s)\big)ds.$$

Hence, we obtain (5) and considering $h(t,u) = h(u)$ we obtain (4). $\square$

Example 2.2 (Parametric families of the jump distribution)**.**

(1) In the case $h(t) \equiv 1$, S becomes a compound Poisson process with drift. The exponent of its Laplace transform is $-\theta\mu t + lt(\varphi(\theta) - 1)$. Note that this is the limiting case obtained for $\alpha \nearrow 1$ in the linear, exponential, and rational structures introduced above.

(2) For the exponential structure, consider jumps with an Erlang-distribution, denoted $\Gamma(n,\nu)$. This is a flexible class of positive random variables which contains the exponential and the χ^2_n-distribution as special cases. Let $V_1 \sim \Gamma(n,\nu)$ with $n \in \mathbb{N}$ and $\nu > 0$. Then

$$\varphi\left(\theta\right) = \mathbb{E}(e^{-\theta V_1}) = \left(\frac{\nu}{\nu+\theta}\right)^n, \quad \theta > -\nu.$$

The tractability of the Erlang-distribution mainly attributes to the following result:

$$\int \frac{a^n}{x(a+bx)^n} dx = \ln\left(\frac{x}{a+bx}\right) + \sum_{i=1}^{n-1} \frac{a^i}{i(a+bx)^i} \tag{6}$$

see Lemma A.1. A small calculation using (6) shows that $\ln \mathbb{E}\left[\exp(-\theta S_t)\right]$ equals

$$\begin{aligned} &-(\mu\theta + l)t + \frac{l}{\beta}\left(\frac{\nu}{\nu+\theta}\right)^n \Bigg[\beta t - \ln\left(\frac{\nu+\alpha\theta}{\nu+\theta+\theta(\alpha-1)e^{-\beta t}}\right) \\ &+ \sum_{i=1}^{n-1} \frac{(\nu+\theta)^i}{i}\left(\frac{1}{(\nu+\theta\alpha)^i} - \frac{1}{(\nu+\theta+\theta(\alpha-1)e^{-\beta t})^i}\right)\Bigg]; \end{aligned} \tag{7}$$

the proof is provided in Lemma A.2. For $n = 1$ we obtain an exponential distribution with parameter $\nu > 0$ and the obvious simplification of Equation (7).

(3) Having a response function with rational structure and exponential(ν)-distributed jumps yields that $\ln \mathbb{E}(\exp(-\theta S_t))$ equals

$$-\mu\theta t + l\left(\frac{\beta\nu\theta(1-\alpha)}{(\nu+\theta)^2}\ln\left(1 + \frac{(\nu+\theta)t}{\beta(\nu+\alpha\theta)}\right) - \frac{\theta t}{\nu+\theta}\right) \tag{8}$$

for $\theta \geq 0$. We provide a proof in the Appendix in Lemma A.3.

One can, of course, replace the distributions we chose in the examples with more general ones, like stable distributions or generalized gamma distributions. In this case one will typically need a one-dimensional numerical integration to provide the Laplace transform of S.

2.1. **Conditionally independent defaults.** If $\mathcal{G} \subset \mathcal{F}$ is an arbitrary σ-algebra then we call $\tau_1, \dots, \tau_d$ *conditionally independent* w.r.t. $\mathcal{G}$ if

$$\mathbb{Q}(\tau_1 > t_1, \dots, \tau_d > t_d | \mathcal{G}) = \prod_{k=1}^{d} \mathbb{Q}(\tau_k > t_k | \mathcal{G}).$$

It is easy to see that this applies in our setup with $\mathcal{G} = \sigma(S_t : t \geq 0)$, and we have

$$\mathbb{Q}(\tau_k > t_k | \mathcal{G}) = e^{-S_{g_k(t_k)}}.$$

We study the unconditional probability $\mathbb{Q}(\tau_k > t)$ in more detail in Proposition 3.2 and the following remark. If S_k is absolutely continuous, i.e. $S_{t,k} = \int_0^t \lambda_{u,k} du$ for some positive process λ_k, then λ_k is called *default intensity* of component k. Such models are called *reduced-form models* and we refer to [10] and [5] for further details. The following immediate result shows how such models appear in our setup.

Lemma 2.2. *Consider $k \in \{1, \dots, d\}$ and $g_k(t) = t$. If h is absolutely continuous, then*

$$\lambda_{t,k} = \mu + \sum_{T_n \leq t} V_n h'(t - T_n),$$

such that λ_k is again a shot-noise process, starting from μ.

If $h'(t)$ is of the form $a \exp(-bt)$ for some constants a and b, this leads to the well-known class of affine processes as shown in [16]. The lemma extends in an obvious way to the case when g_k is absolutely continuous.

2.2. **Simulation.** In this section we provide an efficient algorithm for simulating the model and we show some simulation results. The main advantage of the algorithm is its efficiency in high dimensions. This results from the fact that we have a conditionally i.i.d. structure, meaning that once the shot-noise process is simulated, the effort increases (almost) linearly in the dimension. This is a quite convenient situation. The algorithm works as follows: We fix a time horizon $T > 0$ and use the fact that conditional on the number of jumps of a Poisson process its jump times are equal in distribution to the order statistics of i.i.d. uniform random variables on $[0, T]$, see [32, p. 17].

Algorithm 2.1 (Simulating dependent default times)**.** Simulation of one path of a shot-noise process S on $[0, T]$ and the corresponding vector of default times $(\tau_1, \dots, \tau_d)$.

(1) Draw the number of jumps on $[0, T]$, abbreviated N, from a Poisson(lT)-distribution.
(2) Simulate N i.i.d. $\mathcal{U}[0, T]$ random variables $U_1, \dots, U_N$ and set $T_i := U_{i:N}$, $i = 1, \dots, N$, where $U_{i:N}$ is the ith order statistic.
(3) Simulate N i.i.d. random variables $V_1, \dots, V_N$ (jump heights) according to the chosen jump size distribution.
(4) Compute the path $S_t = \mu t + \sum_{i=1}^{N} V_i h(t - T_i)$, $t \in [0, T]$.
(5) Simulate the default thresholds as i.i.d. exponential(1)-random variables $E_1, \dots, E_d$ and determine $\tau_1, \dots, \tau_d$ according to Equation (3). In case not all default times are triggered, increase T.

An illustration is given in Figure 1. It clearly shows clustering of defaults around the jumps of S. After each jump, the probability for additional defaults decreases as time goes on. The bivariate copula behind the model is illustrated in Figure 2 by means of scatterplots for various parameter constellations.

3. Joint distribution of defaults and further properties

In this section we further elaborate on the distributional properties of the model and we derive general results on the pricing of derivatives. We then compute the joint distribution of the default times. Thereafter we study the separation property and deduce the coefficients of tail dependence.

