PRICING BASKET DEFAULT SWAPS IN A TRACTABLE SHOT-NOISE MODEL

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Abstract. We value CDS spreads and kth-to-default swap spreads in a tractable shot noise model. The default dependence is modelled by letting the individual jumps of the default intensity be driven by a common latent factor. The arrival of the jumps is driven by a Poisson process. By using conditional independence and properties of the shot noise processes we derive tractable closed-form expressions for the default distribution and the ordered survival distributions in a homogeneous portfolio. These quantities are then used to price and study CDS spreads and kth-to-default swap spreads as function of the model parameters. We study the kth-to-default spreads as function of the CDS spread, as well as other parameters in the model. All calibrations lead to perfect fits.

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1. Introduction

In recent years the market for portfolio credit derivatives, which are derivatives with a payoff linked to the credit loss in a portfolio, has seen a rapid growth and increased liquidity. This has been followed by an intense research for understanding and modelling the main feature driving these products, namely default dependence. The current credit crisis undermines the necessity of models which can calibrate to market data on one side and also capture contagion effects on the other side. The model proposed in this paper is an affine model with a jump component. This allows to introduce a high dependence between different obliger which is of immense importance for practical applications.

Affine models have been widely used in modelling of interest rates and credit risk, but typically the jump component plays a minor role. However, the main driver of contagion effects[1] is the jump component. In this paper we concentrate on the jump component only, while the results easily can be enriched by adding a diffusion component.

As an illustration of the tractability of the model we show a number of small numerical studies. We consider a basket of obligors and calibrate this portfolio to a the given CDS-spread. Under the calibrated model we price kth-to-default swaps and study the influence of the parameters. This shows that on the one side it is easy to calibrate the model to CDS spreads and on the other hand that the computation of kth-to default swaps is highly tractable in our framework.

The rest of this paper is organized as follows. In Section 2 we introduce the shot noise model that is used in the paper. Section 3 derives the formulas that are needed for pricing portfolio credit derivatives and this is the main contribution of the paper. Next, Section 4 concretizes the results of Section 3 into an explicit and tractable example of the model. Section 5 gives a short recapitulation of a credit default swap (CDS) and its portfolio extension, the kth-to default swap. Finally, in Section 6 we use the results of Section 4, for numerical investigation of a number of properties of kth-to-default spreads. We study the k-th-to default spreads as function of the average CDS spread in the portfolio and investigate how some of the model parameters affect the kth-to default spreads when keeping the CDS spread constant.

2. The model

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ where the filtration $\mathbb{F}$ satisfies the usual conditions. $\mathbb{Q}$ is a martingale measure equivalent to the objective measure $\mathbb{P}$.

Let $\{X_{i,j}, Y_j : 1 \leq i \leq m, j \geq 1\}$ be independent nonnegative random variables where $X_{i,j}$ has distribution function F_i and Y_j has distribution function F_Y. Furthermore, let M be a Poisson process with constant intensity ρ and denote its jump times by $S_1, S_2, \dots$.

[1] Contagion is the effect that a default of a company leads to an increased default probability of related firms and came most apparent after the default of Lehman Brothers.

Let $\lambda_i = (\lambda_{t,i})_{t\geq 0}$, $1 \leq i \leq m$ be m processes where λ_i satisfies the SDE

$$\begin{aligned} d\lambda_{t,i} &= -\delta_i \lambda_{t,i} dt + dC_{t,i} \\ C_{t,i} &= \sum_{j=1}^{M_t} Y_j X_{i,j} \end{aligned} \tag{2.1}$$

The intuitive interpretation of (2.1) is that at the each jump time S_j of M, the process λ_i jumps by the amount $Y_j X_{i,j}$. Otherwise it decays exponentially with rate δ_i. This process is a Markovian shot-noise process (compare, e.g. Dassios & Jang (2003) and Gaspar & Schmidt (2008)). Furthermore, the dependence structure of the multivariate shot noise process $(\lambda_1, \lambda_2, \dots, \lambda_m)$ is determined by the process M and the random variables $\{Y_j : j \geq 1\}$. If Y_1 is constant, then $\lambda_{t,1}, \dots, \lambda_{t,m}$ are independent conditional on M.

Consider a portfolio consisting of m obligors. The default time of obligor i is denoted by τ_i. Let $E_1, \dots, E_m$ be independent random variables, exponentially distributed with parameter one, which also are independent of the processes $\{\lambda_i\}$. We define the default time τ_i as

$$\tau_i = \inf\left\{ t > 0 : \int_0^t \lambda_{s,i} ds \geq E_i \right\},$$

which implies that τ_i have default intensities λ_i, see e.g. Lando (2004) or McNeil, Frey & Embrechts (2005). This framework is typically called conditional independent modelling of default times.

Let $\mathcal{G}_t = \sigma(\lambda_{s,i} : 0 \leq s \leq t, 1 \leq i \leq m)$. Then, for $T > t$, it is easy to see that

$$\mathbb{Q}\left[\tau_i > t \,|\, \mathcal{G}_T\right] = \exp\left(- \int_0^t \lambda_{s,i} ds \right). \tag{2.2}$$

The following lemma will be useful.