3.1. **Pricing.** For the valuation of derivatives, assume that $\mathbb{Q}$ is the risk-neutral measure used for pricing. Denote by $(r_t)_{t \geq 0}$ the default free short rate. We assume that r is bounded from below. Single-name credit derivatives can be valued with the following result. Denote $\mathcal{H}_t := \sigma(\mathbb{1}_{\{\tau_k \leq s\}} : 0 \leq s \leq t, k = 1, \dots, d)$ the information about the default times. Market factors like r and S are assumed to be $\mathcal{G}$-measurable

Proposition 3.1. *Suppose that Y is a $\mathcal{G}$-measurable random variable. Then*

$$\mathbb{E}\left(e^{-\int_t^T r_u du} Y \mathbb{1}_{\{\tau_k > T\}} \mid \mathcal{G} \vee \mathcal{H}_t\right) = \mathbb{1}_{\{\tau_k > t\}} e^{-\int_t^T r_u du} e^{-(S_{T,k} - S_{t,k})} Y.$$

Proof. See [5, p. 146]. □

More generally, for portfolio products the following result gives the joint survival function. For fixed dimension d, we use the abbreviation

$$H_k(s) = H_k(s_k, \dots, s_d, s) = \sum_{j=k}^{d} h(s_j - s), \tag{9}$$

such that for $d = 2$ we have $H_1(s) = h(s_1 - s) + h(s_2 - s)$ and $H_2(s) = h(s_2 - s)$.

Proposition 3.2. *The conditional survival probability of the default times equals*

$$\mathbb{Q}(\tau_1 > t_1, \dots, \tau_d > t_d \mid \mathcal{G} \vee \mathcal{H}_t) = \mathbb{1}_{\{\tau_1 > t, \dots, \tau_d > t\}} \prod_{k=1}^d e^{-(S_{t_k,k} - S_{t,k})}.$$

Moreover, set $\tilde{s}_k := g_k(t_k)$ and denote by $s_1 \leq \dots \leq s_d$ their ordered list, and $s_0 := 0$. Then

$$\mathbb{Q}(\tau_1 > t_1, \dots, \tau_d > t_d) = \exp\bigg(-\mu \sum_{k=1}^d s_k - l s_d + l \sum_{k=1}^d \int_{s_{k-1}}^{s_k} \varphi(H_k(s))\, ds \bigg)$$

with H_k as defined in (9).

Proof. The first result follows from Proposition 3.1 using conditional independence. For the computation of the unconditional joint survival probability, observe that taking $t = 0$ leads to

$$\mathbb{Q}\left(\tau_1 > t_1, \dots, \tau_d > t_d\right) = \mathbb{E}\left(e^{-\sum_{k=1}^d S_{g_k(t_k)}}\right) = \mathbb{E}\left(e^{-\sum_{k=1}^d S_{s_k}}\right),$$

where $s_1 \leq \dots \leq s_d$. Consider the case $\mu = 0$ and $d = 2$. Then

$$\begin{aligned}
&\mathbb{E}(e^{-S_{s_1} - S_{s_2}}) \\
&= \mathbb{E}\bigg(\exp\Big(-\sum_{n=1}^{N_{s_1}} V_n(h(s_1 - T_n) + h(s_2 - T_n)) - \sum_{n=N_{s_1}+1}^{N_{s_2}} V_n h(s_2 - T_n)\Big)\bigg) \\
&= \mathbb{E}\bigg(\exp\Big(-\sum_{n=1}^{N_{s_1}} V_n(h(s_1 - T_n) + h(s_2 - T_n))\Big)\bigg) \cdot \mathbb{E}\bigg(\exp\Big(-\sum_{n=N_{s_1}+1}^{N_{s_2}} V_n h(s_2 - T_n)\Big)\Big| \mathcal{F}_{s_1}\bigg).
\end{aligned} \tag{10}$$

Regarding the conditional expectation, we use the fact that the increments of $(\sum_{i=1}^{N_t} V_n)$ over $(0, s_1]$ and $(s_1, s_2]$ are independent and stationary and obtain that

$$\begin{aligned}
\mathbb{E}\bigg(\exp\Big(-\sum_{n=N_{s_1}+1}^{N_{s_2}} V_n h(s_2 - T_n)\Big)\Big| \mathcal{F}_{s_1}\bigg) &= \mathbb{E}\bigg(\exp\Big(-\sum_{n=1}^{N_{s_2 - s_1}} V_n h(s_2 - s_1 - T_n)\Big)\bigg) \\
&= \exp\left(-l(s_2 - s_1) + \int_0^{s_2 - s_1} \varphi(h(s_2 - s_1 - s)) ds \right) \\
&= \exp\left(-l(s_2 - s_1) + \int_{s_1}^{s_2} \varphi(h(s_2 - s)) ds \right) \\
&= \exp\left(-l(s_2 - s_1) + \int_{s_1}^{s_2} \varphi(H_2(s)) ds \right).
\end{aligned}$$

Regarding the remaining term in (10), we apply Equation (5) from Proposition 2.1 with the function $\tilde{h}(t, u) = h(s_2, t, u) := h(u) + h(s_2 - t + u)$. Note that

$$\tilde{h}(s_1, s_1 - T_n) = h(s_1 - T_n) + h(s_2 - T_n)$$

which is exactly the term appearing in (10). Thus, (5) gives

$$\begin{aligned}\mathbb{E}\Big(\exp\Big(-\sum_{n=1}^{N_{s_1}} V_n \tilde{h}(s_1, s_1 - T_n)\Big)\Big) &= \exp\Big(-ls_1 + l\int_0^{s_1} \varphi\big(\tilde{h}(s_1, s_1 - s)\big) ds\Big) \\ &= \exp\Big(-ls_1 + l\int_0^{s_1} \varphi\big(h(s_1 - s) + h(s_2 - s)\big) ds\Big) \\ &= \exp\Big(-ls_1 + l\int_0^{s_1} \varphi\big(H_1(s)\big) ds\Big).\end{aligned}$$

For the case $\mu > 0$ we have to multiply this expression by $\exp(-\mu(s_1 + s_2))$ and obtain the result by induction. □

Remark 3.1. In particular, the above proposition yields

$$\mathbb{Q}\big(\tau_k > t\big) = \exp\Big(-\mu g_k(t) - l\int_0^{g_k(t)} \big(1 - \varphi(h(s))\big)\, ds\Big) = \mathbb{E}(e^{-S_{g_k(t)}}),$$

compare also Proposition 2.1. Therefore, two default times τ_i and τ_j have the same distribution if and only if $g_i = g_j$.

In the exponential structure we are able to exploit the multiplicative structure of h, i.e. $\exp(-\beta(t+s)) = \exp(-\beta t)\exp(-\beta s)$. A similar calculation is possible in the linear structure as well. Regarding models with exponential or rational structure, we study Example 2.2(2) in more detail. In this case the model has an exponential structure where the jumps have an Erlang-distribution. The following corollary summarizes the results in this case.

Corollary 3.3. *Assume that $h(t) = a + b\exp(-\beta t)$ with $\beta > 0$, $a + b > 0$ and that $V_1 \sim \Gamma(n, \nu)$ with $n \in \mathbb{N}$ and $\nu > 0$. Consider $t_1, \dots, t_d \geq 0$ and denote by $s_1 \leq \dots \leq s_d$ the ordered list of $g_1(t_1), \dots, g_d(t_d)$. Let $a_k := \nu + a(d - k + 1)$, $b_k := b\sum_{i=1}^k \exp(-\beta s_i)$ and*

$$F_k(t) := \frac{1}{(a_k)^n}\Bigg(\ln\Big(\frac{t}{a_k + b_k t}\Big) + \sum_{i=1}^{n-1} \frac{1}{i}\Big(\frac{a_k}{a_k + b_k t}\Big)^i\Bigg),$$

$k = 1, \dots, d$. Then

$$\mathbb{Q}\left(\tau_1 > t_1, \dots, \tau_d > t_d\right) = \exp\Big(-\mu\sum_{k=1}^d s_k - ls_d + \frac{l\nu^n}{\beta}\sum_{k=1}^d \Big(F_k(e^{\beta s_k}) - F_k(e^{\beta s_{k-1}})\Big)\Big).$$

To apply this result to Example 2.2(2), set $a = \alpha$ and $b = 1 - \alpha$ with $\alpha \in [0, 1]$. In the case where we have a rational structure as in Example 2.2(3), the integration can be traced back to finding the roots of a polynomial which can be solved efficiently by numerical methods.