Lemma 2.1. *Let $H(x) = \delta^{-1}(1 - e^{-\delta x})$. Then*

$$\int_0^t \lambda_{s,i} ds = \lambda_{0,i} H(t) + \sum_{j=1}^{M_t} Y_j X_{i,j} H(t - S_j).$$

Proof. First, observe that the solution of the SDE (2.1) is given by

$$\lambda_{t,i} = \sum_{j=0}^{M_t} Y_j X_{i,j} \exp\big(- \delta(t - S_j)\big),$$

where we use $Y_0 = 1$, $X_0 = \lambda_{0,i}$ and $S_0 = 0$ to simplify the notation. Hence

$$\begin{aligned}
\int_0^t \lambda_{s,i} ds &= \int_0^t \sum_{j=0}^{M_s} Y_j X_{i,j} e^{-\delta(s-S_j)} ds \\
&= \int_0^t \sum_{j=0}^{M_t} Y_j X_{i,j} e^{-\delta(s-S_j)} 1_{\{S_j \leq s\}} ds \\
&= \sum_{j=0}^{M_t} Y_j X_{i,j} e^{\delta S_j} \int_0^t e^{-\delta s} 1_{\{S_j \leq s\}} ds \\
&= \sum_{j=0}^{M_t} Y_j X_{i,j} e^{\delta S_j} \frac{1}{\delta} \left(e^{-\delta S_j} - e^{-\delta t} \right) \\
&= \sum_{j=0}^{M_t} Y_j X_{i,j} \frac{1}{\delta} \left(1 - e^{-\delta(t-S_j)} \right)
\end{aligned}$$

and the conclusion follows. □

Note that Lemma 2.1 purely results from the shot-noise assumptions and can easily be generalized to non-exponential decay, compare for example Gaspar & Schmidt (2008). It moreover holds for arbitrary random variables η_j replacing $Y_j X_{i,j}$. Furthermore, Lemma 2.1 holds both for inhomogeneous and homogeneous credit portfolios.

3. Pricing credit derivatives in the homogeneous model

Consider a portfolio consisting of m obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$ and identical recovery rates $\phi_1 = \phi_2 = \ldots = \phi_m = \phi$. The credit loss L_t for this portfolio at time t, in percent of the nominal portfolio value at $t = 0$, is given by

$$L_t = \frac{1-\phi}{m} \sum_{i=1}^{m} 1_{\{\tau_i \leq t\}} = \frac{1-\phi}{m} N_t$$

where $N_t = \sum_{i=1}^{m} 1_{\{\tau_i \leq t\}}$ counts the number of defaults in the portfolio.

It is well known that in order to price portfolio credit derivatives – such as basket default swaps or CDO tranches – on portfolios with homogeneous recoveries, it is enough to find the distribution $\{\mathbb{Q}(N_t = k)\}_{k=0}^{m}$ at different time points t, see e.g. Herbertsson (2008), Frey & Backhaus (2008). Furthermore, to price CDS spreads we need the individual default distributions $\mathbb{Q}(\tau_i \leq t)$.

In order to simplify computations further, one often assumes that the portfolio is homogeneous, which means that all default times are exchangeable. To this regard, consider

$$\mathbb{Q}(N_t = k) = \sum_{M \subset \{1,\ldots,m\}, |M|=k} \mathbb{Q}\Big(\tau_i \leq t : i \in M, \tau_i > t : i \notin M \Big).$$

Exchangeability means that the probability on the r.h.s. does only depend on the number of defaults being smaller than t. Then

$$\mathbb{Q}\left(N_t=k\right)=\binom{m}{k}\mathbb{Q}\left(\bigcap_{i=1}^{k}\{\tau_i\leq t\},\bigcap_{i=k+1}^{m}\{\tau_i>t\}\right) \tag{3.1}$$

which reduces the computations to find $\mathbb{Q}\big(\bigcap_{i=1}^{k}\{\tau_i\leq t\},\bigcap_{i=k+1}^{m}\{\tau_i>t\}\big)$.

In this paper, we only consider exchangeable portfolios in our model given by (2.1). We therefore make the following assumption in (2.1)

$$\lambda_{i,0}=\lambda_0,\quad \delta_i=\delta \quad \text{and} \quad F_i=F \quad \text{for} \quad 1\leq i\leq m. \tag{3.2}$$

This implies that the default times are exchangeable and have the same distribution. We can then state the following useful lemma.

Lemma 3.1. *Under* (3.2) *we have that*

$$\mathbb{Q}\left(N_t=k\right)=\binom{m}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^jG(t,m-k+j). \tag{3.3}$$

where

$$G(t,k):=\mathbb{E}\left(e^{-\sum_{i=1}^{k}\int_0^t\lambda_{s,i}ds}\right). \tag{3.4}$$

Proof. Note that (3.2), conditional independency and (2.2) in (3.1) yields

$$\begin{aligned}\mathbb{Q}\left(N_t=k\right)&=\binom{m}{k}\mathbb{E}\left(\prod_{i=1}^{k}\left(1-e^{-\int_0^t\lambda_{s,i}ds}\right)e^{-\sum_{i=k+1}^{m}\int_0^t\lambda_{s,i}ds}\right)\\&=\binom{m}{k}\mathbb{E}\left(\left(1+\sum_{j=1}^{k}\sum_{\emptyset\neq T\subset\{1,\dots,k\},|T|=j}(-1)^je^{-\sum_{i\in T}\int_0^t\lambda_{s,i}ds}\right)e^{-\sum_{i=k+1}^{m}\int_0^t\lambda_{s,i}ds}\right)\end{aligned}$$

Recall that $G(t,k)=\mathbb{E}\left(e^{-\sum_{i=1}^{k}\int_0^t\lambda_{s,i}ds}\right)$.

Then the homogeneity assumption (3.2) implies that for any $T\subset\{1,\dots,k\}$ with $|T|=j$

$$\mathbb{E}\left(e^{-\sum_{i\in T}\int_0^t\lambda_{s,i}ds}e^{-\sum_{i=k+1}^{m}\int_0^t\lambda_{s,i}ds}\right)=G(t,j+m-k)$$

and this and the above observation renders that

$$\mathbb{Q}\left(N_t=k\right)=\binom{m}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^jG(t,m-k+j).$$

which concludes the lemma. □

Thus, in order to find $\mathbb{Q}\left(N_t=k\right)$ it is sufficient to compute $\{G(t,,m-k+j)\}_{j=0}^{k}$ for any $k=1,\dots,m$. Throughout we denote by X a prototype for $X_{i,j}$, for example $X_{1,j}$ and similarly Y for Y_j. Furthermore, recall that $\varphi_X(z)=\mathbb{E}\left(e^{-zX}\right)$ is the Laplace transform of the non-negative random variable X. The following result gives the necessary quantities for finding $\{G(t,j)\}$.

Proposition 3.2. *Under* (3.2) *we have that*

$$G(t,k)=e^{-k\lambda_0 H(t)-\rho t}\cdot\exp\left(\rho t\int_{\mathbb{R}}\int_0^1\Big(\varphi_X\big(yH(tz)\big)\Big)^k dzF_Y(dy)\right) \tag{3.5}$$

Proof. First, by Lemma 2.1

$$\sum_{i=1}^k\int_0^t\lambda_{s,i}ds=\sum_{i=1}^k\Big(\lambda_{0,i}H(t)+\sum_{j=1}^{M_t}Y_jX_{i,j}H(t-S_j)\Big). \tag{3.6}$$

To compute the right hand side in (3.6) we use the following observations: Conditional on $M_t=\ell$ the jump times $\{S_i\}_{i=1}^{\ell}$ are distributed like the order statistics of uniform random variables over the interval, see for example p.502 in Rolski, Schmidli, Schmidt & Teugels (1999). More precisely, let $\eta_1,\eta_2,\dots,\eta_\ell$ be ℓ independent random variables all with distribution $U[0,t]$, then $\mathcal{L}(S_1,\dots,S_\ell|M_t=\ell)=\mathcal{L}(\eta_{1:\ell},\dots,\eta_{\ell:\ell})$ where $\{\eta_{n:\ell}\}_{n=1}^{\ell}$ is the ordering of $\{\eta_n\}_{n=1}^{\ell}$. Thus,

$$\begin{aligned}\mathbb{E}\left(e^{-\sum_{i=1}^k\sum_{j=1}^{M_t}Y_jX_{i,j}H(t-S_j)}\,\Big|\,M_t=\ell\right)&=\mathbb{E}\left(e^{-\sum_{i=1}^k\sum_{j=1}^{\ell}Y_jX_{i,j}H(t(1-\eta_{j:\ell}))}\right)\\&=\mathbb{E}\left(e^{-\sum_{i=1}^k\sum_{j=1}^{\ell}Y_jX_{i,j}H(t(1-\eta_j))}\right)\end{aligned} \tag{3.7}$$

where the last equality follows because all $Y_j,X_{i,j}$ are independent of $\eta_1,\dots,\eta_j$ and since all $X_{i,j}$ are exchangeable as they are independent and have identical distributions. By (3.2) we have that $\lambda_{0,i}=\lambda_0$ and thus,

$$\begin{aligned}&\mathbb{E}\Bigg(\exp\Big(-\sum_{i=1}^k\Big(\lambda_{0,i}H(t)+\sum_{j=1}^{M_t}Y_jX_{i,j}H(t-S_j)\Big)\Big)\Big|M_t=\ell\Bigg)\\&\qquad=e^{-k\lambda_0H(t)}\mathbb{E}\left(\exp\Big(-\sum_{i=1}^k\sum_{j=1}^{\ell}Y_jX_{i,j}H(t(1-\eta_j))\Big)\right).\end{aligned} \tag{3.8}$$

Next, we compute the expectation in (3.8). First,

$$\begin{aligned}&\mathbb{E}\left(e^{-\sum_{i=1}^k\sum_{j=1}^{\ell}Y_jX_{i,j}H(t(1-\eta_j))}\,\Big|\,Y_1=y_1,\dots,Y_\ell=y_\ell,\eta_1=z_1,\dots,\eta_\ell=z_\ell\right)\\&\quad=\mathbb{E}\left(\prod_{i=1}^k\prod_{j=1}^{\ell}e^{-y_jX_{i,j}H(t(1-z_j))}\right)=\prod_{i=1}^k\prod_{j=1}^{\ell}\mathbb{E}\left(e^{-y_jX_{i,j}H(t(1-z_j))}\right)\\&\quad=\prod_{i=1}^k\prod_{j=1}^{\ell}\varphi_X\big(y_jH(t(1-z_j))\big)\\&\quad=\prod_{j=1}^{\ell}\Big(\varphi_X\big(y_jH(t(1-z_j))\big)\Big)^k,\end{aligned} \tag{3.9}$$

where we used that $\{Y_j\}$ and $\{\eta_j\}$ are independent of $\{X_{i,j}\}$. Hence, by (3.9)

$$\begin{aligned}
&\mathbb{E}\left(e^{-\sum_{i=1}^{k}\sum_{j=1}^{\ell} Y_j X_{i,j} H(t(1-\eta_j))}\right)\\
&\qquad= \mathbb{E}\left(\prod_{j=1}^{\ell}\Big(\varphi_X\big(Y_j H(t(1-\eta_j))\big)\Big)^k\right)\\
&\qquad= \prod_{j=1}^{\ell}\mathbb{E}\left(\Big(\varphi_X\big(Y_j H(t(1-\eta_j))\big)\Big)^k\right)\\
&\qquad= \left[\mathbb{E}\left(\Big(\varphi_X\big(Y_1 H(t(1-\eta_1))\big)\Big)^k\right)\right]^{\ell}\\
&\qquad= \left[\int_{\mathbb{R}}\int_0^1 \big(\varphi_X\big(yH(tz)\big)\big)^k\, dz F_Y(dy)\right]^{\ell}
\end{aligned}$$

where the second equality follows from (3.9) and the last equality is due to the fact that $1-\eta$ is uniformly distributed on $[0,1]$. Finally, using the above results together with (3.8) and the definition of $G(t,k)$ in (3.4) and (3.6) we get

$$\begin{aligned}
G(t,k) &= \mathbb{E}\left(e^{-\sum_{i=1}^{k}\int_0^t \lambda_{s,i} ds}\right)\\
&= \sum_{\ell=0}^{\infty}\mathbb{E}\left(e^{-\sum_{i=1}^{k}\int_0^t \lambda_{s,i} ds}\,\middle|\, M_t=\ell\right)\mathbb{Q}\left(M_t=\ell\right)\\
&= \sum_{\ell=0}^{\infty} e^{-\rho t}\frac{(\rho t)^{\ell}}{\ell!}e^{-k\lambda_0 H(t)}\left[\int_{\mathbb{R}}\int_0^1 \big(\varphi_X\big(yH(tz)\big)\big)^k\, dz F_Y(dy)\right]^{\ell}\\
&= e^{-\rho t - k\lambda_0 H(t)}\cdot\exp\left(\rho t\int_{\mathbb{R}}\int_0^1 \big(\varphi_X\big(yH(tz)\big)\big)^k dz F_Y(dy)\right)
\end{aligned}$$

which concludes the proposition.