Proof. Note that

$$H_k(s) = a(d - k + 1) + e^{\beta s} b\sum_{i=1}^k e^{-\beta s_i} = a_k - \nu + e^{\beta s} b_k$$

where the function H_k is defined in (9). As the jump sizes have an Erlang-distribution, we have that $\varphi(\theta) = \nu^n(\nu+\theta)^{-n}$, such that

$$\begin{aligned}\int_{s_{k-1}}^{s_k} \varphi(H_k(s))ds &= \int_{s_{k-1}}^{s_k} \frac{\nu^n}{(a_k + b_k e^{\beta s})^n} ds \\ &= \frac{\nu^n}{\beta (a_k)^n} \int_{e^{\beta s_{k-1}}}^{e^{\beta s_k}} \frac{(a_k)^n}{x(a_k + b_k x)^n} dx \\ &= \frac{\nu^n}{\beta}\big(F_k(e^{\beta s_k}) - F_k(e^{\beta s_{k-1}})\big)\end{aligned}$$

by (6). Applying Proposition 3.2, we conclude. □

3.2. **Separation of dependence structure from marginal laws.** When setting up a multivariate model, it is extremely convenient if the marginal laws can be separated from the dependence structure. On a theoretical level, this reflects the decomposition of a distribution into the univariate marginal laws and the copula, see [34]. For practical applications, it is very common to start with given univariate marginals and to combine them in a suitable way. Such a separation is especially convenient for the calibration of the model, which can then be done in two separate steps: first, the calibration of the univariate marginal laws, followed by the calibration of the dependence structure in a second step.

In this paragraph we analyse this property in our model, generalizing [26] to the case of inhomogeneous portfolios. We consider fixed marginal distributions $p_k : [0,\infty) \to [0,1)$, $k = 1,\dots,d$, and assume that $p_1,\dots,p_d$ are continuous.

Definition 3.1. We say that the *separation condition* holds, if there exist (a) a stochastic process F which is a distribution function on $[0,\infty)$ for each fix $\omega \in \Omega$, (b) a vector of i.i.d. $\mathcal{U}[0,1]$-distributed random variables $U_1,\dots,U_d$, independent of F, (c) deterministic and strictly increasing functions $g_1,\dots,g_d$, and (d) pre-specified distribution functions $p_1,\dots,p_d$ on $[0,\infty)$, such that for $k = 1,\dots,d$ one has

(i) $\tau_k = \inf\{t \geq 0 : F_{g_k(t)} \geq U_k\}$,
(ii) $\mathbb{Q}(\tau_k \leq t) = p_k(t)$, for all $t \geq 0$.

Note that (i) can equivalently be written using i.i.d. exponential(1) triggers $E_1,\dots,E_d$ and $\tau_k = \inf\{t \geq 0 : -\ln(1 - F_{g_k(t)}) \geq E_k\}$. If the separation condition holds, the default times are conditionally independent. Moreover, any appropriate marginal distribution p_k can be matched and the dependence can be specified in a separate step by choosing the stochastic process F. In our setup, we have $F_{g_k(t)} = 1 - \exp(-S_{g_k(t)})$. Condition (ii) is equivalent to

$$\mathbb{Q}(\tau_k \leq t) = \mathbb{Q}(F_{g_k(t)} \geq U_k) = \mathbb{E}(F_{g_k(t)}), \quad t \geq 0. \tag{11}$$

Lemma 3.4 shows how a given marginal distribution is matched. Therefore, the separation condition holds.

Lemma 3.4 (Given univariate marginal laws)**.** *Let* $p : [0,\infty) \to [0,1)$ *be a non-decreasing, continuous function such that* $p(t) > 0$ *for all* $t > 0$*. Define* $I(t) := \mathbb{E}\big(\exp(-S_t)\big)$ *and assume that* $\lim_{t\to\infty} I(t) = 0$.[5] *Then* $\mathbb{Q}(\tau_k \leq t) = p(t)$ *for all* $t \geq 0$ *is in our model equivalent to*

$$g_k(t) = I^{-1}\big(1 - p(t)\big), \quad \forall t \geq 0. \tag{12}$$

Moreover, $t \mapsto g_k(t)$ *defined as in (12) is increasing and the condition* $\lim_{t\to\infty} p(t) = 1$ *implies* $\lim_{t\to\infty} g_k(t) = \infty$.

[5] This holds for all commonly used jump size distributions and non-degenerate h.

Proof. Recall that we assumed $\mu \geq 0$, $\mathbb{Q}(V_1 > 0) = 1$, and h being non-negative and increasing, which ensured that S is a non-decreasing processes. Hence $\mathbb{E}\big(\exp(-S_t)\big)$ is non-increasing in t. We prove that it is even strictly decreasing. By Proposition 2.1 we have that

$$I(t) = \exp\left(-\mu t - l\int_0^t \big(1 - \varphi(h(s))\big)\, ds\right).$$

$\mathbb{Q}(V_1 > 0)$ implies that $1 - \varphi(h(s)) = 1 - \mathbb{E}(\exp(-h(s)V_1)) > 0$ for $s > 0$. Together with our assumptions $l > 0$ and $\mu \geq 0$ we obtain that I is strictly decreasing. Moreover, we also assumed that $I(t) \to 0$ with $t \to \infty$. Hence, as noted in (11) we have that

$$p(t) = \mathbb{Q}(\tau_k \leq t) = 1 - \mathbb{E}(e^{-S_{g_k(t)}}) = 1 - I(g_k(t)),$$

which is then equivalent to $g_k(t) = I^{-1}(1 - p(t))$. Finally, we observe that $I^{-1}(\epsilon) \to \infty$ as $\epsilon \to 0$ and we conclude. □

Example 3.1. For the case $h(t) \equiv 1$, $g(t)$ can easily be calculated. Note that the function $I(t)$ is precisely the Laplace transform of $(S_t)_{t\geq 0}$ at 1, which we computed in Proposition 2.1 and is well known for the compound Poisson case. Hence, we have

$$\ln\big(I(t)\big) = -\mu t - lt\mathbb{E}\big(1 - e^{-V}\big) = -\mu t - lt\big(1 - \varphi(1)\big), \quad t \geq 0.$$

Inverting this expression yields

$$g(t) = I^{-1}\big(1 - p(t)\big) = \frac{\ln\left(1 - p(t)\right)}{-\mu - l\big(1 - \varphi(1)\big)} = \frac{-\ln\left(1 - p(t)\right)}{\mu + l\big(1 - \varphi(1)\big)}.$$

For more involved specifications of the response function, the inverse of $I(t)$ can not easily be calculated analytically. Numerically, however, for given $t \mapsto p(t)$, it is easy to solve

$$I\big(g(t)\big) - 1 + p(t) = 0$$

for $g(t)$, using, for example, the bisection method.