□

As already mentioned, the quantity $\mathbb{Q}(N_t=k)$ is central for pricing portfolio credit derivatives, and the fact that we are able to derive $\mathbb{Q}(N_t=k)$ explicitly up to the quantity $\int\int_0^1(\varphi_X(yH(tz)))^k dz F_Y(dy)$ is remarkable. Depending on φ_X this quantity can be computed explicitly. However, in the following section we show that also the numerical computation of this quantity is very feasible.

4. An explicit example

In this section we give an tractable and explicit example of the model presented in (2.1) under the assumption (3.2). To be more specific, we assume that

$$Y \in \{y_1, y_2\} \quad \text{where} \quad \mathbb{Q}(Y=y_1)=q \quad \text{and} \quad X \sim \chi^2(2) \tag{4.1}$$

where $y_1, y_2 \geq 0$. Hence, Y is a two-point distributed random variable and X has chi-squared distribution with 2 degrees of freedom. This result can be generalized in a number of ways. First, Y could have a finite number of states. Second, any distribution for X with has an closed form expression for its Laplace transform still leads to tractable formulas. We chose the stated formulation for simplicity and it is remarkable that it provides a good fit in our numerical examples. We can now state the following lemma,

Lemma 4.1. *Under* (3.2) *and* (4.1) *we have that*

$$G(t,k) = \exp\Big(- k\lambda_0 H(t) + \rho t \left[qI(y_1,k,t) + (1-q)I(y_2,k,t) - 1 \right] \Big) \tag{4.2}$$

where

$$I(y,k,t) := \int_0^1 \frac{1}{\left(1 + 2y\delta^{-1}\left(1 - e^{-\delta t z}\right)\right)^k} dz. \tag{4.3}$$

Proof. Recall that if $X \sim \chi^2(2)$ then $\varphi_X(s) = (1+2s)^{-1}$ and since $H(x) = \delta^{-1}\left(1 - e^{-\delta x}\right)$ we have

$$\varphi_X\left(yH(tz)\right) = \frac{1}{1 + 2y\delta^{-1}\left(1 - e^{-\delta t z}\right)}. \tag{4.4}$$

Since Y is a two-point distributed random variable where $Y \in \{y_1, y_2\}$ and $\mathbb{Q}\left(Y = y_1\right) = q$ we get

$$\int_{\mathbb{R}} \int_0^1 \Big(\varphi_X\big(yH(tz)\big) \Big)^k dz F_Y(dy) = qI(y_1,k,t) + (1-q)I(y_2,k,t) \tag{4.5}$$

where we define $I(y,k,t)$ as

$$I(y,k,t) := \int_0^1 \frac{1}{\left(1 + 2y\delta^{-1}\left(1 - e^{-\delta t z}\right)\right)^k} dz.$$

Finally, plugging (4.5) into (3.5) in Proposition 3.2 yields (4.2). □

It is possible to obtain analytical expressions for the integrals $I(y,k,t)$, however as k increases these become quite long and tedious. In practice we evaluate $I(y,k,t)$ using numerical quadrature. However, for $k=1$, we can simplify (4.2) as stated in the following lemma.

Lemma 4.2. *Under* (3.2) *and* (4.1) *we have that*

$$\mathbb{Q}\left(\tau_i > t\right) = e^{-\lambda_0 H(t) + ct} \left[1 + 2y_1\delta^{-1}(1 - e^{-\delta t})\right]^{\frac{\rho q}{\delta + 2y_1}} \left[1 + 2y_2\delta^{-1}(1 - e^{-\delta t})\right]^{\frac{\rho(1-q)}{\delta + 2y_2}} \tag{4.6}$$

where c is given by

$$c = \rho\left(\frac{q}{1 + 2y_1\delta^{-1}} + \frac{1-q}{1 + 2y_2\delta^{-1}} - 1 \right).$$

Proof. From (2.2) and (3.4) we get

$$\mathbb{Q}\left(\tau_i > t\right) = \mathbb{E}\left(\exp\left(- \int_0^t \lambda_{s,i} ds \right) \right) = G(t,1)$$

and by Lemma 4.1 with $k = 1$ we have

$$G(t,1) = e^{-\lambda_0 H(t)} \exp\left(\rho t \left[qI(y_1,1,t) + (1-q)I(y_2,1,t) - 1\right]\right) \tag{4.7}$$

where $I(y,1,t)$ is given by (4.3) with $k = 1$ viz.

$$I(y,1,t) := \int_0^1 \frac{1}{1 + 2y\delta^{-1}\left(1 - e^{-\delta t z}\right)} dz. \tag{4.8}$$

Furthermore, note that (see e.g. p.171 in Råde & Westergren (1995))

$$\int \frac{1}{b + ce^{az}} dz = \frac{z}{b} - \frac{1}{ab} \ln |b + ce^{az}|$$

and this observation with (4.8) yield

$$I(y,1,t) = \frac{1}{1 + 2y\delta^{-1}} + \frac{1}{t(\delta + 2y)} \ln\left[1 + 2y\delta^{-1}(1 - e^{-\delta t})\right]. \tag{4.9}$$

Next, some calculations renders

$$\begin{aligned}\rho t\big(qI(y_1,1,t) + (1-q)I(y_2,1,t) - 1\big) &= \rho t\left(\frac{q}{1 + 2y_1\delta^{-1}} + \frac{1-q}{1 + 2y_2\delta^{-1}} - 1\right) \\ &+ \frac{\rho q}{\delta + 2y_1} \ln\left[1 + 2y_1\delta^{-1}(1 - e^{-\delta t})\right] + \frac{\rho(1-q)}{\delta + 2y_2} \ln\left[1 + 2y_2\delta^{-1}(1 - e^{-\delta t})\right]\end{aligned}$$

and plugging this into (4.7) yields (4.6). □

It is straightforward to generalize Lemma 4.1 and Lemma 4.2 to the following setup

Corollary 4.3. *Assume that*

$$Y \in \{y_1, y_2, \ldots, y_M\} \quad \text{where} \quad \mathbb{Q}(Y = y_j) = q_j \quad \text{and} \quad X \sim \chi^2(2) \tag{4.10}$$

and $y_j \geq 0$ *for each* $j = 1, \ldots, M$. *Then under* (3.2) *it holds that*

$$G(t,k) = \exp\left(-k\lambda_0 H(t) + \rho t\left[\sum_{j=1}^{M} q_j I(y_j,k,t) - 1\right]\right)$$

with $I(y,k,t)$ *defined by* (4.3). *Furthermore,*

$$\mathbb{Q}(\tau_i > t) = e^{-\lambda_0 H(t) + ct} \prod_{j=1}^{M} \left[1 + 2y_j\delta^{-1}(1 - e^{-\delta t})\right]^{\frac{\rho q_j}{\delta + 2y_j}}$$

where c *is given by*

$$c = \rho\left(\sum_{j=1}^{M} \frac{q_j}{1 + 2y_j\delta^{-1}} - 1\right).$$

5. Pricing CDS and basket default swaps

In this section we give a short description of the single-name CDS spread and kth-to default swaps. We will focus on a homogeneous portfolio described by the model (2.1) with condition (3.2). First, Subsection 5.1 presents formulas for the single-name CDS spread in this model. Then, Subsection 5.2 outlines the kth-to-default swap. In the sequel all computations are assumed to be made under a risk-neutral martingale measure $\mathbb{Q}$. Typically such a $\mathbb{Q}$ exists if we rule out arbitrage opportunities. Further, we assume the that risk-free interest rate is a deterministic constant given by r.

5.1. **Pricing the single-name CDS.** In this subsection we give a short description of a single-name credit default swap, which is one of our calibration instruments.

Consider a obligor $\boldsymbol{C}$ with default time τ and recovery rate R. A single-name credit default swap (CDS) with maturity T where the reference entity is obligor $\boldsymbol{C}$, is a bilateral contract between two counterparties, $\boldsymbol{A}$ and $\boldsymbol{B}$, where $\boldsymbol{B}$ promises to pay $\boldsymbol{A}$ the credit losses $(1-R)$ at τ if the obligor defaults before time T. As compensation for this, **A** pays $S\Delta$ to the protection seller $\boldsymbol{B}$, at $0 < t_1 < t_2 < \ldots < t_N = T$ or until $\tau < T$. We assume that the payment dates are equidistant, i.e. $\Delta = t_n - t_{n-1}$ for any n. The CDS spread S is determined so that expected discounted cashflows between $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal when the CDS contract is settled at $t = 0$. Assuming a constant interest rate r and deterministic recovery rate implies that S is given by

$$S = \frac{(1-R)\int_0^T e^{-rs} dF(s)}{\Delta \sum_{n=1}^{n_T} e^{-rt_n}(1-F(t_n))}$$

where $F(t) = \mathbb{Q}(\tau \le t)$ is the distribution functions of the default time for the obligor $\boldsymbol{C}$.

In practice, if $\tau \in [t_n, t_{n+1}]$, then $\boldsymbol{A}$ will also pay $\boldsymbol{B}$ the accrued default premium up to τ, see e.g. in Herbertsson & Rootzén (2008). In this paper we have ignored the accrued payments in our model of the CDS spread. The effect of ignoring the accrued premium is very small, see e.g. p.428 in McNeil et al. (2005).

5.2. **Pricing kth-to-default swaps.** A k^{th}-to default swap is a generalization of a the single-name credit default swap, to a portfolio of m obligors. It pays protection at the kth default in the portfolio. To be more specific, consider a basket of m bonds each with notional N, issued by m obligors with default times $\tau_1, \tau_2, \ldots, \tau_m$ and recovery rates $R_1, R_2, \ldots, R_m$. Further, let $T_1 < \ldots < T_m$ be the ordering of $\tau_1, \tau_2, \ldots, \tau_m$. A kth-to-default swap with maturity T on this basket is a bilateral contract between two counterparties, $\boldsymbol{A}$ and $\boldsymbol{B}$, where $\boldsymbol{B}$ promises $\boldsymbol{A}$ to pay the credit losses that $\boldsymbol{B}$ suffers at T_k if $T_k < T$. Just as in the CDS, $\boldsymbol{A}$ pays be $\boldsymbol{B}$ a fee up to the default time T_k or until T, whichever comes first. The payments dates are identical to those in the CDS case and the fee is $S^{(k)}\Delta$ where Δ is as previously $t_n - t_{n-1}$ and we assumed equidistant payment dates. The main difference lies in the default payment at T_k. If $T_k < T$, $\boldsymbol{B}$ pays $\boldsymbol{A}$ $N(1-R_i)$ if it was obligor i which defaulted at time T_k. However, in a homogeneous portfolio $R_1 = R_2 = \ldots = R_m = R$ so the payment at T_k is always $N(1-R)$. The kth-to-default spread $S^{(k)}$ is expressed in bp per annum and determined so that the expected discounted cash-flows between $\boldsymbol{A}$ and $\boldsymbol{B}$ coincide at $t = 0$. Assuming the same conditions as in the CDS, we therefore have

$$S^{(k)} = \frac{(1-R)\int_0^T e^{-rs} dF_k(s)}{\Delta \sum_{n=1}^{n_T} e^{-rt_n}(1 - F_k(t_n))}.$$