3.3. **Tail dependence.** An important measure for joint extremes are the coefficients of upper and lower tail dependence. In the present context, especially a positive lower tail dependence coefficient among the default times is important to adequately consider the risk of joint early defaults.

We assume a homogeneous structure concerning the univariate marginal laws, i.e. $g_k(t) \equiv g(t)$ for $k = 1, \dots, d$. The results in this section can, however, be extended to inhomogeneous marginals but become notationally cumbersome. The results on tail dependence coefficients for CIID models from [26] show that the coefficient of lower tail dependence λ_l of a pair of default times (τ_i, τ_j) is related to the market frailty $F_{g(t)}$ via

$$\lambda_l = \lim_{t\downarrow 0} \mathbb{Q}(\tau_i \leq t\,|\tau_j \leq t\,) = \lim_{t\downarrow 0} \frac{\mathbb{Q}(\tau_i \leq t, \tau_j \leq t)}{\mathbb{Q}(\tau_j \leq t)} = \lim_{t\downarrow 0} \frac{\mathbb{E}((F_{g(t)})^2)}{\mathbb{E}(F_{g(t)})}. \tag{13}$$

In our framework we are able to derive an explicit expression for the lower tail dependence coefficient. Interestingly, this is related to the behavior of h at zero and the expected size of the jumps as well as the ratio of the drift μ over the jump intensity l.

Proposition 3.5 (Lower tail dependence)**.** *Assume that h is continuous at zero and g strictly increasing around zero.*

(i) *If $h(0) > 0$ or $\mu > 0$ then the coefficient of lower tail dependence λ_l is*

$$\lambda_l = \frac{1 + \varphi(2h(0)) - 2\varphi(h(0))}{\mu/l + 1 - \varphi(h(0))} > 0.$$

(ii) *If $h(0) = 0$ and $\mu = 0$ and moreover h is strictly increasing around zero then*

$$\lambda_l = 2 - 2 \lim_{t \downarrow 0} \frac{\varphi'(2t)}{\varphi'(t)}, \tag{14}$$

provided existence of this limit.

Both cases have a very appealing interpretation: in the first case, where $h(0) > 0$ or $\mu > 0$, we necessarily have positive tail dependence.[6] The special case $h(0) = 1$, $\mu = 0$ corresponds to a Lévy-frailty model where the frailty process is of compound Poisson type. In this case, the lower tail dependence coefficient can be expressed via the Laplace exponent Ψ of the compound Poisson process and is given by $\lambda_l = 2 - \Psi(2)/\Psi(1)$, see [25]. The connection to the present case is established by $\Psi(x) = l\big(1 - \varphi(x)\big)$.

In the second case, where $h(0) = 0$ and $\mu = 0$, the jumps V_n are continuously absorbed into S over time, but we still can have positive lower tail dependence. Interestingly, this is closely related to the upper tail dependence coefficient of an Archimedean copula with generator φ, which is given by precisely the same limit as we have in (14). Knowing this, we can immediately conclude that if $\mathbb{E}(V) < \infty$, the limit equals zero, which follows from our formula as then $\lim_{t \downarrow 0} \varphi'(t) = \lim_{t \downarrow 0} \varphi'(2t) = -\mathbb{E}(V)$. Interpreted differently, if the jumps have finite expectation and $\mu = 0$, the slope of h is not strong enough to produce lower tail dependence. Moreover, the limit in (14) is known for all distributions of V that are commonly used as a mixture variable for extendible Archimedean copulas, respectively their associated generators φ, see e.g. [6].

Proof. The strict monotonicity of g around zero gives that

$$\lim_{t \downarrow 0} \frac{\mathbb{E}((F_{g(t)})^2)}{\mathbb{E}(F_{g(t)})} = \lim_{t \downarrow 0} \frac{\mathbb{E}((F_t)^2)}{\mathbb{E}(F_t)}$$

by substitution. Proposition 2.1 allows us to compute

$$\begin{aligned}
\mathbb{E}(F_t) &= \mathbb{E}(1 - \exp(-S_{g(t)})) = 1 - \exp\Big(-\mu t - l \int_0^t \big(1 - \varphi\big(h(s)\big)\big) ds \Big), \\
\mathbb{E}((F_t)^2) &= \mathbb{E}\Big(1 + e^{-2S_{g(t)}} - 2e^{-S_{g(t)}}\Big) \\
&= 1 + \exp\Big(-2\mu t - l \int_0^t \big(1 - \varphi\big(2h(s)\big)\big) ds \Big) - 2\exp\Big(-\mu t - l \int_0^t \big(1 - \varphi\big(h(s)\big)\big) ds \Big).
\end{aligned}$$

With l'Hospital's rule we obtain that

$$\lambda_l = \lim_{t \downarrow 0} \frac{-2\mu - l(1 - \varphi(2h(t))) - 2\big(-\mu - l(1 - \varphi(h(t)))\big)}{\mu + l(1 - \varphi(h(t)))}.$$

If we have that $h(0) > 0$ or $\mu > 0$ we can directly compute the limit and obtain that

$$\lambda_l = \frac{l - 2l\varphi(h(0)) + l\varphi(2(h(0))}{\mu + l - l\varphi(h(0))}.$$

Strict convexity of the Laplace transform φ now yields that $\lambda_l > 0$.

In the case where $h(0) = 0$ and $\mu = 0$ we apply l'Hospital's rule again and strict monotonicity of h around zero gives

$$\lambda_l = 2 - 2 \lim_{t \downarrow 0} \frac{\varphi'(2t)}{\varphi'(t)}$$

if the limit exists and we conclude. □

[6]This is due to φ being the Laplace transform of the positive random variable V and, hence, being completely monotone by Bernstein's theorem, see [3].

3.4. **Singular component.** An interesting property of the model is the flexibility to interpolate between absolutely continuous distributions and distributions with a singular component. The latter is the result of jumps in S, respectively F, which implies $\mathbb{Q}\big(p_1(\tau_1) = p_2(\tau_2) = \ldots = p_k(\tau_k)\big) > 0$, $k = 2, \ldots, d$, where p_i is the marginal law of τ_i. This is caused by the positive probability of multiple triggers to be activated jointly via a jump of S, respectively F. On the contrary, whenever S, respectively F, is continuous, the probability of multiple triggers to be jointly activated is zero, as $\mathbb{Q}(U_1 = \ldots = U_k) = 0$. This flexibility is illustrated in Figure 2 by means of scatterplots of the model-implied survival copula. The latter is obtained by transforming the marginals to $\mathcal{U}[0,1]$. Figure 3 illustrates the flexibility of the model to interpolate between independence and comonotonicity.

4. Applications to credit portfolios

To enhance tractability for large portfolios, one may consider the following approximation via homogeneous groups. To this end, we assume that the portfolio can be divided into M groups such that the default times are conditionally independent *and* in each group conditionally i.i.d..

More precisely, consider i.i.d. $(E_{ij})_{i,j\in\mathbb{N}}$, exponential(1)-distributed and independent of the shot-noise process S. For $1 \le m \le M$ denote the number of entities in group m by n_m, and set $\boldsymbol{n} := (n_1, \ldots, n_M)^\top \in \mathbb{N}^M$. In a *homogeneous group* approach, we assume that

$$\tau_{km} = \inf\{t \ge 0 : S_{g_m(t)} \ge E_{km}\}.$$

For the approximation of the fraction of defaulted companies up to time t we consider a sequence $(\boldsymbol{n}^k)$ of elements of $\mathbb{N}^M$ such that the number of entities in each group converges to infinity and the fraction of entities in each group converges to a constant.

(A1) Assume that $(\boldsymbol{n}^k)_{k\in\mathbb{N}} \subset \mathbb{N}^M$ is a sequence such that $\min(n_1^k, \ldots, n_M^k) \to \infty$ as $k \to \infty$ and that there exists $\boldsymbol{\alpha} \in \mathbb{R}^M$ with $(n_1^k + \cdots + n_M^k)^{-1}\boldsymbol{n}^k \to \boldsymbol{\alpha}$, as $k \to \infty$.

It is immediate that $\boldsymbol{\alpha}$ lies in the unit simplex, i.e. $\alpha_m \in [0,1]$ and $\alpha_1 + \cdots + \alpha_M = 1$. We define the average number of defaulted companies until time t by

$$L^k(t) := \frac{1}{n_1^k + \cdots + n_M^k} \sum_{m=1}^{M} \sum_{i=1}^{n_m^k} \mathbb{1}_{\{\tau_{im} \le t\}},$$

and denote

$$L(t) := \sum_{m=1}^{M} \alpha_m (1 - e^{-S_{g_m(t)}}).$$

An application of the Glivenko–Cantelli theorem, see, e.g., [20], gives the following result on uniform convergence of L^k to L. It is a generalization of Lemma 2.2 in [26] to the case of inhomogeneous groups.

Proposition 4.1. *Assume that (A1) holds. Then*

$$\lim_{k\to\infty} \sup_{t\ge 0} \mid L^k(t) - L(t) \mid = 0$$

$\mathbb{Q}$-*almost surely.*

Proof. To prove the claim we condition on the path of S and then follow the steps from the classical Glivenko–Cantelli theorem, taking additional care on the non-identical distribution

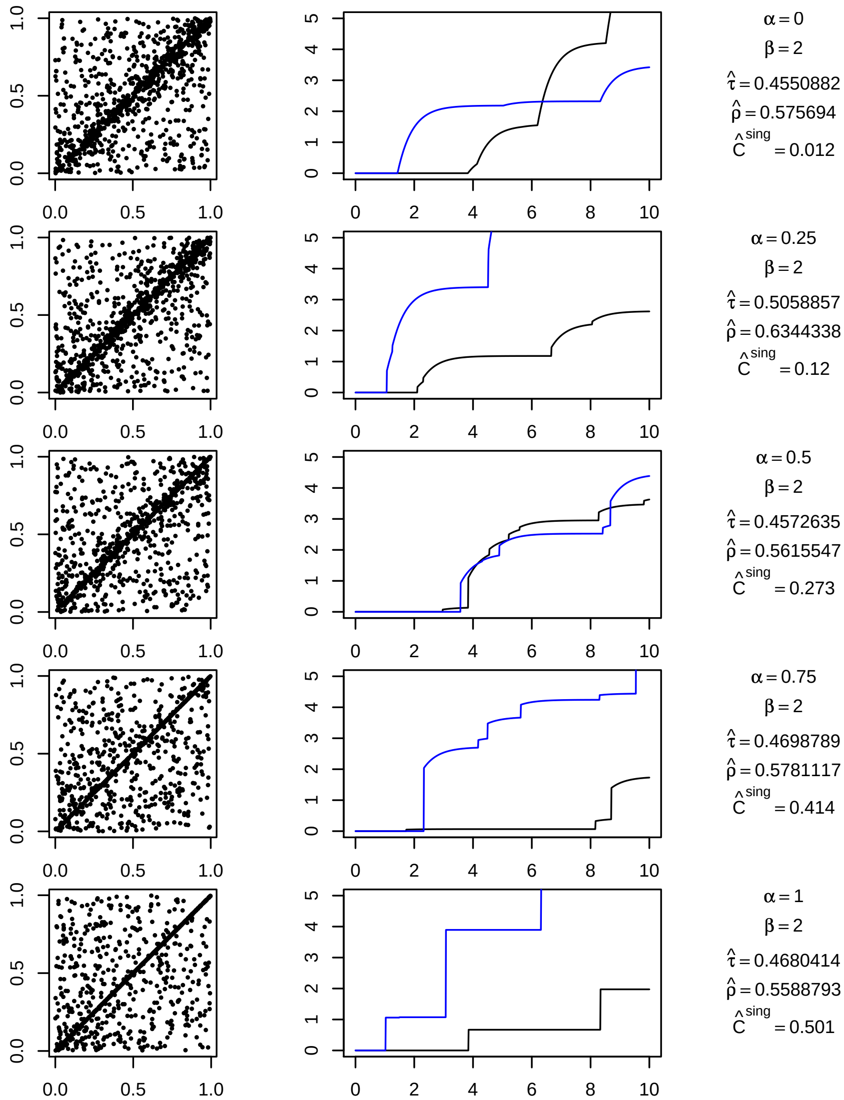

FIGURE 2. The bivariate model-implied survival copula is illustrated by means of scatterplots (left) for different parameters (α, β); exemplarily for the exponential structure, i.e. Example 2.1(3). Moreover, two sample paths illustrate the respective shot-noise process S for each parameter constellation (middle). Finally, the (empirical) measures of dependence Kandall's tau and Spearman's rho (right) illustrate the level of implied dependence and demonstrate the flexibility of the ansatz. Most notably, the model interpolates between an absolutely continuous copula (for $\alpha = 0$, where the process S is continuous) and a Cuadras–Augé copula (for $\alpha = 1$, where the process S is a compound Poisson process).

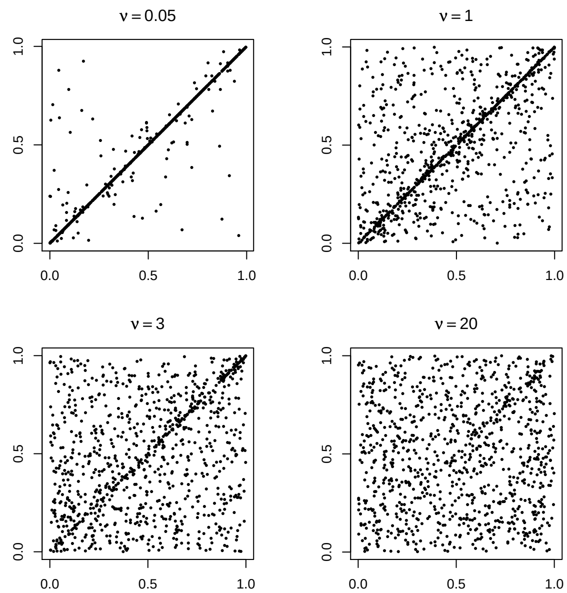

FIGURE 3. Scatterplots from the bivariate implied survival copula of the model from Example 2.1(3). Jumps are exponentially distributed with parameter ν, the other parameters are $\alpha = 0.5$, $\beta = 2$, and $l = 0.5$. With decreasing ν, the expected size of jumps increases such that more joint defaults occur, i.e. more samples fall on the diagonal. The limit $\nu \nearrow \infty$ implies independence, whereas $\nu \searrow 0$ implies comonotonicity. This illustrates the range of possible implied dependence structures within this model specification.