Here $F_k(t) = \mathbb{Q}(T_k \leq t)$ is the distribution functions of the ordered default times. The rest of the notation are the same as in the CDS contract. We also observe that $1 - F_k(t) = \mathbb{Q}(T_k > t) = \mathbb{Q}(N_t < k)$, so the kth-to-default spread in a homogeneous model is completely determined by the distribution for N_t. To be more specific, recall that

$$\mathbb{Q}(T_k > t) = \mathbb{Q}(N_t < k) = \sum_{j=0}^{k-1} \mathbb{Q}(N_t = j)$$

where $\mathbb{Q}(N_t = j)$ is computed by using Lemma 3.1. Furthermore, note that for $k \leq m - 1$ we have

$$\mathbb{Q}(T_{k+1} > t) = \mathbb{Q}(T_k > t) + \mathbb{Q}(N_t = k)$$

which is useful from computational point of view when finding the survival distribution $F_k(t)$ for several $k = 1, 2, \ldots, \ell$ where $\ell \leq m$.

6. Some numerical examples

In the following we illustrate the remarkable tractability of this simple model. It is easily able to capture average CDS spreads of homogenous portfolios and the parameters give an intuitive interpretation. We will in this section use the model (4.1) presented in Section 4.

Subsection 6.2 studies the kth-to default spreads as function of the average CDS spread in the portfolio. Further, in Subsection 6.2 we investigate how the decay rate δ affect the kth-to default spreads when keeping the CDS spread constant. In the sequel the interest is set to 3%, the recovery is 40% and the maturity for the credit derivatives are 5 years with quarterly payment frequency. Finally, we will in this section for simplicity set $\lambda_0 = 0$ in all computations.

6.1. The kth-to-default spreads as function of the CDS spread.

Consider a homogeneous portfolio with 10 obligors, satisfying the model specified in (2.1), (3.2) and (4.1). We let the average CDS spread S vary between 50 bps to 300 bps in steps of 10 bps. We calibrate the model CDS spread against each such spread S and then compute the kth-to default swap for $1 \leq k \leq 7$. For each fixed spread S_n say, the initial parameters are chosen to be the calibrated parameters for the previous S_{n-1}. For the first spread , i.e. for S_1, the initial parameters are chosen to $y_1 = 0.009, y_2 = 0.05, q = 0.55, \rho = 0.3$ and $\delta = 0.75$. The results for the basket spreads are displayed in Figure 1 and 2. As expected, the kth-to default swap spreads are increasing with S. Note that the curves turn from a concave to a convex shape as k increases. Furthermore, in the calibrations the absolute error never exceeds one thousand of a bp.

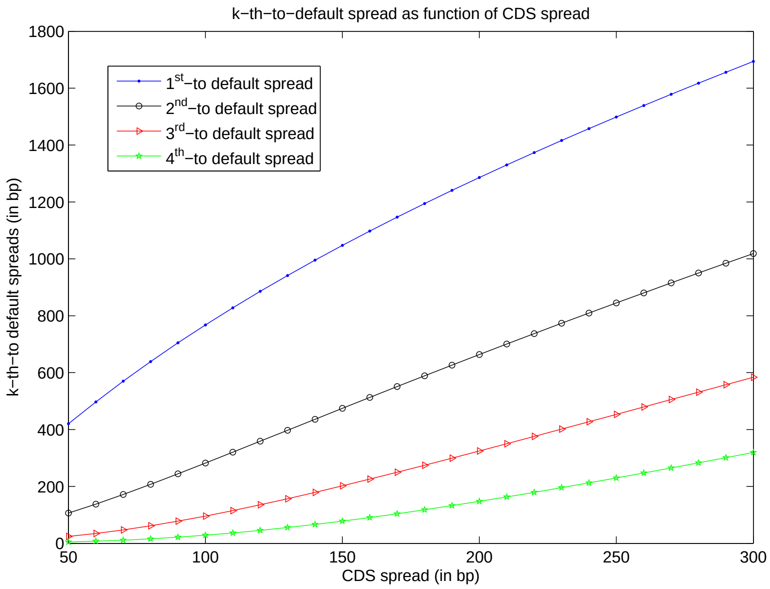

Figure 1. The different kth-to-default spreads for $k \leq 4$ as a function of the CDS-spread. The portfolio consists of 10 obligors.

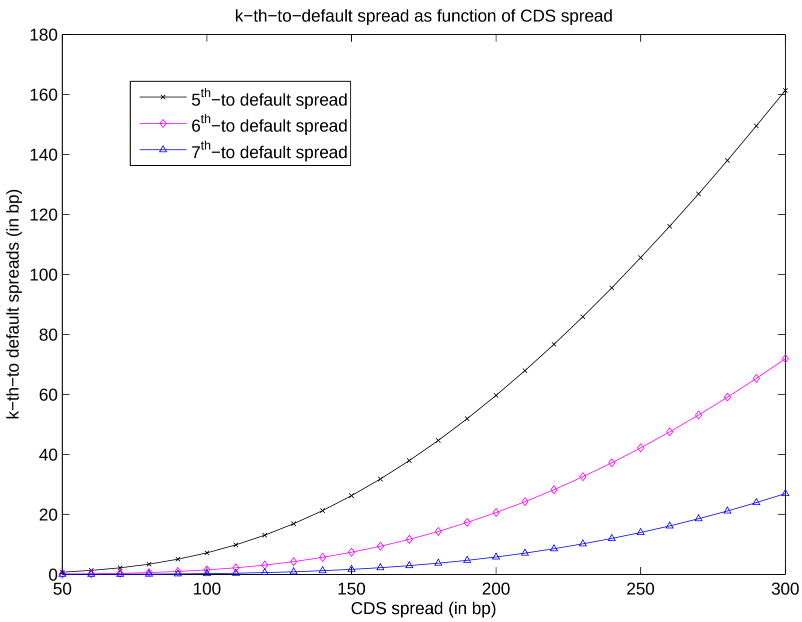

Figure 2. The different kth-to-default spreads for $5 \leq k \leq 7$ as a function of the CDS-spread. The portfolio consists of 10 obligors.