over the groups. Observe that $\{\tau_{km} \le t\} = \{S_{g_m(t)} \ge E_{km}\}$. Moreover, by the independence of S and the exponentially distributed random variables we obtain that

$$\begin{aligned}\sup_{t\ge 0} \mid L^k(t) - L(t) \mid &= \sup_{t\ge 0} \Big| \frac{1}{n_1^k + \cdots + n_M^k} \sum_{m=1}^{M} \sum_{k=1}^{n_m^k} \mathbb{1}_{\{S_{g_m(t)} \ge E_{km}\}} - \sum_{m=1}^{M} \alpha_m (1 - e^{-S_{g_m(t)}}) \Big| \\ &\le \sup_{t\ge 0} \Big| \frac{1}{n_1^k + \cdots + n_M^k} \sum_{m=1}^{M} \sum_{k=1}^{n_m^k} \mathbb{1}_{\{t \ge E_{km}\}} - \sum_{m=1}^{M} \alpha_m (1 - e^{-t}) \Big|\end{aligned}$$

with equality if $S_{g_m(t)}$, $m = 1, \ldots, M$ are continuous and have image $\mathbb{R}_+$.

Observe that

$$\frac{1}{n_1^k + \cdots + n_M^k} \sum_{m=1}^{M} \sum_{k=1}^{n_m^k} \mathbb{1}_{\{t \ge E_{km}\}} = \sum_{m=1}^{M} \alpha_m \frac{n_m}{\alpha_m (n_1^k + \cdots + n_M^k)} \frac{1}{n_m^k} \sum_{k=1}^{n_m^k} \mathbb{1}_{\{t \ge E_{km}\}}.$$

The strong law of large numbers yields that $(n_m^k)^{-1} \sum_{k=1}^{n_m} \mathbb{1}_{\{t \ge E_{km}\}} \to (1 - e^{-t})$ as $k \to \infty$. Moreover, (A1) guarantees that $\frac{n_m}{\alpha_m (n_1^k + \cdots + n_M^k)} \to 1$ as $k \to \infty$. This gives pointwise convergence for each t. Together with the continuity of the exponential distribution we obtain uniform convergence, see [20, Proposition 3.24], and the claim follows. □

4.1. **Valuation of CDOs.** Synthetic CDOs are swap-like credit derivatives that allow to exchange potential losses within a certain loss interval (called tranche) of some credit portfolio against periodic premium / insurance payments. The pricing problem related to CDO tranches requires one to determine the "fair spread" / "fair premium rate" for each tranche. Computed with this spread, the expected discounted default leg (present value of the losses) agrees with the expected discounted premium leg of the respective tranche (present value of the premium payments). As part of the contractual specifications, one agrees upon the premium frequency (usually quarter-yearly), the contracts notional, the upper- and lower attachment points of the tranche (in percent of the CDOs overall notional), the handling of accrued interest, and the maturity of the contract. Being more specific concerning contractual specifications is not relevant for the following considerations.

Given the assumptions of a homogeneous and deterministic loss given default across the portfolio and discount factors being independent of default times, all expected discounted payment streams can be expressed as expectations of functions of the number of defaults up to time t. To this aim, we have to compute

$$\mathbb{E}\big(f(L^k(t))\big), \quad t \geq 0, \tag{15}$$

where the non-linear function f depends on the tranches attachment points and the loss given default. The advantage of the present situation is that Proposition 4.1 provides us with a very convenient approximation of the loss distribution. Hence, we could specify the model such that the density of S_t is known and obtain (via density transformation) the density of the approximated portfolio loss distribution. The required expectation (15) could then conveniently be computed as a single integral. This observation constitutes the main advantage of conditionally independence models and clearly justifies their popularity.

Alternatively, we can take advantage of the fact that the Laplace transform of the shot-noise process can be calculated explicitly. Therefore, we can employ Laplace inversion techniques from option pricing, even if the density of the portfolio loss is not known. A common assumption in the CDO literature is a large homogeneous portfolio assumption which we consider here for simplicity, i.e., we let $M = 1$.

More precisely, given the number of defaults up to time t, the pricing of a CDO tranche, say j, requires to compute the expected loss affecting it, $\mathbb{E}(L_j^k(t))$, with

$$L_j^k(t) = \min\big(\max\big(0, (1-R)L^k(t) - l_j\big), u_j - l_j\big), \quad t \geq 0, \tag{16}$$

where l_j and u_j denote the lower and upper attachment points of the tranche, respectively, and the recovery rate R as assumed above. Under the approximation of Proposition 4.1 together with $M = 1$, $\mathbb{E}(L_j^k(t))$ can be rewriten as

$$\begin{aligned}\mathbb{E}(L_j^k(t)) &\approx \mathbb{E}\Big(u_j - l_j + \big(l_j - (1-R)L(t)\big)^+ - \big(u_j - (1-R)L(t)\big)^+\Big)\\ &= u_j - l_j + (1-R)\mathbb{E}\left(\left(e^{-S_{g(t)}} - \left(1 - \frac{l_j}{1-R}\right)\right)^+ - \left(e^{-S_{g(t)}} - \left(1 - \frac{u_j}{1-R}\right)\right)^+\right).\end{aligned}$$

The expectation now essentially equals the difference of two call options with strikes $1 - l_j/(1-R)$ and $1 - u_j/(1-R)$ on the asset $\exp(-S_{g(t)})$. This observation, together with the fact that the Laplace transform of the model is known, allows for efficient CDO pricing schemes without approximating the density of the portfolio loss distribution. Following the approach of Raible [30], inverse Laplace transforms allow to compute the prices of those options, where the inversion integrals can be approximated numerically to a high precision by applying an algorithm suggested by Talbot [37].

Example 4.1. We investigate CDO spreads computed with our model in a small case study. This allows us to understand how the parameters influence the model-implied dependence structure as well as CDO tranche spreads. Moreover, it demonstrates the viability of our ansatz. Each valuation, implemented with inverse Laplace techniques on a standard PC, requires only a fraction of a second. The (homogeneous) marginal laws are taken to be exponential with rate 0.5% in all cases, corresponding to a five year survival probability of 97.53%. The respective functions $t \mapsto g(t)$ are computed via simple bisection. The interest rate is fixed at $r = 1\%$, the recovery at $R = 40\%$. As our base case we consider a compound Poisson specification with $\mu = 1$, jump intensity $l = 1$, and $\Gamma(n, \nu)$ distributed jumps.