The calibrated "implied" parameters $y_1, y_2, q, \rho, \delta$ as function of the CDS spread are displayed in Figure 3, which also shows the implied five-year default probability as function of the CDS spread. From this figure we conclude that the implied y_1, y_2 and ρ are increasing with the CDS spread. This observation is intuitively clear from the model setup in (2.1). Furthermore, q is also decreasing with an increasing CDS spread, although on a narrow interval.

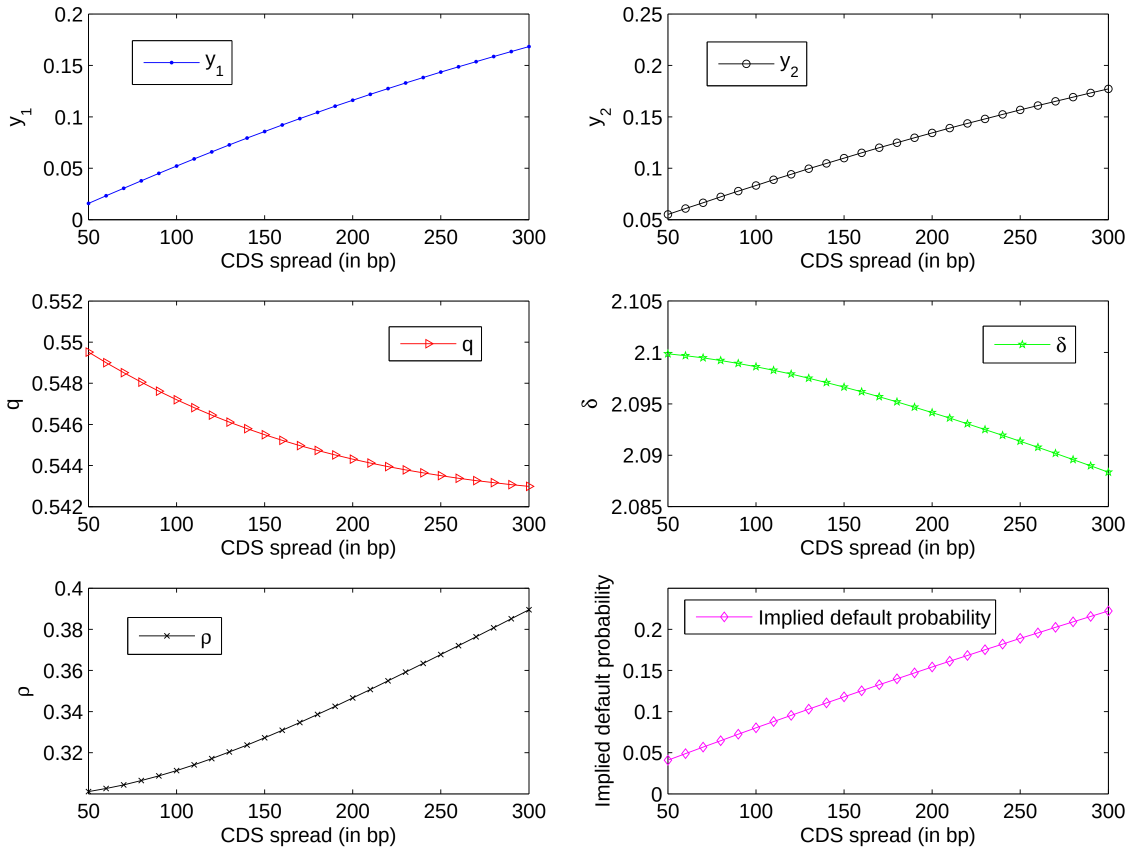

Figure 3. The calibrated parameters and the implied five year default probability as function of the CDS-spread.

6.2. **The kth-to-default spreads as function of the decay rate δ.** In this subsection we investigate how the basket defaults spreads are affected by the decay rate δ in the model given by (2.1) and (3.2).

We again consider a homogeneous portfolio with 10 obligors, satisfying (2.1) and (3.2). The average CDS spread is 150 bps. We vary the decay rate δ between 0.5 to 28 in steps of 0.5. For each δ, the model is calibrated so that the CDS spread is maintained at 150 bps, and we then compute the kth-to-default swap for $1 \leq k \leq 7$. Hence, in each calibration, δ is fixed, and the rest of the parameters are found so that the model CDS spread will be 150 bp. For each fixed δ_n say, the initial parameters are chosen to be the calibrated parameters for the previous ρ_{n-1}. For the first δ used, i.e. for $\delta_1 = 0.5$, the initial parameters are chosen to $y_1 = 0.1, y_2 = 0.3, q = 0.75, \rho = 0.04$. In the calibration for each δ, the absolute calibration error (i.e. model CDS spread minus market CDS spread) never exceeds one thousand of a bp.

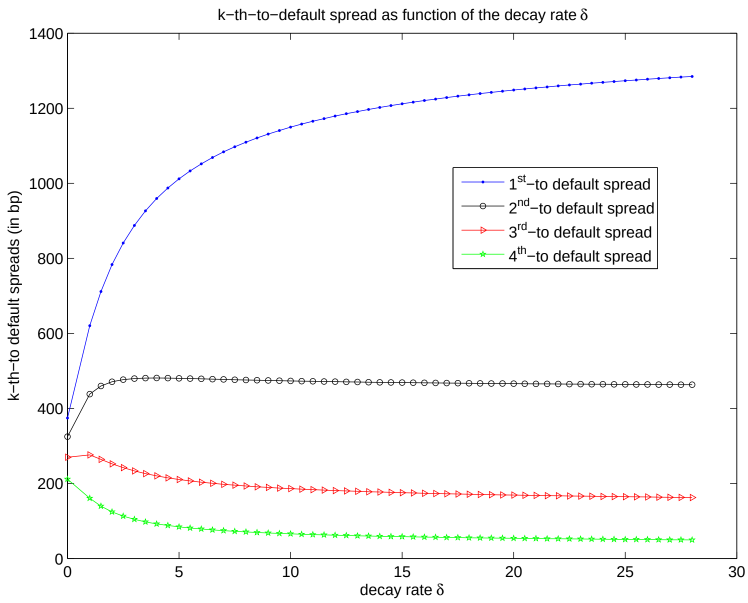

Figure 4. The kth-to-default spreads for $k \leq 4$ as function of the decay rate δ. The CDS spread is 150 bps and the portfolio consists of 10 obligors.

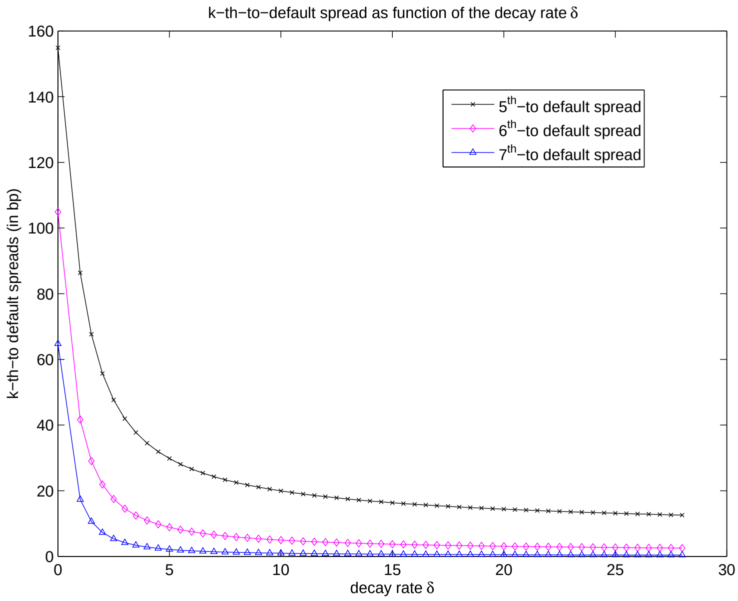

Figure 5. The kth-to-default spreads for $5 \leq k \leq 7$ as function of the decay rate δ. The CDS spread is 150 bps and the portfolio consists of 10 obligors.

From Figure 4 and Figure 5 we see that the kth-to- default swaps tend to converge to constant for each $k > 2$ as δ increases.

Another interesting observation is that the calibrated parameters y_1, y_2, ρ all increase as δ increase, see Figure 6. Intuitively this is clear, since as δ increase, the individual default probability will decrease. Hence, since we are holding the CDS spread constant, some of the parameters of y_1, y_2, ρ must increase in order to compensate for the "marginal" loss effect on the default distribution for the obligor, as δ increase.

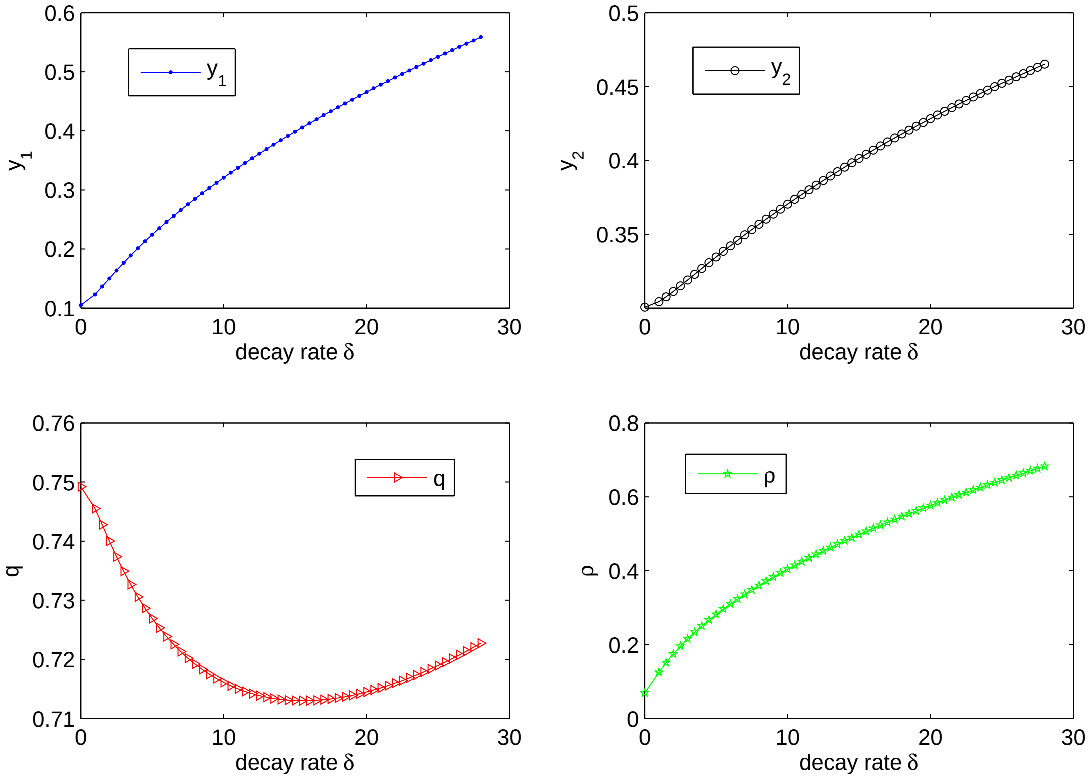

Figure 6. The calibrated parameters as a function of the decay rate δ. The CDS spread is 150 bps.

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