Model	μ	l	n	ν	α	β	upfront	t2	t3	t4	t5
Example	1	1	1	1.5	-	-	16.61	32.67	30.06	27.53	22.34
2.2(1)	1	1	2	3	-	-	15.57	34.26	33.10	31.38	26.20
Example	1	1	2	3	0.5	1	20.93	37.52	33.65	28.73	17.92
2.2(2)	1	1	2	3	0.75	1	17.83	35.83	33.87	31.12	23.66
	1	1	2	3	0.25	1	25.41	37.31	27.45	18.14	6.37
	0.5	1	2	3	0.5	1	14.19	61.56	55.11	47.03	29.44
	1	1.5	2	3	0.5	1	17.26	50.77	45.45	38.75	24.16
	1	1	3	3	0.5	1	17.46	36.19	35.22	33.25	25.78
	1	1	2	2	0.5	1	17.78	35.80	33.86	31.13	23.73
Example	1	1	-	1.5	0.5	1	21.43	33.72	28.61	24.04	16.05
2.2(3)	1	1	-	1.5	0.75	1	18.63	33.36	29.87	26.57	20.17
	1	1	-	1.5	0.25	1	25.55	31.74	23.03	16.41	7.72
	0.5	1	-	1.5	0.5	1	14.88	55.54	47.18	39.71	26.61
	1	1.5	-	1.5	0.5	1	17.88	45.71	38.77	32.58	21.76
	1	1	-	3	0.5	1	25.66	31.61	22.77	16.11	7.46

Table 1. CDO tranche spreads (5 year maturity, quarter-yearly premium frequency) for various dependence structures. For the equity tranche, the upfront payment is quoted in %, while a running spread of 500 bps is fixed. The spreads of the remaining tranches t2,. . .,t5 are quoted in bps. To compare the entries across the model specifications, note that $\Gamma(1, \nu) = exponential(\nu)$. The tranche segmentation is done according to the *iTraxx Europe* convention.

We draw the following conclusions from Table 1: (1) In the compound Poisson case 2.2(1), if we move from $\Gamma(1, 1.5)$ to $\Gamma(2, 3)$ (note that the expected jump magnitute remains the same), the level of dependence increases slightly such that the equity spread is reduced and the senior tranches trade at higher spreads. Cases with higher jumps (i.e. $\nu \searrow$) or more frequent jumps (i.e. $l \nearrow$) also increase the level of dependence, however, these obvious cases are not presented. (2) In the case of an exponential response function 2.2(2), the level of dependence is increasing in α, with the compound Poisson case as limit for $\alpha \nearrow 1$. Altering β has only little influence on CDO spreads (we investigated this sensitivity but did not report numerical results for the sake of brevity). Moreover, the level of dependence is increasing in the jump intensity l and decreasing in the drift μ. (3) In the case 2.2(3) of a rational response function and exponentially distributed jumps, we note that the model-implied dependence (and thus spreads of senior tranches) is again increasing in α, decreasing in μ, and increasing in the expected number l and size $1/\nu$ of jumps. The limit for $\alpha \nearrow 1$ is again the compound Poisson specification.

5. Application to insurance portfolios

The shot-noise portfolio default model can easily be modified to suit various actuarial applications and we refer to [31] for a general account on this topic; related approaches include [22] and [8]. Spoken generally, a portfolio of insurance claims can be modeled via its claim arrival times and the associated loss magnitudes.[7] The claim arrival times, as given below in (17), have stylized statistical properties that allow for an actuarial interpretation: often, multiple losses are caused by the same adverse event, e.g., a flood, a hurricane, etc. This corresponds to a jump of S in our model. The resulting claim arrival times, however, might not be immediate at such an event. This effect is governed via the function h, allowing us to model various shapes of loss reporting pattern. Often, the reporting period is limited, say to T^*, and we may cover this case by letting $h(T^*) = 1$.

Moreover, in actuarial applications, we often have hierarchical dependence structures, since groupings by geographic regions or exposures to specific risk factors occur quite naturally. Such effects can be modeled via multiple driving shot-noise processes as we outline in the following. Note that depending on the application, $\mathbb{Q}$ can be the real-world probability measure or a risk-neutral measure used for pricing.

Approximation of the loss function. We consider N risk factors, indexed by $n = 1, \dots, N$. The claim amounts $Z_1, Z_2, \dots$ occur at the arrival times $\tau_1, \tau_2, \dots$. We assume that for each risk factor n there exists a shot-noise process S_n. As an immediate extension of Proposition 4.1 we obtain an approximation of the losses if the number of insured contracts is sufficiently large. We assume that the individual risk characteristics can be grouped into M groups: consider i.i.d. $(E_{ij})_{i,j\in\mathbb{N}}$, exponential(1)-distributed and independent of the shot-noise processes $\mathbf{S} := (S_1, \dots, S_N)$. For each group m the risk characteristics is denoted by $\boldsymbol{\theta}_m \in \mathbb{R}^N$. Assume that

$$\tau_{km} = \inf\big\{t \geq 0 : \boldsymbol{\theta}_m \mathbf{S}_t \geq E_{km}\big\}. \tag{17}$$

We assume that (A1) holds and consider the loss process

$$L^k(t) := \frac{1}{n_1^k + \dots + n_M^k} \sum_{m=1}^{M} \sum_{i=1}^{n_m^k} \mathbb{1}_{\{\tau_{im} \leq t\}} Z_{im}.$$

Assume that in each group the claim sizes have expectation $\bar{Z}_m < \infty$. The limit of the expectation of the losses conditional on S is

$$L(t) := \sum_{m=1}^{M} \alpha_m (1 - e^{-\boldsymbol{\theta}_m \mathbf{S}_t}) \bar{Z}_m.$$

An analogous argument to Proposition 4.1 now yields that

$$\lim_{k\to\infty} \sup_{t\geq 0} |L^k(t) - L(t)| = 0$$

[7] The most straightforward approach is to take i.i.d. loss sizes; this even allows for the generalization of the LHP approximation if the loss severity has existing expectation. More advanced might be a setting where all loss magnitudes between two events are stochastically dependent. This can be achieved, e.g., via an Archimedean dependence structure induced by taking the variable V as mixing variable in a [27] type conditionally i.i.d. model. See [2] for an overview of this topic.

$\mathbb{Q}$-almost surely.

Catastrophe bonds. The so-called CAT bonds are a common product for insurers. Instead of buying reinsurance on their claims it enables the insurer to pass their risk to investors. A CAT bond offers a coupon payment c at payment dates $t_1, \dots, t_K$ and the repayment of the principal 1 at t_K if no trigger event happend. In the case of a trigger event happend, the coupons are ceased and a fraction δ of the principal is payed back.

As an example we consider as trigger event if the loss process L crosses a barrier B and assume zero interest rates. In this case the payment at t_k would be

$$f_k(L_{t_k}) = \begin{cases} c + \mathbb{1}_{\{k=K\}}, & \text{if } L_{t_k} \le B, \\ \delta \mathbb{1}_{\{k=K\}}, & \text{if } L_{t_k} > B. \end{cases}$$

for $k = 1, \dots, K$. The value of the CAT bond then computes to the expectation of discounted payoffs, i.e.

$$\sum_{k=1}^{T} \mathbb{E}\Big(f_k(L_{t_k}) \Big).$$

The expectations can now be computed either by the above approximation result or by means of a Monte-Carlo simulation based on Algorithm 2.1.

For more information on CAT bonds we refer to [7] or [23]. Our model also extends the approach in [8], where shot-noise Cox processes in an exponential structure and $\alpha = 0$ (see Example 2.1(3)) have been applied to derivatives on a catastrophe index.

Non-proportional reinsurance contracts. A third application for our framework is the pricing of non-proportional reinsurance contracts, written on an insurance portfolio. From a structural point of view, such contracts bear close similarities to CDOs: both are bilateral contracts where the payment streams depend (in a non-linear way) on the accumulated loss / number of claims within the reference portfolio. Consequently, pricing and risk management requires the portfolio loss distribution (or an approximation thereof). In the present context, we might either use the LHP approximation or, if a more complicated structure is postulated and the assumptions behind the approximations are not justified, implement the pricing problem by means of a Monte-Carlo simulation.

6. Conclusion and outlook

We have presented a new bottom-up multivariate default model which allows for interesting statistical properties concerning the model-implied dependence structure among default times. This includes positive lower tail dependence, the possibility for joint defaults, and time-inhomogeneous innovation; while the model is still numerically tractable. This renders the model suitable for many applications. We exemplarily treated the pricing of CDOs and sketched various actuarial applications. An interesting open problem is the estimation of the model from historical data and the detailed analysis of the insurance applications.

Appendix A. Explicit computations

For the reader's convenience we provide the following calculations used in the text:

Lemma A.1. *We have that*

$$\int \frac{a^n}{x(a+bx)^n} dx = \ln\Big(\frac{x}{a+bx}\Big) + \sum_{i=1}^{n-1} \frac{a^i}{i(a+bx)^i}$$

for $a, b > 0$ *and* $x > \max\{-b^{-1}a, (1-b)^{-1}a\}$.

Proof. We proof the statement by induction. Indeed, we have that

$$\int \frac{a}{x(a+bx)}dx = \ln\Big(\frac{x}{a+bx}\Big).$$

Note that

$$\frac{a^n}{x(a+bx)^n} = \frac{a^{n-1}}{x(a+bx)^{n-1}} - \frac{a^{n-1}b}{(a+bx)^n}$$

such that together with

$$-\int \frac{a^{n-1}b}{(a+bx)^n}dx = \frac{a^{n-1}}{(n-1)(a+bx)^{n-1}}$$

we may conclude. □

Now we can provide the proof of (7).

Lemma A.2. *Consider $\varphi(\theta) = \nu^n(\nu+\theta)^{-n}$ and $h(t) = \alpha+(1-\alpha)(1-e^{-\beta t})$ with $\alpha \in [0,1]$. Then, for all $\theta \geq 0$, $\ln \mathbb{E}(e^{-\theta S_t})$ equals*

$$\begin{aligned} &-(\mu\theta+l)t + \frac{l}{\beta}\Big(\frac{\nu}{\nu+\theta}\Big)^n\Bigg[\beta t - \ln\Big(\frac{\nu+\alpha\theta}{\nu+\theta+\theta(\alpha-1)e^{-\beta t}}\Big) \\ &+\sum_{i=1}^{n-1}\frac{(\nu+\theta)^i}{i}\Big(\frac{1}{(\nu+\theta\alpha)^i} - \frac{1}{(\nu+\theta+\theta(\alpha-1)e^{-\beta t})^i}\Big)\Bigg]. \end{aligned}$$

Proof. We start from Proposition 2.1 and consider $\beta \neq 0$. The main step is to compute

$$\begin{aligned} \int_0^t \varphi(\theta h(s))ds &= \int_0^t \frac{\nu^n}{(\nu+\theta(\alpha+(1-\alpha)(1-e^{-\beta s})))^n}ds \\ &= \frac{1}{\beta}\int_{e^{-\beta t}}^1 \frac{\nu^n}{x(\nu+\theta(\alpha+(1-\alpha)(1-x)))^n}dx \\ &= \frac{\nu^n}{\beta(\nu+\theta)^n}\int_{e^{-\beta t}}^1 \frac{(\nu+\theta)^n}{x(\nu+\theta+\theta(\alpha-1)x)^n}dx. \end{aligned}$$

Using $a = \nu+\theta$ and $b = \theta(\alpha-1)$ in Lemma A.1 yields that this expression equals

$$\begin{aligned} &\frac{\nu^n}{\beta(\nu+\theta)^n}\Bigg\{\beta t + \ln\big(\nu+\theta+\theta(\alpha-1)e^{-\beta t}\big) - \ln(\nu+\alpha\theta) \\ &+\sum_{i=1}^{n-1}\frac{(\nu+\theta)^i}{i}\Big[\frac{1}{(\nu+\theta+\theta(\alpha-1))^i} - \frac{1}{(\nu+\theta+\theta(\alpha-1)e^{-\beta t})^i}\Big]\Bigg\}. \end{aligned}$$

With (4) we obtain the claim. □

If V_1 is exponentially distributed with parameter ν, then $\varphi(\theta) = \mathbb{E}(\exp(-\theta V_1)) = \nu(\nu+\theta)^{-1}$. We now proof Equation (8).

Lemma A.3. *Consider $\varphi(\theta) = \nu(\nu+\theta)^{-1}$ and $h(t) = \alpha+(1-\alpha)\frac{t}{t+\beta}$. Then, for all $\theta \geq 0$, $\ln \mathbb{E}(e^{-\theta S_t})$ equals*

$$-\mu\theta t + l\Big(-\frac{\theta t}{\nu+\theta} + \frac{\beta\nu\theta(1-\alpha)}{(\nu+\theta)^2}\ln\Big(1+\frac{(\nu+\theta)t}{\beta(\nu+\alpha\theta)}\Big)\Big)$$

Proof. As in the previous lemma we compute

$$\begin{aligned}\int_0^t \varphi(\theta h(s))ds &= \int_0^t \frac{\nu}{(\nu+\alpha\theta+\theta(1-\alpha)\frac{s}{s+\beta})}ds \\ &= \nu\int_0^t \frac{\beta+s}{\beta(\nu+\alpha\theta)+(\nu+\theta)s}ds. \end{aligned} \tag{18}$$

It is easy to check that

$$\int \frac{a+s}{b+cs}ds = \frac{1}{c}\Big(s+(a-\frac{b}{c})\ln(b+cs)\Big).$$

Hence with $a=\beta$, $b=\beta(\nu+\alpha\theta)$ and $c=\nu+\theta$ we obtain

$$(18) = \frac{\nu}{\nu+\theta}\left(t+\Big(\beta-\beta\frac{\nu+\alpha\theta}{\nu+\theta}\Big)\ln\Big(\frac{\beta(\nu+\alpha\theta)+(\nu+\theta)t}{\beta(\nu+\alpha\theta)}\Big)\right).$$

We use (2.1) and obtain that $\ln\mathbb{E}(e^{-\theta S_t})$ equals

$$\begin{aligned}&-\mu\theta t - lt + \frac{l\nu}{\nu+\theta}\left(t+\beta\Big(\frac{\theta(1-\alpha)}{\nu+\theta}\Big)\ln\Big(1+\frac{(\nu+\theta)t}{\beta(\nu+\alpha\theta)}\Big)\right)\\ &\qquad = -\mu\theta t + l\left(-\frac{\theta t}{\nu+\theta}+\frac{\beta\nu\theta(1-\alpha)}{(\nu+\theta)^2}\ln\Big(1+\frac{(\nu+\theta)t}{\beta(\nu+\alpha\theta)}\Big)\right)\end{aligned}$$

and we conclude. □

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